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Unitary SK_1 for a graded division ring and its quotient division ring

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ABSTRACT

Let E be a graded division ring finite-dimensional over its center with torsion-free abelian grade group, and let $q(\mathsf{E})$ be its quotient division ring. Let τ be a degree-preserving unitary involution on E. We prove that $SK_1(\mathsf{E},\tau)\cong SK_1(q(\mathsf{E}),\tau)$.

1. Introduction

Let D be a division algebra finite-dimensional over its center K. Let τ be a unitary involution on D, i.e., τ is an antiautomorphism of D with $\tau^2 = id$ such that $\tau|_K \neq id$. By definition,

$$SK_1(D,\tau) = \Sigma'_\tau(D)/\Sigma_\tau(D),$$

where

$$\Sigma_\tau(D) = \big\langle \big\{ a \in D^* \mid a = \tau(a) \big\} \big\rangle$$

and

$$\Sigma'_\tau(D) = \big\{ a \in D^* \mid Nrd_D(a) = \tau\big(Nrd_D(a)\big) \big\}.$$

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(Other notation has been used in past for what we call $\mathrm{SK}_1(D,\tau)$, including $\mathrm{USK}_1(D)$ in [5], $\mathrm{SK}_1\mathrm{U}(\tau, D)$ in [6], and $\mathrm{SUK}_1(D)$ in [9].)

Analogously, suppose $\mathsf{E} = \bigoplus_{\gamma\in\Gamma_\mathsf{E}} \mathsf{E}_\gamma$ is a graded division algebra (i.e., a graded ring in which all nonzero homogeneous elements are units) of finite rank over its center Z, and with torsion-free abelian grade group Γ_E. Suppose E has a unitary graded involution τ, which is a ring antiautomorphism with $\tau(\mathsf{E}_\gamma) = \mathsf{E}_\gamma$ for each γ, such that $\tau^2 = id$ and $\tau|_\mathsf{Z} \neq id$. Then there is a reduced norm map $Nrd_\mathsf{E} : \mathsf{E} \to \mathsf{Z}$, and one can define $\Sigma_\tau(\mathsf{E})$, $\Sigma'_\tau(\mathsf{E})$, and $\mathrm{SK}_1(\mathsf{E},\tau)$ just as we did above for D. Any such E has a ring of (central) quotients, $q(\mathsf{E}) = \mathsf{E} \otimes_\mathsf{Z} q(\mathsf{Z})$, where $q(\mathsf{Z})$ is the quotient field of the integral domain Z, and it is known that $q(\mathsf{E})$ is a division ring finite-dimensional over its center, which is $q(\mathsf{Z})$. The unitary graded involution τ on E extends canonically to a unitary involution on $q(\mathsf{E})$, also called τ. In this paper we will prove the following:

Theorem 1.1. *Let E be a graded division algebra (with torsion-free abelian grade group) finite-dimensional over its center, and let τ be a unitary graded involution on E. Let $q(\mathsf{E})$ be the quotient division ring of E and let τ denote also the extension of τ to a unitary involution of $q(\mathsf{E})$. Then,*

$$\mathrm{SK}_1(\mathsf{E},\tau) \cong \mathrm{SK}_1\big(q(\mathsf{E}),\tau\big).$$

This theorem complements a result in [2]: Suppose D is a division algebra finite-dimensional over its center $K = Z(D)$, and suppose K has a Henselian valuation v. It is well known that v has a unique extension to a valuation on D. The filtration on D induced by the valuation yields an associated graded ring $\mathsf{gr}(D)$ which is a graded division algebra of finite rank over its center. If τ is a unitary involution on D which is compatible with the valuation, there is an induced graded involution $\widetilde{\tau}$ on $\mathsf{gr}(D)$. It was shown in [2, Th. 3.5] that if the restriction of v to the τ-fixed field K^τ is Henselian and K is tamely ramified over K^τ, then τ is compatible with v, $\widetilde{\tau}$ is unitary, and

$$\mathrm{SK}_1(D,\tau) \cong \mathrm{SK}_1\big(\mathsf{gr}(D),\widetilde{\tau}\big). \tag{1.1}$$

This theorem served to focus attention on unitary SK_1 for graded division algebras. The graded division algebra $\mathsf{gr}(D)$ has a significantly simpler structure than the valued division algebra D; notably $\mathsf{gr}(D)$ has a much more tractable group of units (as they are all homogeneous). Consequently, SK_1 calculations are often substantially easier for $\mathsf{gr}(D)$ than for D. This was demonstrated in [2] and [7], where many formulas for $\mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau})$ were proved. By that approach new results on $\mathrm{SK}_1(D,\tau)$ were obtained, as well as new and simpler proofs of results that had previously been proved by calculations that were often quite complicated. Now, by virtue of the theorem proved here, all the results proved in [2] and [7] on $\mathrm{SK}_1(\mathsf{E},\tau)$ for a graded division algebra E carry over to yield corresponding results for $\mathrm{SK}_1(q(\mathsf{E}),\tau)$.

One of the key results for unitary SK_1 is the "Stability Theorem," which says that the unitary SK_1 is unchanged on passage from a division algebra D to a rational division algebra over D. (This was originally proved in [10, §23].) We will show at the end of Section 4 that the Stability Theorem is a quick corollary of our theorem.

When the torsion-free abelian grade group Γ_E of a graded division algebra E is finitely-generated (hence a free abelian group), E has a concrete description as an iterated twisted Laurent polynomial ring over the division ring E_0, as follows: Take any homogeneous $x_1,\dots,x_n$ in the group E^* of units of E such that $\Gamma_\mathsf{E} = \mathbb{Z}\,deg(x_1) \oplus \dots \oplus \mathbb{Z}\,deg(x_n)$. Then,

$$\mathsf{E} = \mathsf{E}_0\big[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}; \sigma_1, \dots, \sigma_n\big],$$

i.e., $\mathsf{E} = \mathsf{E}_n$, where for $i = 1, 2, \dots, n$, $\mathsf{E}_i = \mathsf{E}_{i-1}[x_i, x_i^{-1}; \sigma_i]$, which is a Laurent polynomial ring over E_{i-1} with multiplication twisted by the relation $x_i c = \sigma_i(c) x_i$ for all $c \in \mathsf{E}_{i-1}$. Each σ_i is a graded (i.e., degree-preserving) automorphism of E_{i-1}, and σ_i is completely determined by its action on E_0 and on $x_1, x_2, \dots, x_{i-1}$. To assure that E is finite-dimensional over its center, it is assumed that some power of σ_i is an inner automorphism of E_{i-1}. The quotient division ring $q(\mathsf{E})$ is the iterated twisted rational division algebra

$$A_n = \mathsf{E}_0(x_1, \dots, x_n; \sigma_1, \dots, \sigma_n), \tag{1.2}$$

which is also the quotient division algebra of the iterated twisted polynomial ring $\mathsf{E}_0[x_1, \dots, x_n; \sigma_1, \dots, \sigma_n]$.

In [12] the second author gave formulas for $\mathrm{SK}_1(A_n, \tau_n)$ for the iterated twisted rational division algebras A_n as in (1.2) above, for a unitary involution τ_n on A_n arising from a unitary graded involution on the iterated twisted polynomial ring $\mathsf{E}_0[x_1, \dots, x_n; \sigma_1, \dots, \sigma_n]$. The crucial identity

$$\Sigma'_{\tau_n}(A_n) = \big(\Sigma'_{\tau_n}(A_n) \cap \mathsf{E}_0\big) \cdot \Sigma_{\tau_n}(A_n) \tag{1.3}$$

is given in that paper, with a much too brief and incomplete sketch of proof. (More detailed proofs of this identity and other results in [12] are given in this author's habilitation, which, regrettably, was never published. A complete proof of the $n = 1$ case was given in [10].) As will be seen in Section 4 below, the proof of this identity, stated there as (4.1), constitutes the bulk of the proof of our theorem. Our proof of this identity is by induction on n, as is done in [12], but the proof differs substantially, in that it is carried out as much as possible by calculations in the divisor group $Div(T)$ of the twisted polynomial ring in one variable $T = \mathsf{E}_0[x_1; \sigma_1]$. This $Div(T)$ is a free abelian group, and calculations with it are significantly simpler than with the quotient division ring $q(T)$. The style of proof here is analogous to the approach in [1, §5] to proving the corresponding result for the nonunitary $\mathrm{SK}_1(\mathsf{E})$ of a graded division algebra.

Th. 1.1 above, when combined with (1.1), provides a unifying perspective for understanding why there is such a great similarity between the formulas for $\mathrm{SK}_1(D, \tau)$ for D a division algebra over a Henselian field, as in [9] and [13] and the formulas for $\mathrm{SK}_1(\mathsf{E}_0(x_1, \dots, x_n; \sigma_1, \dots, \sigma_n), \tau_n)$ given in [10] and [12]: The formulas in each case coincide with formulas for $\mathrm{SK}_1(\mathsf{E}, \tau)$ for a related graded division algebra E.

Th. 1.1 and formula (1.1) are unitary analogues to results for nonunitary SK_1 for division algebras over Henselian fields, graded division algebras, and their associated quotient division algebras given in [1, Th. 4.8, Th. 5.7]. This is another manifestation of the philosophy that results about SK_1 ought to have corresponding results for the unitary SK_1. From the perspective of algebraic groups, this philosophy is motivated by the fact that division algebras are associated with algebraic groups of inner type A_n, while division algebras with unitary involution are associated with algebraic groups of outer type A_n (see, e.g., [5, Ch. VI]).

2. Graded division algebras and unitary involutions

In this section we recall some basic known facts about graded division algebras and unitary involutions which will be used in the proof of Th. 1.1. A good reference for the properties of graded division algebras stated here without proof is [3].

Let Γ be a torsion-free abelian group. A ring E is a *graded division ring* (with grade group in Γ) if E has additive subgroups E_γ for $\gamma \in \Gamma$ such that $\mathsf{E} = \bigoplus_{\gamma \in \Gamma} \mathsf{E}_\gamma$ and $\mathsf{E}_\gamma \cdot \mathsf{E}_\delta = \mathsf{E}_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$, and each $E_\gamma \setminus \{0\}$ lies in E^*, the group of units of E. The grade group of E is

$$\Gamma_{\mathsf{E}} = \big\{\gamma \in \Gamma \bigm| \mathsf{E}_\gamma \neq \{0\}\big\},$$

a subgroup of Γ. For $a \in \mathsf{E}_\gamma \setminus \{0\}$ we write $deg(a) = \gamma$. A significant property is that $\mathsf{E}^* = \bigcup_{\gamma \in \Gamma_{\mathsf{E}}} E_\gamma \setminus \{0\}$, i.e., every unit of E is actually homogeneous. (Γ_{E} torsion-free is needed for this.) Thus, E is not a division ring if $|\Gamma_{\mathsf{E}}| > 1$. But, E has no zero divisors. (This also depends on Γ_{E} being torsion-free.) However, E_0 is a division ring, and each E_γ ($\gamma \in \Gamma_{\mathsf{E}}$) is a 1-dimensional left- and right-E_0-vector space.

