FLOW IN CHANNELS WITH NON-SMOOTH WALLS

V. Mityushev

Department of Mathematics, Pedagogical Academy,
ul.Podchorazych 2, Krakow, 30-084, Poland
vmityu@yahoo.com

Introduction. Three-dimensional Poiseille and Couette flows enclosed by wavy walls is investigated in [1, 2] by symbolic-numerical algorithms. The wave amplitude is proportional to the mean clearance of the channel multiplied by a small dimensionless parameter ε. A perturbation expansion in terms of the powers of ε of the full steady Navier – Stokes equations yields a cascade of boundary value problems which are solved at each step in closed form. The supremum value of ε for which the expansion converges, is determined as a function of the Reynolds number $\mathcal{R}e$. The analytical-numerical algorithm is applied to compute the velocity in the channel to $O(\varepsilon^4)$. Even in the first order approximation $O(\varepsilon)$, new results are obtained which complement the triple deck theory and its modifications. In particular, the incipient separation-detachment is discussed using the Prandtl – Schlichting criterion of starting eddies. The value ε_e for which eddies start in the channel, is analytically deduced as a function of $\mathcal{R}e$ as well as analytical formulas for the coordinates of the separation points. These analytical formulas show that ε_e in 3D channels is always less than ε_e in 2D channels. For non-smooth channels, a criterion of infinitesimally small ε_e is deduced. The critical value of ε up to which bifurcation of solutions can occur is estimated.

The present note is devoted to application of the results [1, 2] to Moffat's eddies [3] described by linear Stokes equations in non-smooth channels.

Structure of the flow. The structure of the flow in curvilinear channels essentially depends on the wave amplitude of the walls. The flow in some channels is separated into a series of flow cells. In the middle part of the channel, the velocity profiles look like a disturbed Poiseuille parabolic profile. The flow near the walls can be similar to the flow in a cavity where developed viscous eddies arise. These eddies are characterized by a change of the vorticity sign. Systematical studies of eddies were performed by Moffatt in his seminal paper [3] where he proved that any flow at the corner between two planes consists of a sequence of self-similar eddies, when the angle between the planes is less than a critical value. Let $\mathbf{u} = \mathbf{u}(x, z) = (u(x, z), w(x, z))$ be the dimensionless velocity vector, and $p = p(x, z)$ the pressure. The fluid is governed by the Stokes equations

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

in the channel bounded by the walls

$$z = b(1 + \varepsilon T(x)), \quad z = -b(1 + \varepsilon T(x)), \tag{2}$$

with the boundary conditions on (2)

$$\mathbf{u} = \mathbf{0}. \tag{3}$$

The solution $\mathbf{u}$ of (1)–(3) belongs to the class of periodic functions with period 2π in x. An overall external gradient pressure is applied along the x-direction. It can be normalized by a constant jump along the x-axis of the periodic cell

$$p(x + \pi, z) - p(x - \pi, z) = -4\pi. \tag{4}$$

One can use well developed methods of complex analysis to solve the problem, since solutions of equations (1) can be written via pair of analytic functions. Here, the results of [1] are used. The problem (1)-(4) had been solved (also for non-symmetric 3D channels) in paper [1] for the infinitely differentiable function $T(x)$. A cascade of boundary value problems were deduced for the Stokes equations for a straight channel to calculate velocities in the form of an ε-expansion.

In the present paper, we restrict ourselves the first-order approximation

$$\mathbf{u}(x, z) = \mathbf{u}_0(x, z) + \mathbf{u}_1(x, z)\varepsilon + O(\varepsilon^2), \quad p(x, z) = p_0(x, z) + p_1(x, z)\varepsilon + O(\varepsilon^2). \tag{5}$$

The representations (5) were justified in [1] where it was proved that (5) holds when ε does not exceed a critical value

$$\varepsilon_c = (b \sup_m [m \max(|\alpha_m|, |\beta_m|])^{-1}, \tag{6}$$

where α_m and β_m denotes the Fourier coefficients of $T(x)$. Hence,

$$T(x) = \sum_{m=1}^{\infty} \alpha_m \cos mx + \beta_m \sin mx. \tag{7}$$

The Prandtl – Schlichting criterion [2] for eddies in 2D smooth channels reads as

$$\frac{\partial u}{\partial n} = 0, \tag{8}$$

where u is the x-component of the velocity $\mathbf{u}$, $\frac{\partial}{\partial n}$ is the normal derivative to the wall. In the next section, equation (8) is solved explicitly up to $O(\varepsilon^2)$.

We have [1]

$$u_1(x, z) = 2b^2 \sum_{m=1}^{\infty} P_m(z)[\alpha_m \cos mx + \beta_m \sin mx], \tag{9}$$

where the function $P_m(z)$ has the form

$$P_m(z) = 2\frac{(bm \cosh bm - \sinh bm) \cosh mz - \sinh bm \sinh mz}{2bm - \sinh 2bm}. \tag{10}$$

Calculating formally $\frac{\partial u}{\partial n}$ at the bottom point $(0, -b - \varepsilon)$ by (8) up to $O(\varepsilon^2)$ we arrive at the formula

$$\frac{\partial u}{\partial n}(0, -b - \varepsilon) = 2b\left[1 + \varepsilon \sum_{m=1}^{\infty} \alpha_m \left(1 + \frac{4bm \sinh^2 bm}{2bm - \sinh 2bm}\right)\right] + O(\varepsilon^2). \tag{11}$$

It follows from formula (11) that eddies arise at any non-smooth channel.

However, the obtained result has to be checked up to $O(\varepsilon^4)$, since lower order formulas can give not real physical results as shown in [2].

References

1. *Malevich A.E., Mityushev V.V., Adler P.M.* Stokes flow through a channel with wavy walls // Acta Mechanica, V. 182, 2006. P. 151-182.

2. *Malevich A.E., Mityushev V.V., Adler P.M.* Couette flow in channels with wavy walls // Acta Mechanica, V. 197, 2008. P. 247–283.

3. *Moffat H.K.* Viscous and resistive eddies near a sharp corner // J. Fluid Mech., V. 18, 1964. P. 1–18.