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This book presents the results of the research of the sparse underdetermined systems of linear algebraic equations and their applications. The methods of decomposition and the theory of graphs partitioning are applied to construct solutions of underdetermined systems with special sparse matrices. Numerous examples of decomposition algorithms for different types of sparsity are considered. Some of these algorithms are implemented in Wolfram Mathematica using new technologies for constructing analytical and numerical solutions.

Table 46. Fig. 64. Bibl. 60.

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CONTENTS

LIST OF REFERENCE SYMBOLS

$\emptyset$ − empty set
$\in$ − set membership
$\exists$ − existential quantor
∞ − infinity
$\oplus$ − direct sum
$\subseteq$ − subset
$\subset$ − proper subset
$\bigcup$ − set union
$\bigcap$ − set intersection
$\setminus$ − set difference
$X \times Y \stackrel{\text{def}}{=} \{(x,\ y) : x \in X,\ \ y \in Y\}$ − cartesian product of set X and Y
$\square$ − sign of end of proof
$\vee$ − disjunction
$\wedge$ − conjunction
$|I|$ − number of elements in I (cardinality of I)
Z_+ − set of positive integers
$\mathbf{R}$ − set of real numbers
R^n − set of vectors of length n with real components
$rank(A)$ − rank of matrix A
$det\ A$ − square matrix A determinant
A^T − transpose of A
A^{-1} − inverse matrix for non-singular matrix A
$f(n) = O(q(n))$, if there's such a constant c > 0, that $f(n) \le cq(n)$, for sufficiently large $n \in Z_+$
$\| x \| \stackrel{\text{def}}{=} \sqrt{x^T x}$ − Euclidean norm of vector $x \in R^n$
SLAE − System of Linear Algebraic Equations
SLP − Sensor Location Problem
CAS – Computer Algebra System

INTRODUCTION

Sparse matrix technology is an important research area in applied mathematics. This direction of research not only is used as an computing means in many applications, but is a separate area of theoretical research in the area of sparse matrix analysis. In addition with the theoretical research in the area of sparse matrix analysis also research in the field of advanced information technologies for constructing numerical solutions for sparse systems of linear algebraic equations and inequalities and their applications is conducted. Present bibliography in monography "Sparse linear systems and their applications" contains only the works that reflect the direction of research, developed in the monography and does not pretend to be exhaustive. The problems, models and methods of network flow programming which constitute a class of network optimization problems of linear and nonlinear programming are an important application of the obtained results for sparse underdetermined systems of linear algebraic equations. It is clear that a book of this size can not cover all the variety of research on sparse matrix analisys and its applications. So, in the selection of material, the author adhered to the following principle: to include only those results on solving sparse linear systems, which are applicable for construction of efficient algorithms for solving of linear and nonlinear problems in the network flow programming or have a relatively new annex to solution of graph and network optimization problems. The approach in which the general optimization principles are used to the sparse structure of problem restrictions [11–13, 44, 45, 54, 55] is perspective in the constructive network optimization theory. Important is research combinatorial properties of embedded network structures of restrictions of the extremal problems and development and use modern advances in technology of construction their numerical solutions [2, 45, 53]. It is necessary to emphasize the importance of the combinatorial properties of the support [10–12, 45, 53] for different types of sparse matrices in the systems of linear algebraic equations, and the latest advances in technologies of construction of their numerical solutions.

In this monography we consider the application of the graph theory for construction the solutions of linear systems with rectangular sparse matrices, namely of linear underdetermined sparse systems. Different types of sparsity are investigated. The decomposition theory for a graph or a multigraph will be applied to construct the solutions of linear systems with rectangular sparse matrices with different types of sparsity. Sparse systems of these types appear

in non-homogeneous network flow programming problems [2, 12, 21, 22, 26, 27, 44, 45, 47 – 49] in the form of restrictions and can be characterized as systems with a large sparse sub-matrix representing the embedded network structure. We develop direct methods for finding solutions of systems of these types. These algorithms are based on the theoretic-graph specificities of the structure of the support for the graph or the multigraph or generalized multigraph and on the properties of the basis of the solution space of homogeneous sparse systems of special types. The decomposition of the graphs or the multigraphs or, as a result, the decomposition of sparse underdetermined systems is one of the main steps in the solution of linear systems with rectangular sparse matrices. The work on this monography was caused, mainly, by the analysis of problems of non-homogeneous network flow optimization [2, 12, 21, 22, 26, 27, 44, 45, 47–49] with additional constraints of the general type in the case of large data. It was also caused by the new applications of the constructive theory to the solution of the linear underdetermined sparse systems in the Sensor Location Problem for the graphs and multigraphs [6, 46, 50, 52, 53]. The general idea of the decomposition theory for construction the solutions of linear systems with rectangular sparse matrices is based on the following key steps:

