A NOTE ON THE FRACTIONAL BLACK–SCHOLES EQUATION

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The Black -- Scholes equation

$$\frac{\partial w}{\partial t} = rw - rx\frac{\partial w}{\partial x} - \frac{1}{2}\nu^2 x^2 \frac{\partial^2 w}{\partial x^2} \tag{1}$$

is a basis of the financial mathematical model intended for option assessment [1]. In other words, this is a mathematical model of the stock market, in which the price of shares is a stochastic process. Despite on some debatable assumptions, the Black – Scholes equation undoubtedly accelerated progress in the financial market development: before 1973, nobody knew that the option as such may be assessed.

Upon changing the variables it is possible to show that the Black – Scholes equation may be reduced to a simple equation of diffusion.

However, recently an intensive analysis has been given to the generalization of the diffusion equation [2] in relation to nonlocal processes [3]:

$$D_\tau^\alpha w = \chi D_\xi^\beta w, \qquad \left\langle |\xi|^\beta \right\rangle = \chi \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)} \tau^\alpha, \tag{2}$$

here $D_\tau^\alpha w$ is a fractional derivative [4] depending on time $\tau, D_\xi^\beta w$ is a fractional derivative depending on the coordinate ξ, and $\Gamma(z)$ is the Euler gamma-function.

Considering the casual nature of option change, suppose $\partial w/\partial x$ may not exist, however. there exists fractional derivative $D_\xi^\beta w$. Employing the analogue of the Taylor formula, as well as the analogue of the mean square derivation formula (2), we obtain the fractional generalization of the Black–Scholes equation:

$$D_t^\alpha w = \Gamma(1+\alpha) r^\alpha w - r^\alpha x^\alpha D_x^\alpha w - \frac{1}{1+\alpha} \nu^{1+\alpha} x^{1+\alpha} D_x^{1+\alpha} w. \qquad (3)$$

In the case when the nonlocality parameter $\alpha \to 1$, the fractional Black – Scholes equation (3) turns into the simple Black – Scholes equation (1). Actually, we have not one but an infinite multitude of fractional Black – Scholes equations for different values of the nonlocality parameter $0 < \alpha < 1$.

References

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4. *Samko S. G., Kilbas A. A., Marichev O. I.* Integrals and Derivatives of the Fractional Order and Some of Their Applications. New York: Dover, 1993.