Let M be any graded left E-module, i.e., M is a left E-module with additive subgroups M_γ such that $\mathsf{M} = \bigoplus_{\gamma\in\Gamma} \mathsf{M}_\gamma$ and $\mathsf{E}_\gamma \cdot \mathsf{M}_\varepsilon \subseteq \mathsf{M}_{\gamma+\varepsilon}$ for all $\gamma, \varepsilon \in \Gamma$. Then, M is a free E-module with homogeneous base, and any two such bases have the same cardinality, which is called the dimension, $dim_{\mathsf{E}}(\mathsf{M})$; M is therefore said to be a graded vector space over E.

Let $\mathsf{Z} = Z(\mathsf{E})$, the center of E, which is a graded subring of E. Indeed, Z is a *graded field*, i.e., a commutative graded division ring. Then E is a left (and right) graded Z-vector space, and we write $[\mathsf{E}:\mathsf{Z}]$ for $dim_{\mathsf{Z}}(\mathsf{E})$. In this paper we work exclusively with finite-dimensional graded division algebras, i.e., those E with $[\mathsf{E}:\mathsf{Z}] < \infty$. Clearly Z_0 is a field, and E_0 is a finite-dimensional Z_0-algebra. Moreover, Γ_{Z} is a subgroup of Γ_{E}, and it is easy to verify the "fundamental equality"

$$[\mathsf{E}:\mathsf{Z}] = [\mathsf{E}_0:\mathsf{Z}_0]|\Gamma_{\mathsf{E}}:\Gamma_{\mathsf{Z}}|. \tag{2.1}$$

We have $\mathsf{Z}_0 \subseteq Z(\mathsf{E}_0) \subseteq \mathsf{E}_0$. Let $\mathcal{G}(Z(\mathsf{E}_0)/\mathsf{Z}_0)$ be the Galois group for the finite-degree field extension $Z(\mathsf{E}_0)$ of Z_0. There is a well-defined canonical map

$$\theta_{\mathsf{E}} : \Gamma_{\mathsf{E}} \to \mathcal{G}\big(Z(\mathsf{E}_0)/\mathsf{Z}_0\big) \quad \text{given by} \quad \theta_{\mathsf{E}}\big(deg(a)\big) : c \mapsto aca^{-1} \quad \text{for all } a \in \mathsf{E}^*,\ c \in \mathsf{E}_0.$$

Clearly $\Gamma_{\mathsf{Z}} \subseteq ker(\theta_{\mathsf{E}})$, so $|im(\theta_E)| \leqslant |\Gamma_{\mathsf{E}}:\Gamma_{\mathsf{Z}}| < \infty$. Moreover, the fixed field of $im(\theta_{\mathsf{E}})$ is Z_0. Hence, $Z(\mathsf{E}_0)$ is Galois over Z_0 with abelian Galois group $\mathcal{G}(Z(\mathsf{E}_0)/\mathsf{Z}_0) = im(\theta_{\mathsf{E}})$.

Since Z is a commutative ring with no zero divisors, it has a quotient field $q(\mathsf{Z})$. Then E has its ring of central quotients

$$q(\mathsf{E}) = \mathsf{E} \otimes_{\mathsf{Z}} q(\mathsf{Z}).$$

Because E is a free, hence torsion-free Z-module, the canonical map $\mathsf{E} \to q(\mathsf{E})$, $a \mapsto a \otimes 1$, is injective. Therefore, we view E as a subring of $q(\mathsf{E})$. Note that $q(\mathsf{E})$ has no zero divisors since E has none. Furthermore, $q(\mathsf{E})$ is a $q(\mathsf{Z})$-algebra with $[q(\mathsf{E}):q(\mathsf{Z})] = [\mathsf{E}:\mathsf{Z}] < \infty$. Hence, $q(\mathsf{E})$ is a division ring, called the quotient division algebra of E. Clearly, $Z(q(\mathsf{E})) = q(\mathsf{Z})$. The index of E is defined to be $ind(\mathsf{E}) = \sqrt{[\mathsf{E}:\mathsf{Z}]} = ind(q(\mathsf{E})) \in \mathbb{Z}$.

It is known that E is an Azumaya algebra over Z, and hence from general principles that there is a reduced norm map $Nrd_{\mathsf{E}} : \mathsf{E} \to \mathsf{Z}$. In fact, by [1, Prop. 3.2(i)], Nrd_{E} coincides with the restriction to E of the usual reduced norm $Nrd_{q(E)}$ on $q(\mathsf{E})$. Also, by [1, Prop. 3.2(iv)], for $a \in \mathsf{E}_0$, we have

$$Nrd_{\mathsf{E}}(a) = N_{Z(\mathsf{E}_0)/\mathsf{Z}_0}\big(Nrd_{\mathsf{E}_0}(a)\big)^{\lambda} \quad \text{where } \lambda = ind(\mathsf{E})/\big(ind(\mathsf{E}_0)\cdot\big[Z(\mathsf{E}_0):\mathsf{Z}_0\big]\big), \tag{2.2}$$

where Nrd_{E_0} is the reduced norm for E_0 and $N_{Z(\mathsf{E}_0)/\mathsf{Z}_0}$ is the field norm from $Z(\mathsf{E}_0)$ to Z_0.

A graded involution on the graded division algebra E is a ring antiautomorphism $\tau : \mathsf{E} \to \mathsf{E}$ such that $\tau^2 = id_{\mathsf{E}}$ and $\tau(\mathsf{E}_\gamma) = \mathsf{E}_\gamma$ for each $\gamma \in \Gamma$. Such a τ is said to be *unitary* (or of the second kind) if $\tau|_{\mathsf{Z}} \neq id_{\mathsf{Z}}$ where $\mathsf{Z} = Z(\mathsf{E})$. Assuming τ is unitary, let $\mathsf{F} = \mathsf{Z}^\tau = \{c \in \mathsf{Z} \mid \tau(c) = c\}$, which is a graded subfield of Z with $[\mathsf{Z}:\mathsf{F}] = 2$. It follows from the fundamental equality that either $[\mathsf{Z}_0:\mathsf{F}_0] = 2$ and $\Gamma_{\mathsf{Z}} = \Gamma_{\mathsf{F}}$ or $\mathsf{Z}_0 = \mathsf{F}_0$ and $|\Gamma_{\mathsf{Z}}:\Gamma_{\mathsf{F}}| = 2$. The second case, where the involution induced by τ on E_0 is not unitary, tends to be uninteresting, but it can occur. We write τ also for the induced involution $\tau \otimes id_{q(\mathsf{F})}$ on $q(\mathsf{E}) = \mathsf{E} \otimes_{\mathsf{F}} q(\mathsf{F})$. Recall that for all $a \in \mathsf{E}$ we have

$$Nrd_{\mathsf{E}}\big(\tau(a)\big) = \tau\big(Nrd_E(a)\big), \tag{2.3}$$

since this equality holds for $Nrd_{q(\mathsf{E})}$. (For, if $a \in q(\mathsf{E})$, then $Nrd_{q(\mathsf{E})}(a)$ is determined by the minimal polynomial p_a of a over $Z(q(\mathsf{E}))$, and $p_{\tau(a)} = \tau(p_a)$.)

If τ' is another unitary graded involution on E, we write $\tau \sim \tau'$ if $\tau|_{\mathsf{Z}} = \tau'|_{\mathsf{Z}}$. In particular, for any $c \in \mathsf{E}^*$, if $\tau(c)c^{-1} \in \mathsf{Z}^*$, then $\tau' = int(c) \circ \tau$ is a unitary graded involution on E, and $\tau' \sim \tau$. Here $int(c)$ denotes the inner automorphism $a \mapsto cac^{-1}$ of E. Since c is homogeneous, $int(c)$ is clearly a graded (i.e., degree-preserving) automorphism of E.

For a unitary graded involution τ on E, set $S_\tau(\mathsf{E}) = \{a \in \mathsf{E}^* \mid \tau(a) = a\}$, the set of symmetric units of E, and set

$$\Sigma_\tau(\mathsf{E}) = \langle S_\tau(\mathsf{E})\rangle \quad \text{and} \quad \Sigma'_\tau(\mathsf{E}) = \{a \in \mathsf{E}^* \mid Nrd_{\mathsf{E}}(a) \in S_\tau(\mathsf{E})\}.$$

Then, by definition, $\mathrm{SK}_1(\mathsf{E}, \tau) = \Sigma'_\tau(\mathsf{E})/\Sigma_\tau(\mathsf{E})$.

We recall a few fundamental facts about unitary involutions on ungraded division algebras which have analogues for graded division algebras:

Lemma 2.1. *Let D be a division algebra finite-dimensional over its center, and let τ and τ' be unitary involutions on D.*

(a) *Suppose $\tau \sim \tau'$ (i.e., $\tau|_{Z(D)} = \tau'|_{Z(D)}$). Then $\Sigma'_{\tau'}(D) = \Sigma'_\tau(D)$ and $\Sigma_{\tau'}(D) = \Sigma_\tau(D)$, so $\mathrm{SK}_1(D, \tau') = \mathrm{SK}_1(D, \tau)$.*
(b) *$[D^*, D^*] \subseteq \Sigma_\tau(D)$, where $[D^*, D^*] = \langle aba^{-1}b^{-1} \mid a, b \in D^*\rangle$.*

For a proof of (a), see [8, Lemma 1], and for (b) see [5, Prop. 17.26, p. 267]. Part (b) was originally proved by Platonov and Yanchevskiĭ. See [2, Remark 4.1(ii), Lemma 2.3(iv)] for the graded versions of (a) and (b).

3. The divisor group of a twisted polynomial ring

The proof of Th. 1.1, both in the case $\Gamma_{\mathsf{E}} \cong \mathbb{Z}$ and also in the induction argument for $\Gamma_{\mathsf{E}} \cong \mathbb{Z}^n$ will use properties of twisted polynomial rings in one variable over a division ring. In this section we give the properties we need about the divisor group of such a twisted polynomial ring.

Let D be a division ring finite-dimensional over its center K, and let σ be an automorphism of D whose restriction to K has finite order, and let T be the twisted polynomial ring

$$T = D[x; \sigma],$$

consisting of polynomials $\sum_{i=0}^k a_i x^i$ with $a_i \in D$, with the usual addition of polynomials, but multiplication twisted by σ, so that

$$\big(ax^i\big)\big(bx^j\big) = a\sigma^i(b)x^{i+j} \quad \text{for all } a, b \in D,\ i, j \geqslant 0.$$

For the factorization theory of such rings T, [4, Ch. 1] is an excellent reference. Let $A = q(T) = D(x; \sigma)$, the quotient division ring of T, which is a twisted rational function division algebra in one variable. We will make fundamental use of the "divisor group" $Div(T)$ described in [1, §5]: Let $\mathcal{S}$ be the set of isomorphism classes $[S]$ of simple left T-modules S; then

$$Div(T) = \bigoplus_{[S]\in\mathcal{S}} \mathbb{Z}\cdot[S],$$

the free abelian group on $\mathcal{S}$. (Note that in the commutative case when D is a field and $\sigma = id$, then T is a polynomial ring, and $Div(T)$ is its usual divisor group. This is the source of the terminology.) For simple T-modules S, S', we have $ann_T(S)$ and $ann_T(S')$ are maximal two-sided ideals of T, and $S \cong S'$ iff $ann_T(S) = ann_T(S')$. Thus, we could have indexed $Div(T)$ by the maximal two-sided ideals of T; but, the indexing by simple modules is more natural for our purposes. We call an element $\alpha = \sum n_{[S]}[S]$ of $Div(T)$ a *divisor*, and call it an *effective divisor* if every $n_{[S]} \geqslant 0$.