- *Separation of the network part of the equations of the system and the additional part*. The network part of the equations of the system represents a network structure and corresponds to the network part of the main restrictions of the non-homogeneous network flow programming problem [45]. The additional part of the equations of the system corresponds to the additional part of the system of main restrictions and can have a general form. We start the process of solution by considering the network part of the sparse underdetermined system.
- *Introduction of a support of the graph or multigraph for a system.* The term 'support of the graph (multigraph) or network (multinetwork)' (also referred to as network support, or support) is borrowed from the optimization theory [10−12] and is used here for further compatibility with applications in problems of non-homogeneous network flow programming [2, 12, 21, 22, 26, 27, 44, 45, 47−49] and in the Sensor Location Problem for the graphs [6, 46, 50, 52, 53].
- *Construction of the general solution for thenetwork part of the sparse linear system*. We compute a basis of the solution space of the corresponding homogeneous system and interpret the basis vectors as characteristic vectors, entailed by non-support arcs and nodes with variables intensities. Effective

algorithms for finding a partial solution of the network part of the sparse linear non-homogeneous system is obtained.

- *Decomposition algorithms for the system*. The decomposition algorithms for a graph or a multigraph are applied to construct the solutions of systems of linear algebraic equations with rectangular sparse matrices with different types of sparsity.

The book consists of six chapters. In Chapter 1 the properties of the sparsity of fractal-like matrices are investigated and computational algorithms for addition, multiplication and inversion of the fractal-like matrices are developed. Computational estimates of the number of arithmetic operations in the worst case were obtained. A special way of presenting the fractal-like matrix as a vector is offered.

The Chapter 2 of the monography is devoted to the study of sparse underdetermined linear algebraic equations with embedded network structure. The embedded network structure of a sparse system of linear equations is considered and the set of solutions for the network part of this system is constructed. For identification of the set of solutions for the sparse part of the system the properties of one-to-one correspondence between the elements of a forest of trees with special properties and the columns of the supporting matrix of the system are used. We compute a basis of the solution space of the corresponding homogeneous system, generated by the sparse part of the system of equations and interpret the basis vectors as characteristic vectors, entailed by non-supporting arcs and non-supporting nodes with variables intensities. We investigate the support structure for sparse network part of the system and formulate its graph-theoretical properties. An example for construction of the solution of the sparse underdetermined linear system is considered and the developed method of decomposition is applied to this system.

Chapter 3 is devoted to the development of SLAE, which has important applications in problems of finding arc flows and the values of the variables intensities of the nodes on unobservable parts of the graph. That problem is formulated in the following way: find the location of the minimum number of the observed nodes in a graph in order to determine the values of the arc flows and the values of the variable intensities of the nodes of the graph so that the sparse linear system of the special type has an unique solution. That problem is named Sensor Location Problem for a graph (SLP for a graph). For constructing the solution of the SLP for a graph efficient decomposition algorithms for SLAE are applied, which are discussed in Chapter 2 of the

monography. The analytical and numerical solutions in numerous examples of the problem of optimal location of a single sensor in a graph are constructed. Also, an example of optimal location of sensors in a graph for the case of multiple monitored nodes is considered.

The linear underdetermined systems of not full rank are considered in Chapter 4. Combinatorial properties of the embedded network structure of the sparse system are investigated. The decomposition algorithms are based on the graph theoretic specificities of the support structure [12, 45] and properties of the basis of the solution space of the homogeneous system of not full rank. For different types of flows examples of building solutions of sparse systems of not full rank are considered. An implementation in CAS *Wolfram Mathematica* is presented for the decomposition algorithm for the solution of the sparse underdetermined system of not full rank. The computations are performed in the rational arithmetic.