Also, there is a degree homomorphism

$$deg : Div(T) \to \mathbb{Z} \quad \text{given by} \quad deg\big(\sum n_{[S]}[S]\big) = \sum n_{[S]}\, dim_D(S).$$

Note that if M is any left T-module of finite length (equivalently, finite-dimensional as a D-vector space), then M determines an effective Jordan–Hölder divisor

$$jh(M) = \sum_{i=1}^{k} [M_i/M_{i-1}],$$

where $\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M$ is a chain of T-submodules of M with each M_i/M_{i-1} simple. The Jordan–Hölder Theorem shows that $jh(M)$ is well defined.

Every simple left T-module has the form T/Tp for some $p \in T$ with p irreducible, i.e., p has no factorization into a product of terms of positive degree. For nonzero $f \in T$ with $deg(f) > 0$, the division algorithm shows that every nonzero element of T/Tf has the form $s + Tf$ for some $s \in T$ with $deg(s) < deg(f)$. Using this, it is easy to check (cf. [1, Lemma 5.2]) that for nonzero $f, g \in T$ of positive degree

$$T/Tf \cong T/Tg \quad \text{iff} \quad deg(f) = deg(g) \quad \text{and there exist nonzero } s,t \in T \text{ with}$$
$$deg(s) = deg(t) < deg(f) \quad \text{and} \quad ft = sg. \tag{3.1}$$

There is a divisor function

$$\delta : T \setminus \{0\} \to Div(T) \quad \text{given by} \quad \delta(f) = jh(T/Tf).$$

It is easy to check that $\delta(fg) = \delta(f) + \delta(g)$, hence δ extends to a well-defined map $\delta : A^* \to Div(T)$ given by $\delta(fz^{-1}) = \delta(f) - \delta(z)$ for all $f \in T \setminus \{0\}$, $z \in Z(T) \setminus \{0\}$. Clearly, $\delta(T)$ is the monoid of effective divisors in $Div(T)$. Note also that $deg(\delta(f)) = deg(f)$ for all $f \in T \setminus \{0\}$. It is proved in [1, Prop. 5.3] that there is an exact sequence:

$$1 \to [A^*, A^*]D^* \to A^* \xrightarrow{\delta} Div(T) \to 0. \tag{3.2}$$

Let $R = Z(T)$; so R is the polynomial ring $K^\sigma[y]$, where $K^\sigma = \{b \in K \mid \sigma(b) = b\}$ and $y = c^{-1}x^m$, with m minimal such that $\sigma^m|_K = id$ and $c \in D^*$ satisfying $\sigma^m = int(c)$ on D. Note that $q(R) = Z(A)$. This R has its own divisor group $Div(R)$, definable just as with $Div(T)$, and there is a corresponding divisor map $\delta_R : q(R)^* \to Div(R)$. The reduced norm map $Nrd_A : A \to Z(A)$ maps T to R (as T is integral over R, and R is integrally closed), and it is shown in [1, Prop. 5.4] that there is a corresponding "reduced norm" map $DNrd : Div(T) \to Div(R)$ which is injective, and such that the following diagram commutes:

$$\begin{array}{ccc} A^* & \xrightarrow{\ \delta\ } & Div(T) \\ {\scriptstyle Nrd_A}\downarrow & & \downarrow{\scriptstyle DNrd} \\ q(R)^* & \xrightarrow{\ \delta_R\ } & Div(R) \end{array} \tag{3.3}$$

Of course, T is a graded ring, with $T_i = Dx^i$ for all nonnegative $i \in \mathbb{Z}$. Suppose $\rho : T \to T$ is a graded automorphism of T, i.e., a ring automorphism such that $\rho(Dx^i) = Dx^i$ for all i. Then, ρ restricts to a graded automorphism of R and also determines ring automorphisms of A and $q(R)$, all denoted ρ.

Lemma 3.1. *The graded automorphism ρ of T determines an automorphism of $Div(T)$ mapping $\mathcal{S}$ to itself, and also determines an automorphism of $Div(R)$. The automorphisms determined by ρ on each of the terms in diagram* (3.3) *are compatible with the maps in that diagram.*

Proof. Since $\rho(D^*) = D^*$ and $\rho([A^*, A^*]) = [A^*, A^*]$, the exact sequence (3.2) shows that ρ determines a well-defined automorphism ρ of $Div(T)$ given by $\rho\delta(a) = \delta(\rho(a))$ for all $a \in A^*$. Any class in $\mathcal{S}$ has the form $[T/Tp]$ for some irreducible p in T. Then, $\rho(p)$ is also irreducible in F; so, $\rho[T/Tp] = [T/T\rho(p)] \in \mathcal{S}$. For $Div(R)$, ρ is defined analogously by $\rho(\delta_R(q)) = \delta_R(\rho(q))$ for all $q \in q(R)^*$. Since ρ is an automorphism of A, we have $\rho(Nrd_A(a)) = Nrd_A(\rho(a))$ for all $a \in A$. For, $Nrd_A(a)$ is determined by the constant term of the minimal polynomial $p_a \in Z(A)[X]$ of a over $Z(A)$, and $p_{\rho(a)} = \rho(p_a)$. Thus, we have $\rho \circ \delta = \delta \circ \rho$, $\rho \circ \delta_R = \delta_R \circ \rho$, and $\rho \circ Nrd_A = Nrd_A \circ \rho$. Since diagram (3.3) commutes and δ is surjective, it follows that $\rho \circ DNrd = DNrd \circ \rho$. $\square$

Suppose $\tau : T \to T$ is a unitary graded involution. That is, τ is a ring antiautomorphism of T with $\tau^2 = id_T$, $\tau|_R \neq id$, and $\tau(Dx^i) = Dx^i$ for all i. (The last condition is equivalent to: $\tau(D) = D$ and $\tau(x) = dx$ for some $d \in D^*$.) This τ extends to a unitary involution on A given by $\tau(fr^{-1}) = \tau(r)^{-1}\tau(f)$ for all $f \in T$, $r \in R \setminus \{0\}$. Since $\tau([A^*, A^*]) = [A^*, A^*]$ and $\tau(D^*) = D^*$, the exact sequence (3.2) above shows that τ determines an automorphism of order at most 2 of $Div(T)$, also denoted τ. This map $\tau : Div(T) \to Div(T)$ is given by $\tau(\delta(a)) = \delta(\tau(a))$ for all $a \in A^*$. (For a simple left T-module S, we have $S \cong T/Tp$ for some irreducible $p \in T$. Then, $\tau(p)$ is irreducible in T, and $\tau[S] = [T/T\tau(p)]$. It may seem surprising that this is well defined, independent of the choice of p, but the well-definition follows easily from (3.1). In terms of two-sided ideals, $\tau[S]$ is the isomorphism class of simple left T-modules with annihilator $\tau(ann_T(S))$.) Let $Div(T)^\tau = \{\alpha \in Div(T) \mid \tau(\alpha) = \alpha\}$.

Lemma 3.2. *Let τ be a unitary graded involution on T. Suppose $\tau|_D \neq id$ or T is noncommutative. Then,*

$$\delta\big(\Sigma_\tau(A)\big) = Div(T)^\tau.$$

Proof. Suppose first that $\tau|_D \neq id$.

Let $\Omega = \delta(\Sigma_\tau(A)) \subseteq Div(T)$. Since δ maps generators of $\Sigma_\tau(A)$ into $Div(T)^\tau$, we have $\Omega \subseteq Div(T)^\tau$. We must prove that this inclusion is equality. Suppose that $\Omega \subsetneqq Div(T)^\tau$.

Note that if $\alpha \in Div(T)$, say $\alpha = \delta(a)$, then $\alpha + \tau(\alpha) = \delta(a\tau(a)) \in \Omega$. Take any $\eta \in Div(T)^\tau \setminus \Omega$. We can write $\eta = \alpha - \beta$, where α and β are effective divisors. Then,

$$\alpha + \tau(\beta) = \eta + \big(\beta + \tau(\beta)\big) \equiv \eta \pmod{\Omega}.$$

So, $\alpha + \tau(\beta)$ is an effective divisor in $Div(T)^\tau \setminus \Omega$.

Let ξ be an effective divisor in $Div(T)^\tau \setminus \Omega$ of minimal degree. Necessarily $deg(\xi) > 0$, as $\xi \neq 0$. Say $\xi = \delta(z)$ for some $z \in T$. So, $deg(z) = deg(\xi) > 0$. Factor z into irreducibles in T, say $z = p_1 \cdots p_\ell$, and let $\pi_i = \delta(p_i)$. So,

$$\pi_1 + \cdots + \pi_\ell = \eta = \tau(\eta) = \tau(\pi_1) + \cdots + \tau(\pi_\ell).$$

Because the π_i and $\tau(\pi_i)$ are all part of the $\mathbb{Z}$-base of the free abelian group $Div(T)$, we must have $\pi_1 = \tau(\pi_j)$ for some index j. Suppose $j > 1$. Then, $\pi_1 + \pi_j = \pi_1 + \tau(\pi_1) \in \Omega$. Let $\xi' = \xi - (\pi_1 + \pi_j)$. Then, $\xi' \equiv \xi \pmod{\Omega}$. But, ξ' is an effective divisor with $deg(\xi') < deg(\xi)$. This contradicts the minimality of $deg(\xi)$. So, we must have $j = 1$, i.e., $\tau(\pi_1) = \pi_1$. If $\pi_1 \in \Omega$, then $\xi - \pi_1$ is an effective divisor in $Div(T)^\tau \setminus \Omega$ of degree less than $deg(\xi)$. Hence $\pi_1 \in Div(T)^\tau \setminus \Omega$, so in fact $\xi = \pi_1$ by the minimality of $deg(\xi)$.

To simplify notation, let $p = p_1$ and $\pi = \pi_1$. Since p and $\tau(p)$ are irreducible, we have $\pi = [T/Tp]$ and $\tau(\pi) = [T/T\tau(p)]$. Hence, the equality $\pi = \tau(\pi)$ implies that $T/Tp \cong T/T\tau(p)$. Therefore, by (3.1), there exist $f, g \in T \setminus \{0\}$ with $deg(f) = deg(g) < deg(p)$ and

$$pf = g\tau(p). \tag{3.4}$$

Suppose first that $\tau(f) = g$. Then, $pf = \tau(f)\tau(p) = \tau(pf)$, so $pf \in \Sigma_\tau(A)$, hence $\delta(pf) \in \Omega$. Then, $\delta(f) = \delta(pf) - \pi \equiv -\pi \pmod{\Omega}$. Therefore, $\delta(f)$ is an effective divisor in $Div(T)^\tau \setminus \Omega$ with

$$deg\big(\delta(f)\big) = deg(f) < deg(p) = deg(\pi) = deg(\xi).$$

This contradicts the minimality of $deg(\xi)$. So, we must have $\tau(f) \neq g$.

Now suppose that $\tau(f) \neq -g$. Then, let $f' = f + \tau(g) \in T \setminus \{0\}$, and $g' = \tau(f') = \tau(f) + g$. By applying τ to (3.4), we have $p\tau(g) = \tau(f)\tau(p)$, which when added to (3.4) yields $pf' = g'\tau(p)$. But then the preceding argument for the case $\tau(f) = g$ applies for f', as $f' \neq 0$ and $\tau(f') = g'$; it shows that $\delta(f') \in Div(T)^\tau \setminus \Omega$. Since

$$deg\big(f'\big) \leqslant max\big(deg(f), deg\big(\tau(g)\big)\big) = deg(f) < deg(\xi),$$

this contradicts the minimality of $deg(\xi)$.

We thus have $\tau(f) = -g$ while $\tau(f) \neq g$. Hence, $char(D) \neq 2$. Since we have assumed $\tau|_D \neq id$, there is $c \in D^*$ with $\tau(c) = -c$. Then,

$$pfc = g\tau(p)c = c^{-1}cg\tau(p)c = c^{-1}\tau(pfc)c. \tag{3.5}$$

Since $\tau(c) = -c$, the map $\tau' = int(c^{-1}) \circ \tau$ is a unitary involution on A with $\tau' \sim \tau$. Hence, by Lemma 2.1(a), $\Sigma_{\tau'}(A) = \Sigma_\tau(A)$. Since Eq. (3.5) shows $pfc = \tau'(pfc)$, we thus have $pfc \in \Sigma_{\tau'}(A) = \Sigma_\tau(A)$, and hence $\delta(pfc) \in \Omega$. This shows that $\delta(fc) \equiv -\pi \pmod{\Omega}$, and hence $\delta(fc) \in Div(T)^\tau \setminus \Omega$. But, $\delta(fc)$ is an effective divisor, with

$$deg\big(\delta(fc)\big) = deg(f) < deg(\pi) = deg(\xi).$$

Thus, we have a contradiction to the minimality of $deg(\xi)$. The contradiction arose from the assumption that $\Omega \subsetneqq Div(T)^\tau$. So, we must have $Div(T)^\tau = \Omega = \delta(\Sigma_\tau(A))$, as desired.

We have thus far assumed that $\tau|_D \neq id$. Now, suppose that $\tau|_D = id$. Then, D must be commutative, since τ is an antiautomorphism. Our hypothesis now is that T is noncommutative; therefore, $\sigma = int(x)|_D \neq id$. Since $\tau(Dx) = Dx$, there is a nonzero $d \in D$ with $\tau(dx) = \pm dx$. Then let $\widetilde{\tau} = int(dx) \circ \tau$, which is a unitary graded involution of T with $\widetilde{\tau} \sim \tau$ as unitary involutions on A. Hence, $\Sigma_{\widetilde{\tau}}(A) = \Sigma_\tau(A)$ by Lemma 2.1(a). Also, for any $a \in A^*$,

$$\begin{aligned}\widetilde{\tau}\big(\delta(a)\big) &= \delta\big(\widetilde{\tau}(a)\big) = \delta\big(dx\tau(a)(dx)^{-1}\big)\\ &= \delta(dx) + \delta\big(\tau(a)\big) - \delta(dx) = \delta\big(\tau(a)\big) = \tau\big(\delta(a)\big).\end{aligned}$$

Thus, $\widetilde{\tau}$ and τ have the same action on $Div(T)$. Furthermore, $\widetilde{\tau}|_D = int(dx)|_D = \sigma \neq id$. Therefore, the preceding argument shows that the lemma holds for $\widetilde{\tau}$. This yields for τ,

$$\delta(\Sigma_\tau(A)) = \delta\big(\Sigma_{\widetilde{\tau}}(A)\big) = Div(T)^{\widetilde{\tau}} = Div(T)^\tau.$$

Thus, the lemma holds in all cases. □

Remark. The assumption that $\tau|_D \neq id$ or T is noncommutative is definitely needed for Lemma 3.2. For example, suppose that T is commutative and τ is defined by $\tau|_D = id$ and $\tau(x) = -x$. Then, $\delta(x) \in Div(T)^\tau \setminus \delta(\Sigma_\tau(A))$.

Now we combine the action of the graded unitary involution τ on $Div(T)$ with a finite group action. Let H be a finite abelian group which acts on the set $\mathcal{S}$ of isomorphism classes of simple left T-modules. This action induces an action of H on $Div(T)$, making $Div(T)$ into a permutation module for H. There is an associated norm map $N_H : Div(T) \to Div(T)$ given by $N_H(\alpha) = \sum_{h\in H} h\alpha$. Suppose the actions of H and τ on $Div(T)$ are related by

$$\tau(h\alpha) = h^{-1}\tau(\alpha) \quad \text{for all } h \in H,\ \alpha \in Div(T). \tag{3.6}$$

Lemma 3.3. *Let* $I_H(Div(T)) = \langle h\alpha - \alpha \mid h \in H,\ \alpha \in Div(T)\rangle$ *and also let* $Div(T)^{h\tau} = \{\alpha \in Div(T) \mid h\tau(\alpha) = \alpha\}$. *Then,*

$$\{\alpha \in Div(T) \mid N_H(\alpha) = N_H(\tau(\alpha))\} = I_H(Div(T)) + \sum_{h\in H} Div(T)^{h\tau}.$$

Proof. Let G be the semidirect product group $H \rtimes_\psi \langle\tau\rangle$ built using the homomorphism $\psi : \langle\tau\rangle \to Aut(H)$ given by $\psi(\tau)(h) = h^{-1}$. (This is well defined as H is abelian.) More explicitly, $G = H \cup H\tau$ (disjoint union), where the multiplication is determined by that in H together with $\tau h = h^{-1}\tau$ and $\tau^2 = 1$. Note that G is a generalized dihedral group in the terminology of [2, §2.4] since every element of $G \setminus H$ has order 2. The hypothesis (3.6) shows that the actions of τ and H on $Div(T)$ combine to yield a $\mathbb{Z}$-linear group action of G on $Div(T)$. Also, G sends $\mathcal{S}$ to $\mathcal{S}$, since this is true for H and τ. Thus, $Div(T)$ is a permutation G-module. From this group action we build a new action of G on $Div(T)$, denoted $*$, given by

$$h * \alpha = h\alpha \quad \text{and} \quad (h\tau) * \alpha = -h\tau\alpha \quad \text{for all } h \in H,\ \alpha \in Div(T).$$

Let $\widetilde{Div(T)}$ denote $Div(T)$ with this twisted G-action, and let $\widetilde{N_G} : \widetilde{Div(T)} \to \widetilde{Div(T)}$ be the associated norm map, given by

$$\widetilde{N_G}(\alpha) = \sum_{h\in H} h\alpha - \sum_{h\in H} h\tau\alpha = N_H(\alpha) - N_H(\tau\alpha) \quad \text{for all } \alpha \in Div(T).$$

Thus, the lemma describes $ker(\widetilde{N_G})$.

Note that while $Div(T)$ is a permutation G-module, the twisted G-module $\widetilde{Div(T)}$ need not be a permutation G-module, as the twisted action of G does not map $\mathcal{S}$ to itself. Write $\mathcal{S} = \bigcup_{j\in J} \mathcal{O}_j$, where the $\mathcal{O}_j$ are the distinct G-orbits of $\mathcal{S}$ (for the original G-action). Then, $Div(T) = \bigoplus_{j\in J} M_j$, where each $M_j = \bigoplus_{s\in\mathcal{O}_j} \mathbb{Z}s$, which is a G-submodule of $Div(T)$. Let $\mathcal{O}$ be one of the orbits, and let $M = \bigoplus_{s\in\mathcal{O}} \mathbb{Z}s$. Take any $s \in \mathcal{O}$, and let $\mathcal{V} = H\cdot s$, which is an H-orbit within $\mathcal{O}$. Then, $\tau\mathcal{V} = \tau H\cdot s = H\tau\cdot s$, which is also an H-orbit in $\mathcal{O}$, with $\mathcal{V}\cup\tau\mathcal{V} = Hs \cup \tau Hs = Gs = \mathcal{O}$. Let $\mathcal{V} = \{s_1, \dots, s_n\}$. There are two possible cases:

Case I. $\mathcal{V}\cap\tau\mathcal{V} = \varnothing$. Then, $\{s_1, \dots, s_n, -\tau(s_1), \dots, -\tau(s_n)\}$ is a $\mathbb{Z}$-base of M which is mapped to itself by the $*$-action of G. Thus, $\widetilde{M}$ (i.e., M with the twisted G-action) is a permutation G-module. Consequently, $\widehat{H}^{-1}(G, \widetilde{M}) = 0$ (as is true for all permutation G-modules). This means that

$$\begin{aligned} ker(\widetilde{N_G}) \cap M &= I_G(\widetilde{M}) = \langle g * m - m \mid g \in G,\ m \in M\rangle \\ &= \langle hm - m, -h\tau m - m \mid h \in H,\ m \in M\rangle. \end{aligned}$$

Each $hm - m \in I_H(M) \subseteq I_H(Div(T))$, while $-h\tau m - m \in M^{h\tau} \subseteq Div(T)^{h\tau}$. Thus,

$$ker(\widetilde{N_G}) \cap M \subseteq I_H(Div(T)) + \sum_{h\in H} Div(T)^{h\tau}.$$

Case II. $\mathcal{V} \cap \tau\mathcal{V} \neq \varnothing$. Then, $\mathcal{V} = \tau\mathcal{V}$, since they are each H-orbits. So, for each $t \in \mathcal{V}$ there is $k \in H$ with $t = \tau k^{-1} t = k\tau t$; so, $t \in M^{k\tau}$. Thus,

$$ker(\widetilde{N_G}) \cap M \subseteq M = \sum_{h \in H} M^{h\tau} \subseteq \sum_{h \in H} Div(T)^{h\tau}.$$

By combining the two cases, we obtain

$$ker(\widetilde{N_G}) = \sum_{j \in J} \bigl(ker(\widetilde{N_G}) \cap M_j\bigr) \subseteq I_H\bigl(Div(T)\bigr) + \sum_{h \in H} Div(T)^{h\tau}.$$

This proves $\subseteq$ in the lemma, and the reverse inclusion is clear. □

4. Proof of the theorem

We can now prove Th. 1.1.