In Chapter 5 applications of the discussed algorithms in Chapter 4 and in [32, 44] for construction solutions of sparse underdetermined systems in Sensor Location Problem for a multigraph are considered. The graph-theoretical properties of the support of the multigraph for a sparse system of the embedded network structure are investigated and it is used for the decomposition of the support of the multigraph for the whole system. An algorithm for modeling multigraphs for the problem of finding the location of the minimum number of monitored nodes in a multigraph is constructed. The theory of decomposition for the multigraph support is applied in construction the solutions of systems with rectangular sparse matrices for disconnected multigraphs.

In Chapter 6 linear sparse underdetermined systems of full rank for the generalized multigraph are considered by analogy to Chapter 4. Systems of this type can be characterized as systems with a large sparse sub-matrix representing the embedded network structure. The constructed decomposition algorithms for solution of the investigated systems are based on the theoretical-graph specificities of the support structure for the generalized multigraph. The examples for finding the general solutions of full rank sparse underdetermined systems are constructed. A new technology of implementation the decomposition algorithms for solution of the considered sparse underdetermined system of full rank is used. The computations are performed in CAS *Wolfram Mathematica* using the rational arithmetic.

1. FRACTAL-LIKE MATRICES

1.1. Introduction

The currently best lower bound for matrix multiplication algorithm complexity is $\Omega(n^2)$. Still there is little progress towards reaching (or rising) it [20], [58]. The ordinary algorithm with $\Theta(n^3)$ complexity is most commonly used, because asymptotically faster algorithms are impractical for small n. Taking matrix inverse is also hard: in terms of computational complexity it is equivalent to matrix multiplication [4].

For instance, let's consider the set of band matrices. They have zeros outside a diagonal stripe of fixed width. The example of a band matrix is shown in Figure 1.1.

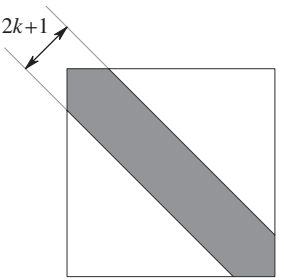

Fig. 1.1. Structure of band matrices

Strictly speaking, $A = (a_{ij}) \in \mathbb{R}_{n\times n}$ is a band matrix iff $\exists k$ such that $a_{ij} = 0$ if $|i - j| > k$. Typically one assumes $1 < k \ll n$. Band matrices allow multiplication and inversion in $O(k^2 n)$ time. Indeed, we should use the ordinary multiplication algorithm or Gauss method correspondingly, taking into account specific zero layout. However, if $k > 1$ almost every matrix from this set doubles its band width after one performs multiplication or inversion on it. If $k = O(\sqrt{n})$ we are able to take products and inverses, using $O(n^2)$ arithmetical operations.

In [37] we built another set of special sparse matrices. Unlike band matrices they have a structure that is invariant under multiplication and inversion. Also they may hold asymptotically more than $n\sqrt{n}$ non-zero entries and still be multiplied or inverted in $O(n^2)$ time.

1.2. Definition of Fractal-Like matrices

We define F_n (the set of n-ordered fractal-like matrices) for each n that is a power of 2, using mathematical induction.

1) $F_{2^0} := \mathbb{R}_{1\times 1}$

2) $\forall k > 0, F_{2^k} := \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A, B, C \in F_{2^{k-1}} \right\}$

Example of the fractal-like matrices are given in Figure 1.2.

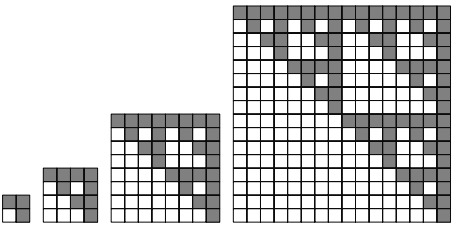

Fig. 1.2. Structure of fractal-like matrices

If $n \to \infty$ the structure of F_n elements reminds Sierpinski triangle, the famous fractal [5].

1.3. Main properties

The set F_n is a linear subspace and a subring of $\mathbb{R}_{n\times n}$ therefore it is a subalgebra of $\mathbb{R}_{n\times n}$. Indeed, let's prove the following theorem.

Theorem 1.3.1. *F_n is an algebra over $\mathbb{R}$, containing E_n. The nonsingular entries of F_n are invertible and their inverses belong to F_n, too.* $\dim F_n = n^{\log_2 3}$.