Proof. Let E be a graded division ring (finite-dimensional over its center) with a unitary graded involution τ, and let $Q = q(\mathsf{E})$. Because $Nrd_Q|_{\mathsf{E}} = Nrd_{\mathsf{E}}$, as noted in Section 2, we have $\Sigma'_\tau(\mathsf{E}) \subseteq \Sigma'_\tau(Q)$. Also, clearly $\Sigma_\tau(\mathsf{E}) \subseteq \Sigma_\tau(Q)$. Therefore, there is a canonical map $\varphi_{\mathsf{E}} : \mathrm{SK}_1(\mathsf{E}, \tau) \to \mathrm{SK}_1(Q, \tau)$. We will show that φ_{E} is an isomorphism.

To see that φ_{E} is injective, give Γ_{E} a total ordering making it into a totally ordered abelian group. Then define a function $\lambda : \mathsf{E} \setminus \{0\} \to \mathsf{E}^*$ as follows: For $c \in \mathsf{E} \setminus \{0\}$, write $c = \sum_{\gamma \in \Gamma_{\mathsf{E}}} c_\gamma$ where each $c_\gamma \in \mathsf{E}_\gamma$. Then set $\lambda(c) = c_\varepsilon$, where ε is minimal such that $c_\varepsilon \neq 0$. It is easy to check that $\lambda(cd) = \lambda(c)\lambda(d)$ for all $c, d \in \mathsf{E} \setminus \{0\}$. It follows that λ extends to a well-defined group epimorphism $\lambda : Q^* \to \mathsf{E}^*$ given by $\lambda(cz^{-1}) = \lambda(c)\lambda(z)^{-1}$ for all $c \in \mathsf{E} \setminus \{0\}$, $z \in Z(\mathsf{E}) \setminus \{0\}$. Clearly $\lambda|_{\mathsf{E}^*} = id$. Note that as τ is a graded involution, we have $\lambda(\tau(c)) = \tau(\lambda(c))$ for all $c \in \mathsf{E} \setminus \{0\}$, and hence for all $c \in Q^*$. Now take any $u \in \Sigma_\tau(Q) \cap \mathsf{E}^*$. Then, $u = s_1 \cdots s_m$ for some $s_i \in Q^*$ with each $\tau(s_i) = s_i$. Then, $u = \lambda(u) = \lambda(s_1) \cdots \lambda(s_m)$ with $\tau(\lambda(s_i)) = \lambda(\tau(s_i)) = \lambda(s_i)$; so, $u \in \Sigma_\tau(\mathsf{E})$. This shows that $\Sigma_\tau(Q) \cap \mathsf{E}^* \subseteq \Sigma_\tau(\mathsf{E})$. Therefore, the canonical map φ_{E} is injective.

To show that φ_{E} is surjective, we will prove the following:

$$\Sigma'_\tau(Q) = \bigl(\Sigma'_\tau(Q) \cap \mathsf{E}_0^*\bigr) \cdot \Sigma_\tau(Q). \tag{4.1}$$

Note that surjectivity of φ_{E} follows immediately from (4.1) because $\Sigma'_\tau(Q) \cap \mathsf{E}_0^* \subseteq \Sigma'_\tau(\mathsf{E})$.

We next reduce to the case of a finitely-generated grade group. For any graded division algebra E, observe that if Δ is a subgroup of Γ_{E}, then E has the graded division subalgebra $\mathsf{E}_\Delta = \bigoplus_{\varepsilon \in \Delta} \mathsf{E}_\varepsilon$, which is finite-dimensional over its center with $\Gamma_{\mathsf{E}_\Delta} = \Delta$. Let $Q_\Delta = q(\mathsf{E}_\Delta) \subseteq Q$. Take any $\gamma \in \Gamma_{Z(\mathsf{E})}$ such that $\tau|_{Z(\mathsf{E})_\gamma} \neq id$ and any $c_1, \ldots, c_m \in \mathsf{E}^*$ such that the c_i span E as a graded $Z(\mathsf{E})$-vector space. Then, the c_i also span Q as a $Z(Q)$-vector space. So, if Δ is any finitely-generated subgroup of Γ_{E} such that $\gamma \in \Delta$ and $deg(c_1), \ldots, deg(c_m) \in \Delta$, then $\tau|_{\mathsf{E}_\Delta}$ is a unitary graded involution and the map $Q_\Delta \otimes_{Z(Q_\Delta)} Z(Q) \to Q$ is surjective. This map is also injective, as its domain is simple since Q_Δ is central simple and finite-dimensional over $Z(Q_\Delta)$. Therefore, $[Q_\Delta : Z(Q_\Delta)] = [Q : Z(Q)]$, and hence $Nrd_{Q_\Delta} = Nrd_Q|_{Q_\Delta}$, so $\Sigma'_\tau(Q_\Delta) = \Sigma'_\tau(Q) \cap Q_\Delta^*$. Now, $\mathsf{E} = \varinjlim \mathsf{E}_\Delta$ and $\Sigma'_\tau(Q) = \varinjlim \Sigma'_\tau(Q_\Delta)$ as Δ ranges over finitely-generated subgroups of Γ_{E} containing γ and $deg(c_1), \ldots, deg(c_m)$; therefore, equality (4.1) holds for E if it holds for each E_Δ. Thus, it suffices to prove (4.1) for E with Γ_{E} finitely-generated.

Henceforward, suppose Γ_{E} is a finitely generated, hence free, abelian group, say $\Gamma_{\mathsf{E}} = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_n$. Take any nonzero $x_i \in \mathsf{E}_{\gamma_i}$ for $i = 1, 2, \ldots, n$. Then, $\mathsf{E} = \mathsf{E}_0[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, an iterated twisted Laurent polynomial ring over the division ring E_0. We prove (4.1) by induction on n.

Suppose first that $n = 1$. Let $D = \mathsf{E}_0$ and $x = x_1$, and let T be the graded subring of E generated by D and x. So, $T = D[x; \sigma]$, a twisted polynomial ring, where $\sigma = int(x)$. Then, $\sigma|_{Z(D)}$ has finite order since $[Z(D):Z(\mathsf{E})_0] < \infty$. To be consistent with the notation of Section 3, let $A = q(T) = q(\mathsf{E}) = Q$. Since τ is a graded unitary involution on E, its restriction $\tau|_T$ is a graded unitary involution on T. Let $Div(T)$ and $\delta : A^* \to Div(T)$ be as in Section 3. If T is commutative, then A is also commutative, so $\mathrm{SK}_1(A,\tau) = 1$ and (4.1) holds trivially. Thus, we may assume that T is noncommutative. Take any $a \in \Sigma'_\tau(A)$. So, $Nrd_A(a) = \tau(Nrd_A(a)) = Nrd_A(\tau(a))$ by (2.3). Recall from [1, Remark 5.1(iv)] that $\delta(Nrd_A(a)) = r\delta(a)$, where $r = ind(A)$. Hence,

$$r\tau\big(\delta(a)\big) = r\delta\big(\tau(a)\big) = \delta\big(Nrd_A\big(\tau(a)\big)\big) = \delta\big(Nrd_A(a)\big) = r\delta(a).$$

Then, $\tau(\delta(a)) = \delta(a)$, as $Div(T)$ is torsion-free. By Lemma 3.2, we have $\delta(a) = \delta(b)$ for some $b \in \Sigma_\tau(A)$. It then follows from the exact sequence (3.2) that $a = bcd$ for some $c \in [A^*, A^*]$ and $d \in D^*$. Recall that $[A^*, A^*] \subseteq \Sigma_\tau(A)$ by Lemma 2.1(b). So, $bc \in \Sigma_\tau(A) \subseteq \Sigma'_\tau(A)$. Hence, $d = (bc)^{-1}a \in \Sigma'_\tau(A) \cap D^*$, so $a = (bc)d \in \Sigma_\tau(A)(\Sigma'_\tau(A) \cap D^*)$. This proves the inclusion $\subseteq$ in (4.1). The reverse inclusion is clear. Thus, (4.1) holds when $n = 1$.

Now assume $n > 1$. By induction (4.1) holds for all graded division algebras with grade group of rank less than n. Let $x = x_1$ as above, and let

$$\begin{aligned}
&\mathsf{Z} = Z(\mathsf{E});\\
&D = \mathsf{E}_0;\\
&T = D[x;\sigma] \subseteq \mathsf{E} \quad \text{where } \sigma = int(x);\\
&R = Z(T);\\
&A = q(T);\\
&\mathsf{B} = A\big[x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}; \sigma_2, \ldots, \sigma_n\big] \subseteq Q \quad \text{where } \sigma_i = int(x_i);\\
&\mathsf{C} = Z(\mathsf{B}).
\end{aligned}$$

For a clearer picture of B and C, let $\mathsf{U} = \mathsf{E}_0[x, x^{-1}; \sigma]$, which is a graded division subalgebra of E with $\mathsf{U}_0 = \mathsf{E}_0$, $\Gamma_\mathsf{U} = \mathbb{Z}\gamma_1$, and $q(\mathsf{U}) = A$. Let $\mathsf{F} = \mathsf{U} \cap \mathsf{Z}$, which is a graded field with $\mathsf{F}_0 = \mathsf{Z}_0$ and $\Gamma_\mathsf{F} = \Gamma_\mathsf{Z} \cap \Gamma_\mathsf{U}$. So,

$$|\Gamma_\mathsf{U} : \Gamma_\mathsf{F}| = |\Gamma_\mathsf{E} \cap \mathbb{Z}\gamma_1 : \Gamma_\mathsf{Z} \cap \mathbb{Z}\gamma_1| \leqslant |\Gamma_\mathsf{E} : \Gamma_\mathsf{Z}| < \infty,$$

hence $[\mathsf{U} : \mathsf{F}] = [\mathsf{U}_0 : \mathsf{F}_0]|\Gamma_\mathsf{U} : \Gamma_\mathsf{F}| = [\mathsf{E}_0 : \mathsf{Z}_0]|\Gamma_\mathsf{U} : \Gamma_\mathsf{F}| < \infty$. Since F is central in U, we have $A = q(\mathsf{U}) = q(\mathsf{F}) \otimes_\mathsf{F} \mathsf{U}$. Hence, $\mathsf{B} = q(\mathsf{F}) \otimes_\mathsf{F} \mathsf{E}$, so $\mathsf{C} = Z(\mathsf{B}) = q(\mathsf{F}) \otimes_\mathsf{F} \mathsf{Z}$, from which it is clear that the rank of B as a free C-module equals $[\mathsf{E} : \mathsf{Z}]$, which is finite. Let

$$\mathsf{E}' = \mathsf{U}\big[x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}; \sigma_2, \ldots, \sigma_n\big] = \mathsf{E},$$

but we grade E' so that $\mathsf{E}'_0 = \mathsf{U}$, $\Gamma_{\mathsf{E}'} = \mathbb{Z}\gamma_2 \oplus \cdots \oplus \mathbb{Z}\gamma_n$, and $\mathsf{E}'_\varepsilon = \bigoplus_{j\in\mathbb{Z}} \mathsf{E}_{j\gamma_1+\varepsilon}$, for each $\varepsilon \in \Gamma_{\mathsf{E}'}$. Now, B is obtained from E' by inverting nonzero elements of F, which are all central in E' and homogeneous of degree 0. So, the grading on E' extends to a grading on B with $\mathsf{B}_0 = A$ and $\Gamma_\mathsf{B} = \Gamma_{\mathsf{E}'} = \mathbb{Z}\gamma_2 \oplus \cdots \oplus \mathbb{Z}\gamma_n$; this grading restricts to a grading on $\mathsf{C} = Z(\mathsf{B})$. Note that B is a graded division ring since $\mathsf{B}_0 = A$ is a division ring and each homogeneous component B_ε of B contains a unit of B (namely any nonzero homogeneous element of E in B_ε). We saw above that $[\mathsf{B} : \mathsf{C}] < \infty$. Also, $q(\mathsf{B}) = Q$, as $\mathsf{E} \subseteq \mathsf{B} \subseteq Q = q(\mathsf{E})$. Our unitary involution τ is a graded involution for E, so also a graded involution for E', hence also a graded involution for B. Moreover, τ is a unitary involution for B, as $\tau|_\mathsf{Z} \neq id$ and $\mathsf{Z} = Z(\mathsf{E}) \subseteq \mathsf{B} \cap Z(Q) \subseteq Z(\mathsf{B})$. Since Γ_B has $\mathbb{Z}$-rank $n-1$, (4.1) holds for τ on B by induction, i.e.

$$\Sigma'_\tau(Q) = \big(\Sigma'_\tau(Q) \cap A^*\big) \cdot \Sigma_\tau(Q).$$

Therefore, to prove (4.1) for E, it suffices to prove

$$\Sigma'_\tau(Q) \cap A^* \subseteq \big(\Sigma'_\tau(Q) \cap D^*\big) \cdot \Sigma_\tau(Q). \tag{4.2}$$

For this, we will work with $Div(T)$ and the action on it by τ and various automorphisms. Note that while τ is unitary as an involution on B, its restriction to A need not be unitary. But, if $\tau|_A$ is not unitary, then $\tau|_{Z(A)} = id$, so $\tau|_{\mathsf{C}_0} = id$, hence C is totally ramified over C^τ, i.e., $\mathsf{C}_0 = (\mathsf{C}^\tau)_0$. But then $\mathrm{SK}_1(\mathsf{B}, \tau) = 1$ by [2, Th. 4.5]. So, $\mathrm{SK}_1(Q, \tau) = \mathrm{SK}_1(\mathsf{B}, \tau) = 1$ since (4.1) holds for B, whence (4.2) and (4.1) for E obviously hold. Thus, we will assume from now on that $\tau|_A$ is unitary.

Assume that T is not commutative.

Since B is a graded division algebra and $\mathsf{C} = Z(\mathsf{B})$, it is known (see Section 2) that $Z(\mathsf{B}_0)$ is Galois over C_0, and there is a well-defined epimorphism $\theta_\mathsf{B} : \Gamma_\mathsf{B} \to \mathcal{G}(Z(\mathsf{B}_0)/\mathsf{C}_0)$ given by $\varepsilon \mapsto int(b_\varepsilon)|_{Z(\mathsf{B}_0)}$ for any nonzero b_ε in B_ε. Here, $\mathsf{B}_0 = A$. Let $H = \mathcal{G}(Z(A)/\mathsf{C}_0)$, which is abelian, as θ_B is surjective. We next define a group action of H on $Div(T)$ so as to be able to use Lemma 3.3.

We use the notation associated to T and $Div(T)$ as in Section 3, including the epimorphism $\delta : A^* \to Div(T)$. Observe that conjugation by an element of E^* yields a degree preserving automorphism of T, and hence a well-defined automorphism of $Div(T)$. That is, for any $y \in \mathsf{E}^*$ and $a \in A^*$, define

$$y \cdot \delta(a) = \delta\big(yay^{-1}\big). \tag{4.3}$$

This is well defined because $ker(\delta) = [A^*, A^*]D^*$ (see (3.2)), which $int(y)$ maps to itself. For any irreducible p of the twisted polynomial ring T, its conjugate ypy^{-1} is also irreducible in T, and $y \cdot [T/Tp] = y \cdot \delta(p) = [T/Typy^{-1}]$. Hence, the action of y maps the distinguished $\mathbb{Z}$-base $\mathcal{S}$ of $Div(T)$ to itself. Therefore, the group action of E^* on $Div(T)$ given by (4.3) makes $Div(T)$ into a permutation E^*-module. Note that if $u \in \mathsf{E}^*$ and $int(u)|_{Z(A)} = id$, then by Skolem–Noether there is $c \in A^*$ with $int(u)|_A = int(c)$. Then, for $a \in A$,

$$u \cdot \delta(a) = \delta\big(cac^{-1}\big) = \delta(c) + \delta(a) - \delta(c) = \delta(a), \tag{4.4}$$

so u acts trivially on $Div(T)$. Now, for any $y \in \mathsf{E}^*$, we have $\tau \circ int(y) \circ \tau \circ int(y) = int(\tau(y)^{-1}y)$. Since τ preserves degrees, we have $\tau(y)^{-1}y \in \mathsf{E}_0 = D \subseteq A$, hence $int(\tau(y)^{-1}y)|_{Z(A)} = id$. Therefore, (4.4) with $u = \tau(y)^{-1}y$ shows that $\tau y \tau y \cdot \alpha = \alpha$ for all $\alpha \in Div(T)$, i.e.,

$$\tau y \cdot \alpha = y^{-1}\tau \cdot \alpha. \tag{4.5}$$

We obtain from the E^*-action an induced action of H on $Div(T)$: For any $h \in H$, choose any $y_h \in \mathsf{E}^*$ such that $int(y_h)|_{Z(A)} = h$. (Such a y_h exists because the composition $\mathsf{E}^* \to \mathsf{B}^* \to \Gamma_\mathsf{B}$ is surjective, as is $\theta_\mathsf{B} : \Gamma_\mathsf{B} \to H$.) Then, for $\alpha \in Div(T)$, set $h \cdot \alpha = y_h \cdot \alpha$. If we had made a different choice of y_h, say y'_h, then as $int(y'^{-1}_h y_h)|_{Z(A)} = h^{-1}h = id$, the calculation in (4.4) shows that $y'_h \cdot \alpha = y_h \cdot \alpha$ for all $\alpha \in Div(T)$. Thus, the action of H on $Div(T)$ is well defined, independent of the choice of the y_h. So, $Div(T)$ is a permutation H-module, and from (4.5), we have $\tau h \cdot \alpha = h^{-1}\tau \cdot \alpha$ for all $h \in H$ and $\alpha \in Div(T)$. Therefore, Lemma 3.3 applies for the norm map N_H of H on $Div(T)$. We interpret the terms appearing in that lemma: First,

$$I_H\big(Div(T)\big) = \delta\big(\big[\mathsf{E}^*, A^*\big]\big). \tag{4.6}$$

For, to see $\subseteq$, take any generator $h \cdot \beta - \beta$ of $I_H(Div(T))$, and take any $b \in A^*$ with $\delta(b) = \beta$. Then,

$$h \cdot \beta - \beta = \delta\big(y_h b y_h^{-1}\big) - \delta(b) = \delta\big(y_h b y_h^{-1} b^{-1}\big) \in \delta\big(\big[\mathsf{E}^*, A^*\big]\big).$$

For the reverse inclusion, take any $z \in \mathsf{E}^*$ and $a \in A^*$. Let $h = int(z)|_{Z(A)}$. Since we could have chosen z for y_h, we have

$$\delta\big(zaz^{-1}a^{-1}\big) = \delta\big(zaz^{-1}\big) - \delta(a) = h \cdot \delta(a) - \delta(a) \in I_H\big(Div(T)\big).$$

This proves (4.6).

For any $h \in H$, we have selected $y_h \in \mathsf{E}^*$ so that $int(y_h)|_{Z(A)} = h$. For any $d \in D = \mathsf{E}_0$, we have $int(dy_h)|_{Z(A)} = int(y_h)|_{Z(A)}$. Since $\mathsf{E}_{deg(y_h)} = \mathsf{E}_0 y_h$, the graded involution τ maps $\mathsf{E}_0 y_h$ to itself. Hence, there is $d_h \in \mathsf{E}_0$ with $\tau(d_h y_h) = \pm d_h y_h$. Then, replace y_h by $d_h y_h$, and set $\tau_h = int(y_h) \circ \tau$. Since $\tau(y_h) = \pm y_h$, the map τ_h is a unitary graded involution on E with $\tau_h \sim \tau$ on E. So, τ_h restricts to a graded involution on T. If $\tau_h|_T$ is not unitary, then $\tau_h|_A$ is not unitary, so $\mathrm{SK}_1(Q, \tau_h) = 1$ by the same argument as given above for τ. But, $\mathrm{SK}_1(Q, \tau_h) = \mathrm{SK}_1(Q, \tau)$ as $\tau_h \sim \tau$ by Lemma 2.1(a), and the triviality of $\mathrm{SK}_1(Q, \tau)$ implies (4.2) and (4.1). Thus, we are done if any $\tau_h|_A$ is not unitary. We therefore assume that each $\tau_h|_A$ is unitary, so $\tau_h|_T$ is a graded unitary involution. Since the action of τ_h on $Div(T)$ coincides with that of $h\tau$, Lemma 3.2 yields

$$Div(T)^{h\tau} = \delta\big(\Sigma_{\tau_h}(A)\big) \quad \text{for each } h \in H. \tag{4.7}$$

(The lemma applies because we are assuming that T is noncommutative.)