Proof. Let's use mathematical induction.

1) Since $\mathbb{R}_{1\times 1} \cong \mathbb{R}$ the theorem statement is true for $n = 1$.

2) Let's prove that it is also true for $n = 2^k, k > 0$, assuming that we've already made a proof for $n = 2^{k-1}$.

Consider two matrices A and B:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \in F_n.$$

$$\forall \alpha, \beta \in \mathbb{R}, \alpha A + \beta B = \begin{pmatrix} \alpha A_1 + \beta B_1 & \alpha A_2 + \beta B_2 \\ 0 & \alpha A_3 + \beta B_3 \end{pmatrix} \in F_n,$$

because $\alpha A_i + \beta B_i \in F_{\frac{n}{2}}$, $i = 1,2,3$.

$$AB = \begin{pmatrix} A_1 B_1 & A_1 B_2 + A_2 B_3 \\ 0 & A_3 B_3 \end{pmatrix} \in F_n,$$

because $A_1 B_1$, $A_1 B_2 + A_2 B_3$, $A_3 B_3 \in F_{\frac{n}{2}}$.

$$E_n = \begin{pmatrix} E_{\frac{n}{2}} & 0 \\ 0 & E_{\frac{n}{2}} \end{pmatrix} \in F_n$$

because $0_{\frac{n}{2}}, E_{\frac{n}{2}} \in F_{\frac{n}{2}}$.

Finally, if $\det A = \det A_1 \cdot \det A_3 \neq 0$ then

$$A^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1} A_2 A_3^{-1} \\ 0 & A_3^{-1} \end{pmatrix} \in F_n,$$

because $A_1^{-1}, A_3^{-1}, -A_1^{-1} A_2 A_3^{-1} \in F_{\frac{n}{2}}$.

$$\dim F_n = 3(\frac{n}{2})^{\log_2 3} = n^{\log_2 3} q.e.d.$$

□

To obtain the number of arithmetical operations required to add, multiply or invert fractal-like matrices through recursive application of above formulas, one may solve recurrence equations. For example, the equation $M_k = 4M_{k-1}$ (for the number of real multiplications required to take product in F_{2^k}) under the initial condition $M_0 = 1$ has the following solution: $M_k = 4^k = n^2$ [37]. More accurate analysis leads to the results listed in Table 1.3.1.

Let's show that in F_n multiplication is as hard as inversion (the same happens in $\mathbb{R}_{n \times n}$ [3]). Indeed, multiplication of A by B may be reduced to 2 inversions:

$$\begin{pmatrix} E & A \\ 0 & -B^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} E & AB \\ 0 & -B \end{pmatrix}$$

(if B is singular we may assume $B := B + \lambda E$ to make it diagonally dominant). As for inversion, it is reduced to 2 multiplications, 2 inversions

Table 1.3.1

Arithmetical cost of operations over fractal-like matrices. ADD/SUB – real addition/subtraction, MUL – real multiplication, DIV – real division, NEG – real sign change

Matrix operation	ADD/SUB count	MUL count	DIV count	NEG count
Addition/subtraction	$n^{\log_2 3}$	0	0	0
Multiplication	$n^2 - n^{\log_2 3}$	n^2	0	0
Inversion	$n^2 - 2n^{\log_2 3} + n$	$n^2 - n$	n	$n^{\log_2 3} - n$

and 1 sign change of lower ordered matrices (see the formula in the above proof). Suppose there exists a multiplication algorithm with $O(n^\alpha)$ complexity ($\alpha \geqslant \log_2 3$). Then we can apply it recursively and finally invert matrices, using $O(n^\alpha)$ operations. It follows from the corresponding recurrence equations for operation count.

Now let's consider the following equality for product of a fractal-like matrix by a vector:

$$Ab = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_1 b_1 + A_2 b_2 \\ A_3 b_2 \end{pmatrix}.$$

The calculation under this formula demands $n^{\log_2 3}$ real multiplications and $n^{\log_2 3} - n$ real additions.

As for solving a linear equation $Ax = b$, $A \in F_n$ the formula

$$x = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_1^{-1}(b_1 - A_2(A_3^{-1} b_2)) \\ A_3^{-1} b_2 \end{pmatrix}$$

yields an algorithm with n divisions, $n^{\log_2 3} - n$ multiplications and $n^{\log_2 3} - n$ additions/subtractions.