We now prove (4.2). For this, take any $a \in \Sigma'_\tau(Q) \cap A^*$, and let $\alpha = \delta(a)$. Since $A = \mathsf{B}_0$ and $Q = q(\mathsf{B})$, the norm formula (2.2) yields

$$Nrd_Q(a) = N_{Z(A)/\mathsf{C}_0}\big(Nrd_A(a)\big)^{\lambda} \quad \text{where } \lambda = ind(Q)/\big(\big[Z(A):\mathsf{C}_0\big] \cdot ind(A)\big).$$

Note that $N_{Z(A)/\mathsf{C}_0}$ is just the norm map N_H for the H-module $Z(A)$. Here, $Div(T)$ and $Div(R)$ are also H-modules, and the maps $\delta_R : Z(A) \to Div(R)$ and $DNrd : Div(T) \to Div(R)$ in diagram (3.3) are H-equivariant. This follows from Lemma 3.1 since for $h \in H$ the action of h on $Z(A)$, $Div(R)$, and $Div(T)$ is induced via $int(y_h)$, which is a graded automorphism of T. Therefore, δ_R and $DNrd$ commute with the norm maps N_H on $Z(A)$, $Div(R)$, and $Div(T)$. Thus, for $\alpha = \delta(a)$ we have, using commutative diagram (3.3),

$$\begin{aligned}\delta_R\big(Nrd_Q(a)\big) &= \delta_R\big(N_{Z(A)/\mathsf{C}_0}\big(Nrd_A(a)\big)^{\lambda}\big) = \lambda\delta_R N_H\big(Nrd_A(a)\big)\\ &= \lambda N_H \delta_R\big(Nrd_A(a)\big) = \lambda N_H\big(DNrd\,\delta(a)\big) = \lambda\, DNrd\big(N_H(\alpha)\big).\end{aligned}$$

Since $a \in \Sigma'_\tau(Q)$, we have $Nrd_Q(a) = Nrd_Q(\tau(a))$. Thus,

$$\begin{aligned}\lambda\, DNrd\big(N_H(\alpha)\big) &= \delta_R\big(Nrd_Q(a)\big)\\ &= \delta_R\big(Nrd_Q\big(\tau(a)\big)\big) = \lambda\, DNrd\big(N_H\big(\tau(\alpha)\big)\big).\end{aligned} \tag{4.8}$$

But, $Div(R)$ is a torsion-free $\mathbb{Z}$-module, and $DNrd$ is injective by [1, Prop. 5.4]. Therefore, Eq. (4.8) yields $N_H(\alpha) = N_H(\tau(\alpha))$. By Lemma 3.3 and (4.6) and (4.7), we thus have

$$\alpha \in I_H\big(Div(T)\big) + \sum_{h\in H} Div(T)^{h\tau} = \delta\big(\big[\mathsf{E}^*, A^*\big]\big) + \sum_{h\in H}\delta\big(\Sigma_{\tau_h}(A)\big).$$

Hence, there exist $b \in [\mathsf{E}^*, A^*]$ and $c_h \in \Sigma_{\tau_h}(A)$ for each $h \in H$ with

$$\delta(a) = \delta(b) + \sum_{h\in H}\delta(c_h) = \delta\bigg(b\prod_{h\in H} c_h\bigg).$$

Then, as $ker(\delta) = [A^*, A^*]D^*$, there exist $d \in D^*$ and $e \in [A^*, A^*]$ such that $a = deb\prod_{h\in H} c_h$. Let $q = eb\prod_{h\in H} c_h$. Since $[A^*, A^*] \subseteq \Sigma_\tau(Q)$ by Lemma 2.1, we have $eb \in \Sigma_\tau(Q)$. But also, each $c_h \in \Sigma_{\tau_h}(A) \subseteq \Sigma_{\tau_h}(Q) = \Sigma_\tau(Q)$, with the last equality given by Lemma 2.1(a), since $\tau_h \sim \tau$ as involutions on Q. Hence, $q \in \Sigma_\tau(Q) \subseteq \Sigma'_\tau(Q)$. So, $d = aq^{-1} \in D^* \cap \Sigma'_\tau(Q)$, whence $a \in (\Sigma'_\tau(Q) \cap D^*) \cdot \Sigma_\tau(Q)$. This proves (4.2), under the assumption that T is noncommutative.

There remains the case where T is commutative. Suppose that $int(x_j)|_D \neq id$ for some $j > 1$. Then, we can rearrange the order of the x_i, viewing the order of generators of E as $x_j, x_2, \dots, x_{j-1}, x_1, x_{j+1}, \dots, x_n$ (and their inverses). Then, we have $T = D[x_j]$, which is not commutative. So, we are back to the previously proved case; hence, (4.2) holds.

The final possibility is that each $D[x_j]$ is commutative for $j = 1, 2, \dots, n$, i.e., $D \subseteq Z(\mathsf{E})$. So, for each $h \in H$, we have $\tau_h|_D = int(y_h)|_D \circ \tau|_D = \tau|_D$. In the basic argument for (4.2), we used the assumption that T is noncommutative only to be able to apply Lemma 3.2 for each τ_h. Now T is commutative, but if $\tau|_D \neq id$ then also each $\tau_h|_D \neq id$, so Lemma 3.2 still applies for each τ_h; then the preceding argument for (4.2) again goes through. Thus, we may assume that $\tau|_D = id$. (Hence, $\mathsf{E}_0 = (Z(\mathsf{E}^\tau))_0$, which implies that $\mathrm{SK}_1(\mathsf{E}, \tau) = 1$ by [2, Th. 4.5]. But, we are not finished because we do not know that $\mathrm{SK}_1(Q, \tau) = 1$.) For any $y \in \mathsf{E}^*$, we have $\tau(y) = cy$ for some $c \in \mathsf{E}_0^* = D^*$, as each E_γ is a τ-stable 1-dimensional E_0-vector space. Then $y = \tau(cy) = c^2 y$, so $c^2 = 1$. Thus, $\tau(y) = \pm y$ for each $y \in \mathsf{E}^*$. If $\tau(x_1) = x_1$, then $\tau|_A = id$, which we saw earlier implies $\mathrm{SK}_1(Q, \tau) = \mathrm{SK}_1(\mathsf{B}, \tau) = 1$, which trivially implies (4.2). Thus, we may assume that $\tau(x_1) = -x_1$. Likewise, we may assume that $\tau(x_j) = -x_j$ for each j. For, if $\tau(x_j) = x_j$, then after reordering the generators of E by interchanging x_1 and x_j, we have $A = D(x_j)$, so again $\tau|_A = id$ and $\mathrm{SK}_1(Q, \tau) = 1$. Similarly, we are done if $\tau(x_1x_2) = x_1x_2$. For, we can then replace x_1 by x_1x_2 as a generator of E, as $\Gamma_\mathsf{E} = \mathbb{Z}(\gamma_1 + \gamma_2) \oplus \mathbb{Z}\gamma_2 \oplus \dots \oplus \mathbb{Z}\gamma_n$. Then $A = D(x_1x_2)$ and $\tau|_A = id$, showing $\mathrm{SK}_1(Q, \tau) = 1$, as before. Thus, we may assume that $\tau(x_1x_2) = -x_1x_2$. Then, $x_2x_1 = (-1)^2\tau(x_1x_2) = -x_1x_2$. Likewise, by reordering the x_i, we are done if $\tau(x_ix_j) = x_ix_j$ for any distinct i and j. So, we may assume that $\tau(x_ix_j) = -x_ix_j$, and hence $x_jx_i = -x_ix_j$, whenever $i \neq j$. If $n \geqslant 3$, we then find that $\tau(x_1x_2x_3) = x_1x_2x_3$. In this case, we can replace x_1 by $x_1x_2x_3$ as a generator of E, as $\Gamma_\mathsf{E} = \mathbb{Z}(\gamma_1 + \gamma_2 + \gamma_3) \oplus \mathbb{Z}\gamma_2 \oplus \dots \oplus \mathbb{Z}\gamma_n$. Then, $A = D(x_1x_2x_3)$, and $\tau|_A = id$, so $\mathrm{SK}_1(Q, \tau) = 1$, as before. The remaining case is that $n = 2$, D is central in E, $\tau|_D = id$, $\tau(x_i) = -x_i$ for $i = 1, 2$, and $x_2x_1 = -x_1x_2$. But then, $Z(\mathsf{E}) = D[x_1^2, x_2^2]$ and $\tau|_{Z(\mathsf{E})} = id$, which is a contradiction as τ is unitary. Thus, in all cases that can occur (4.2) holds, so also (4.1). Hence, $\mathrm{SK}_1(Q, \tau) = \mathrm{SK}_1(\mathsf{E}, \tau)$, as desired. □

Here is a quick consequence of the theorem:

Corollary 4.1 *(Stability Theorem). (See [10].) Let D be a division algebra finite-dimensional over $Z(D)$, and let τ be a unitary involution on D. Then, $\mathrm{SK}_1(D, \tau) \cong \mathrm{SK}_1(D(x), \tau')$, where $D(x)$ is the rational division algebra over D and τ' is the canonical extension of τ to $D(x)$, with $\tau'(x) = x$.*

Proof. Let $\mathsf{E} = D[x, x^{-1}]$, the (untwisted) Laurent polynomial ring in one variable over D. Then, E is a graded division algebra with $\mathsf{E}_0 = D$, $\Gamma_\mathsf{E} = \mathbb{Z}$, and $\mathsf{E}_j = Dx^j$ for all $j \in \mathbb{Z}$. Also, $q(\mathsf{E}) = D(x)$ and $Z(\mathsf{E}) = Z(D)[x, x^{-1}]$, so $[\mathsf{E} : Z(\mathsf{E})] = [D : Z(D)] < \infty$. The extension τ' of τ to $D(x)$ clearly restricts to a unitary graded involution on E. Note that $\mathsf{E}^* = \bigcup_{j\in\mathbb{Z}} D^*x^j$. For $d \in D^*$ and $j \in \mathbb{Z}$, we have $\tau'(dx^j) = \tau(d)x^j$ and $Nrd_\mathsf{E}(dx^j) = Nrd_D(d)x^{mj}$, where $m = ind(D(x)) = ind(D)$. It follows that $\Sigma'_{\tau'}(\mathsf{E}) = \bigcup_{j\in\mathbb{Z}} \Sigma'_\tau(D)x^j$ and $\Sigma_{\tau'}(\mathsf{E}) = \bigcup_{j\in\mathbb{Z}} \Sigma_\tau(D)x^j$, showing that $\mathrm{SK}_1(\mathsf{E}, \tau') \cong \mathrm{SK}_1(D, \tau)$. (This is a special case of the result in [2, Cor. 4.10] that if E is a graded division algebra with unitary graded involution τ, and $\Gamma_\mathsf{E} = \Gamma_{Z(\mathsf{E})^\tau}$, then $\mathrm{SK}_1(\mathsf{E}, \tau) \cong \mathrm{SK}_1(\mathsf{E}_0, \tau|_{\mathsf{E}_0})$.) Since $\mathrm{SK}_1(\mathsf{E}, \tau') \cong \mathrm{SK}_1(D(x), \tau')$ by Th. 1.1, we have $\mathrm{SK}_1(D(x), \tau') \cong \mathrm{SK}_1(D, \tau)$. □

5. Examples

Here are a few examples of $\mathrm{SK}_1(Q, \tau)$ that follow from known results about $\mathrm{SK}_1(\mathsf{E}, \tau)$.

For a field K containing a primitive n-th root of unity ω ($n \geqslant 2$) and any $a, b \in K^*$, let $(\frac{a,b}{K})_\omega$ denote the degree n symbol algebra over K with generators i, j and relations $i^n = a$, $j^n = b$, and $ij = \omega ji$. Note that if K has a nonidentity automorphism η such that $\eta^2 = id$ and $\eta(\omega) = \omega^{-1}$ and if $\eta(a) = a$ and $\eta(b) = b$, then $(\frac{a,b}{K})_\omega$ has a unitary involution τ satisfying $\tau(ci^k j^\ell) = j^\ell i^k \eta(c)$ for all $c \in K$, $k, \ell \in \mathbb{Z}$.