1.4. Optimal storage

We propose a special storage scheme for fractal-like matrices that preserves memory wasting and is convenient to perform arithmetical operations on them. Instead of remembering a 2-dimensional array of size n^2 we may store a row of length $n^{\log_2 3}$. Indeed,

1) $\begin{pmatrix} a \end{pmatrix} \in F_1$ should be represented by a vector $\mathbf{a} = [a]^T$ of length $1 = 1^{\log_2 3}$;

2) $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in F_n$, $n = 2^k$, $k > 0$ should be represented by the vector

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T.$$

Length of the vector $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T$ is equal to

$$3(\frac{n}{2})^{\log_2 3} = n^{\log_2 3},$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are representations of matrices A, B and C correspondingly.

Example. The fractal-like matrix

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

of size $n \times n$, $n = 2^k$, $k = 2$ should be represented by the vector

$$[1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9]^T.$$

2. SPARSE UNDERDETERMINED SYSTEMS

2.1. General type of sparsity

We consider decomposition algorithms for solving sparse linear systems with embedded network structure. We investigate this problem on the base of theoretical-graph specificity of the structure of the support. We also use properties of basis of the solution space of the sparse homogeneous system of linear algebraic equations. One of applications of the given approach is the Sensor Location Problem (SLP). That is the problem of location of the minimum number of sensors in the nodes of the network, in order to determine the arc flows and node intensities.

Let $S = (I,U)$ be a finite oriented connected graph without multiple arcs and loops, where I is a set of nodes and U is a set of arcs defined on $I \times I(|I| < \infty, |U| < \infty)$.

Consider the following linear underdetermined system

$$\sum_{j \in I_i^+(U)} x_{ij} - \sum_{j \in I_i^-(U)} x_{ji} = \begin{cases} a_i, & i \in I \setminus I^* \\ x_i \cdot sign[i], & i \in I^* \end{cases} \tag{2.1.1}$$

$$\sum_{(i,j) \in U} \lambda_{ij}^p x_{ij} = \alpha_p, \quad p = \overline{1,q} \tag{2.1.2}$$

$$sign[i] = \begin{cases} 1, & i \in I^n \\ -1, & i \in I^* \setminus I^n \end{cases}, \quad I^n \subseteq I^* \tag{2.1.3}$$

where $I_i^+(U) = \{j\colon (i,j) \in U\}$, $I_i^-(U) = \{j\colon (j,i) \in U\}$; $a_i, \lambda_{ij}^p, \alpha_p$ – parameters of the system; $x = (x_{ij}, (i,j) \in U;\ x_i, i \in I^*)$ – the vector of unknowns.

The matrix of system (2.1.1) – (2.1.2) has the following block structure:

$$A = \begin{bmatrix} M & R \\ Q & 0 \end{bmatrix} \tag{2.1.4}$$

Here M is a sparse incidence matrix of the graph $S = (I,U)$ of size $|I| \times |U|$. Matrix R has size $|I| \times |I^*|$ and for each column of R there is only one nonzero element equal to $-\,sign[i]$ which corresponds to the i-th line. Q is a matrix of size $q \times |U|$ and consists of the elements λ_{ij}^p for $(i,j) \in U$, $p = \overline{1,q}$.

We assume that $rank\ (A) = |I| + q$ and $rank\ (A) < |U| + |I^*|$ holds.

2.2. Network part of system

We start the solution of system (2.1.1) − (2.1.2) by considering the network part of the system.

Definition 2.2.1. *We call system* (2.1.1) *the network part of the system* (2.1.1) − (2.1.2). *System* (2.1.2) *is called the additional part of the system* (2.1.1) − (2.1.2).

Before we proceed, let us recall the following necessary and sufficient condition of consistency for system (2.1.1) implied by Kronecker-Capelli theorem.

Theorem 2.2.1 (Rank theorem). *If $I^* \neq \emptyset$, then the rank of the matrix of system (2.1.1) for the network $S = (I,U)$ equals $|I|$.*

Remark 2.2.1. We assume, without loss of generality, that the rank of the system (2.1.1) - (2.1.2) is $|I| + q$, where q is a number of equations in the additional part (2.1.2).

Definition 2.2.2. Let's consider any cycle $L = (I_L,U_L)$ of the graph. Let's then construct a vector $\delta = (\delta_{ij}, (i,j) \in U;\ \delta_i, i \in I^*)$ according to the following rules:

- Choose an arbitrary arc from the cycle. Let it be an arc $(\tau,\rho) \in U_L$. This arc sets the cycle detour direction and $\delta_{\tau,\rho} = 1$.
- For cycle's forward arcs, let $\delta_{i,j} = 1$.
- For cycle's backward arcs let $\delta_{i,j} = -1$.
- For nodes form the set I^* that form part of the cycle, let $\delta_i = 0$, for $i \in I_L \cap I^*$.
- For arcs that haven't formed any part of the cycle, let $\delta_{i,j} = 0$, for $(i,j) \in U \setminus U_L$.
- For nodes from the set I^* not included into the cycle, let $\delta_i = 0$, for $i \in I_L \setminus I^*$.

The vector δ constructed according to the described rules we will call the **characteristic vector of the cycle**. Obviously, the characteristic vector depends on the chosen direction.

D e f i n i t i o n 2.2.3. Consider any chain $C = (I_C, U_C)$ of the graph connecting nodes $u, v \in I^*$. The vector δ constructed according to the following rules is the **characteristic vector of the chain with the direction according to a node**:

- Let node u be the beginning of the chain and v be the end. Thus, we define the direction of the chain.
- For the beginning node u of the chain, let $\delta_u = 1$.
- For the last node, let

$$\delta_v = \begin{cases} sign[v] \cdot \delta_{vj}, & (v,j) \in U_C \\ -sign[v] \cdot \delta_{jv}, & (j,v) \in U_C \end{cases}$$

- For forward arcs of the chain, let $\delta_{ij} = sign[u]$.
- For backward arcs of the chain let $\delta_{ij} = -sign[u]$.
- For nodes $i \in I_C \cap I^* \setminus \{u,v\}$ let $\delta_i = 0$.
- For arcs that do not belong to the chain let $\delta_{ij} = 0, (i,j) \in U \setminus U_C$.
- For nodes $i \in I^* \setminus I_C$, let $\delta_i = 0$.

D e f i n i t i o n 2.2.4. Consider any chain $C = (I_C, U_C)$ of the graph connecting two nodes $u, v \in I^*$. The vector δ constructed according to the following rules we will call the **characteristic vector of the chain with the direction according to an arc:**

- Choose any arc $(\tau, \rho) \in U_C$ that define the direction of the chain.
- For the chain's forward arcs, let $\delta_{ij} = 1$.
- For the chain's backward arcs, let $\delta_{ij} = -1$.
- For nodes u and v, let,

$$\delta_u = \begin{cases} sign[u] \cdot \delta_{uj}, & (u,j) \in U_C, \\ -sign[u] \cdot \delta_{ju}, & (j,u) \in U_C. \end{cases} \quad \delta_v = \begin{cases} sign[v] \cdot \delta_{vj}, & (v,j) \in U_C, \\ -sign[v] \cdot \delta_{jv}, & (j,v) \in U_C. \end{cases}$$

- For nodes $i \in I_C \cap I^* \setminus \{u,v\}$, let $\delta_i = 0$.
- For the arcs that do not belong to the chain, let $\delta_{ij} = 0$, where $(i,j) \in U \setminus U_C$.
- For nodes $i \in I^* \setminus I_C$, let $\delta_i = 0$.

Lemma 2.2.1. *The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node and the characteristic vector of a chain with the direction according to an arc satisfy the system*

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} 0, & i \in I \setminus I^* \\ x_i \cdot sign[i], & i \in I^* \end{cases} \tag{2.2.1}$$

Theorem 2.2.2. *Any solution of the system (2.2.1) is a linear combination of characteristic vectors.*

Proof Let $x = (x_{ij}, (i,j) \in U; \quad x_i, i \in I^*)$ be a solution of system (2.2.1). We show that vector x can be represented as the sum of the characteristic vector of the cycle multiplied by some coefficient and some other solution $y = (y_{ij}, (i,j) \in U; \quad y_i, i \in I^*)$ of system (2.2.1), which has a smaller number of nonzero components.