Example 5.1. Let $r_1, \dots, r_m$ be integers with each $r_i \geqslant 2$, let $s = lcm(r_1, \dots, r_m)$ and let $n = r_1 \cdots r_m$. Let L be a field containing a primitive s-th root of unity ω, and suppose L has an automorphism η of order 2 such that $\eta(\omega) = \omega^{-1}$. (For example, take $L = \mathbb{C}$ and η to be complex conjugation.) Let $K = L(x_1, \dots, x_{2m})$, a rational function field over L. For $k = 1, 2, \dots, m$, let $\omega_k = \omega^{s/r_k}$, which is a primitive r_k-th root of unity in L. Let

$$Q = \left(\frac{x_1, x_2}{K}\right)_{\omega_1} \otimes_K \dots \otimes_K \left(\frac{x_{2m-1}, x_{2m}}{K}\right)_{\omega_m}.$$

So, Q is a division algebra over K of exponent s and index n. Extend η to an automorphism of order 2 of K by setting $\eta(x_\ell) = x_\ell$ for $\ell = 1, 2, \dots, 2m$. Each symbol algebra $(\frac{x_{2k-1}, x_{2k}}{K})_{\omega_k}$ has a unitary involution τ_k as described above, with $\tau_k|_K = \eta$. Let $\tau = \tau_1 \otimes \cdots \otimes \tau_m : Q \to Q$. Since the τ_k all agree on K, this τ is a well-defined unitary involution on Q. Let $\mu_\ell(L)$ denote the group of all ℓ-th roots of unity in L. Then,

$$\begin{aligned} \mathrm{SK}_1(Q, \tau) &\cong \{c \in L^* \mid \eta(c^n) = c^n\} / \{c \in L^* \mid \eta(c^s) = c^s\} \\ &\cong \{\xi \in \mu_n(L) \mid \eta(\xi) = \xi^{-1}\} / \mu_s(L). \end{aligned} \tag{5.1}$$

For, let $\mathsf{Z} = L[x_1, x_1^{-1}, \dots, x_{2m}, x_{2m}^{-1}]$, the (commutative) iterated Laurent polynomial ring over L, with its usual multigrading in $x_1, \dots, x_{2m}$; that is, $\Gamma_{\mathsf{Z}} = \mathbb{Z}^{2m}$ with $\mathsf{Z}_{(k_1, \dots, k_{2m})} = L x_1^{k_1} \cdots x_{2m}^{k_{2m}}$. Then, Z is a graded field. Let E be the tensor product of graded symbol algebras,

$$\mathsf{E} = \left(\frac{x_1, x_2}{\mathsf{Z}}\right)_{\omega_1} \otimes_{\mathsf{Z}} \cdots \otimes_{\mathsf{Z}} \left(\frac{x_{2m-1}, x_{2m}}{\mathsf{Z}}\right)_{\omega_m}.$$

Then, the grading on Z extends to a grading on E, with $deg(i_k) = \frac{1}{r_k} deg(x_{2k-1})$ and $deg(j_k) = \frac{1}{r_k} deg(x_{2k})$, where i_k and j_k are the standard generators for $(\frac{x_{2k-1}, x_{2k}}{\mathsf{Z}})_{\omega_k}$. Then, E is a graded division algebra with center Z, and E is totally ramified over Z, i.e., $\mathsf{E}_0 = L = \mathsf{Z}_0$. Clearly $q(\mathsf{Z}) = K$ and $q(\mathsf{E}) = \mathsf{E} \otimes_{\mathsf{Z}} q(\mathsf{Z}) = Q$, and τ restricts to a unitary graded involution on E. Thus, $\mathrm{SK}_1(Q, \tau) \cong \mathrm{SK}_1(\mathsf{E}, \tau)$ by Th. 1.1, and formula (5.1) follows from the corresponding formula for $\mathrm{SK}_1(\mathsf{E}, \tau)$ given in [2, Th. 5.1], as E is totally ramified over Z (i.e., $\mathsf{E}_0 = \mathsf{Z}_0$) and Z is unramified over Z^τ.

Let $F \subseteq K \subseteq N$ be fields with N Galois over K, K Galois over F, and $[K:F] = 2$. Let $Br(K)$ denote the Brauer group of K, and let $Br(N/K)$ denote the relative Brauer group $ker(Br(K) \to Br(N))$. Let

$$Br(N/K; F) = \{[A] \in Br(N/K) \mid cor_{K \to F}[A] = 1\},$$

where $cor_{K \to F}$ is the corestriction mapping $Br(K)$ to $Br(F)$. Recall that the theorem of Albert-Riehm says that $Br(N/K; F)$ consists of the classes of central simple K-algebras A such that N splits A and A has a unitary involution τ such that $K^\tau = F$ (see [5, Th. 3.1, p. 31]). Suppose N is a cyclic Galois extension of K with $[N:K] = n$ and $\mathcal{G}(N/K) = \langle \sigma \rangle$. For $b \in K^*$, let $(N/K, \sigma, b)$ denote the cyclic algebra

$$(N/K, \sigma, b) = \bigoplus_{i=0}^{n-1} N y^i, \quad \text{where } yc = \sigma(c)y \text{ for all } c \in K \text{ and } y^n = b.$$

Let η be the nonidentity F-automorphism of K. Suppose further that N is Galois over F and N/F is *dihedral*. That is, suppose there exists $\rho \in \mathcal{G}(N/F)$ with $\rho|_K = \eta$, $\rho^2 = id_N$, and $\rho\sigma\rho^{-1} = \sigma^{-1}$. So, $\mathcal{G}(N/F) = \langle \sigma, \rho \rangle$, and this group is dihedral if $n \geqslant 3$. Observe that when N/F is dihedral, if $b \in F^*$, then $(N/K, \sigma, b)$ has a unitary involution τ given by $\tau(cy^i) = y^i\rho(c)$ for all $c \in K$, $i \in \mathbb{Z}$. Note that $\tau|_K = \eta$, so $K^\tau = F$ and $[(N/K, \sigma, b)] \in Br(N/K; F)$.

Example 5.2. Let $F \subseteq L$ be fields with $[L : F] = 2$ and L Galois over F with $\mathcal{G}(L/F) = \{id_L, \eta\}$. Let N_1 and N_2 be cyclic Galois extensions of L which are linearly disjoint over L with each N_i dihedral over F as just described. Let $n_j = [N_j : L]$ and let $\mathcal{G}(N_j/L) = \langle \sigma_j \rangle$; extend each σ_j to N_1N_2 so that $\sigma_1|_{N_2} = id_{N_2}$ and $\sigma_2|_{N_1} = id_{N_1}$. Let $K = L(x_1, x_2)$, a rational function field over L, and extend η to K by $\eta(x_i) = x_i$ for $i = 1, 2$ and likewise extend the σ_j to N_1N_2K by $\sigma_j(x_i) = x_i$ for $j = 1, 2$, $i = 1, 2$. Let

$$Q = (N_1K/K, \sigma_1, x_1) \otimes_K (N_2K/K, \sigma_2, x_2),$$

which is a division algebra over K with exponent $lcm(n_1, n_2)$ and degree n_1n_2. As noted above, each $(N_jK/K, \sigma_j, x_j)$ has a unitary involution τ_j with $\tau_j|_K = \eta$. Let $\tau = \tau_1 \otimes \tau_2$, which is a unitary involution on Q. Then,

$$\mathrm{SK}_1(Q, \tau) \cong Br(N_1N_2/L; F)/\big[Br(N_1/L; F) \cdot Br(N_2/L; F)\big]. \tag{5.2}$$

It was shown in [11] that the right expression in (5.2) can be made into any finite abelian group by choosing L to be an algebraic number field and suitably choosing N_1 and N_2. To view Q as a ring of quotients, first take $\mathsf{Z} = L[x_1, x_1^{-1}, x_2, x_2^{-1}] \subseteq K$, so Z is a commutative twice-iterated Laurent polynomial ring over L, and we give Z its usual grading by multi-degree in x_1 and x_2, as in the preceding example. Let $\mathsf{E} = N_1N_2[y_1, y_1^{-1}, y_2, y_2^{-1}] \subseteq Q$, where y_1 and y_2 are the standard generators of the symbol algebras of Q. This E is a twisted iterated Laurent polynomial ring with $y_1^{n_1} = x_1$, $y_2^{n_2} = x_2$, $y_1y_2 = y_2y_1$ and for all $c \in N_1N_2$, $y_1c = \sigma_1(c)y_1$ and $y_2c = \sigma_2(c)y_2$. We extend the grading on $\mathsf{Z} = Z(\mathsf{E})$ to E by setting $deg(y_1) = (\frac{1}{n_1}, 0)$ and $deg(y_2) = (0, \frac{1}{n_2})$; so, $\Gamma_{\mathsf{E}} = \frac{1}{n_1}\mathbb{Z} \times \frac{1}{n_2}\mathbb{Z}$. We can see that E is a graded division algebra by noting that $\mathsf{E}_0 = N_1N_2$, a field, and each homogeneous component E_γ of E is a 1-dimensional E_0-vector space containing a unit of E. This E is semiramified since $[\mathsf{E}_0 : \mathsf{Z}_0] = [N_1N_2 : L] = n_1n_2 = |\Gamma_{\mathsf{E}} : \Gamma_{\mathsf{Z}}|$; indeed, it is decomposably semiramified in the terminology of [7], since $\mathsf{E} = (N_1\mathsf{Z}/\mathsf{Z}, \sigma_1, x_1) \otimes_{\mathsf{Z}} (N_2\mathsf{Z}/\mathsf{Z}, \sigma_2, x_2)$ which expresses E as a tensor product of semiramified graded cyclic algebras. Since τ on Q clearly restricts to a unitary graded involution on E (recall that $\tau(y_1) = y_1$ and $\tau(y_2) = y_2$), Th. 1.1 shows that $\mathrm{SK}_1(Q, \tau) \cong \mathrm{SK}_1(\mathsf{E}, \tau)$. But further, let $K' = L((x_1))((x_2))$, a twice-iterated Laurent power series field over L, and let $D = (N_1K'/K', \sigma_1, x_1) \otimes_{K'} (N_2K'/K', \sigma_2, x_2)$, which is a central simple division algebra over K'. Then, the standard rank 2 Henselian valuation v on K' has associated graded ring $\mathrm{gr}(K') = \mathsf{Z}$, and for the unique extension of v to D we have $\mathrm{gr}(D) = \mathsf{E}$. Because each N_jK' is dihedral over $F((x_1))((x_2))$, there is a unitary involution $\widehat{\tau}$ on D built just as for τ on Q. This $\widehat{\tau}$ is compatible with the valuation on D, and the involution on E induced by $\widehat{\tau}$ is clearly τ. Thus, $\mathrm{SK}_1(E, \tau) \cong \mathrm{SK}_1(D, \widehat{\tau})$ by (1.1) above. But $\mathrm{SK}_1(D, \widehat{\tau})$ was computed in [9, Th. 5.6] (with another proof given in [7, Th. 7.1(ii)]), and the formula given there combined with the isomorphisms stated here yield (5.2).

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