Consider graph $R = \{I,T,S\}$ where T is the set of the arcs that correspond to nonzero components of the vector x, $T = \{(i,j) : (i,j) \in U, x_{ij} \neq 0\}$ and S is the set of the nodes that correspond to nonzero components of the vector x, $S = \{i : i \in I^*, x_i \neq 0\}$.

We find in R the following structural elements: a cycle or a chain between two nodes from the set S.

Possible situations are:

- At least one cycle exists in the graph R.
- A chain $C = (I_C,U_C)$ that connects nodes $i_1,i_2 \in S$ exists in the graph R.

At least one cycle exists in the graph R. Then we choose any one of them $L = (I_L,U_L)$. Let's denote by $\delta = (\delta_{ij}, (i,j) \in U; \quad \delta_i, i \in I^*)$ the characteristic vector of the cycle L. Let $(i_0,j_0) \in U_L$ be any of the cycle's arcs. Without loss of generality, it can be considered a forward arc of the cycle. We represent components of the vector x as

$$\begin{cases} x_{ij} & = x_{i_0j_0}\delta_{ij} + x'_{ij}, (i,j) \in U \\ x_i & = x_{i_0j_0}\delta_i + x'_i, \quad i \in I^* \end{cases} \tag{2.2.2}$$

where the vector $x' = (x'_{ij}, (i,j) \in U; \quad x'_i, i \in I^*)$, where $x'_{ij} = x_{ij} - x_{i_0j_0}\delta_{ij}$, for $(i,j) \in U$, and $x'_i = x_i - x_{i_0j_0}\delta_i$, for $i \in I^*$. It also appears to be

a solution of system (2.2.1) and contains at least one non zero component less. Thus, we have reduced the number of elements of the set T.

A chain $C = (I_C,U_C)$ that connects nodes $i_1,i_2 \in S$ exists in the graph R. Let $(i_0,j_0) \in U_C$ be any of the chain's arcs. Without loss of generality, it can be considered forward. Let's denote by $\delta = (\delta_{ij},\ (i,j) \in U;\ \delta_i,\ i \in I^*)$ the characteristic vector of the chain with the direction according to an arc (i_0,j_0). By analogy, we can construct a characteristic vector of the chain C with the direction according to node $i_1 \in S$. We represent vector x, the solution of system (2.2.1), as (2.2.2) (or analogical formulas if the characteristic vector with the direction according to the node). Obviously, vector x' also appears to be a solution of system (2.2.1) and it has at least one nonzero component less. Thus, we have reduced the set T and broken the chain into two parts: i_1 belongs to one of them and i_2 to the other.

We then apply this process to the vector x' and all successive vectors x'', x''' etc, constructed according to the rules in (2.2.2) while some cycles exist in R or chains, connecting the nodes from S exist in the graph R. During each step, we get at least one more zero component in the successive vectors x', x'' etc.

We prove that if no chains or no cycles exist in the graph R then the system

$$\sum_{j\in I_i^+(T)} x_{ij} - \sum_{j\in I_i^-(T)} x_{ji} = \begin{cases} 0, & i \in I \setminus S \\ x_i \cdot sign[i], & i \in S \end{cases} \tag{2.2.3}$$

has only trivial solution.

Let graph R be consisting of s connected components. Then system (2.2.3) splits into s independent systems.

Consider any connected component $R^k = \big(I(T^k),T^k\big)$, where T^k is set of arcs of the connected component R^k, $I(T^k)$ is set of nodes of the connected component R^k and $S^k = S \cap I(T^k)$, and S is the set of the nodes that correspond to nonzero components of the last vector of x', x'' etc. sequence. It has no cycles, so the connected component R^k is a tree and therefore $|T^k| = |I(T^k)| - 1$. Since graph R^k has no chain of a considered type, we have $|S^k| \leqslant 1$.

If $|S^k| = 0$ then there is one equation more than the number of variables. Let's denote by A^k the block of the matrix of system (2.2.3) for the mentioned connected component that corresponds to the incidence matrix of the tree R^k, for which we have $rank(A^k) = |T^k|$, the corresponding subsystem has only a trivial solution. If $|S^k| = 1$ and thus $S^k = \{i_k\}$ then A^k

is the incidence matrix of the tree T^k plus one column with one nonzero element (the signum of the node i_k). In this case, $rank(A^k) = |T^k| + 1$ which is equal to the number of unknowns of the subsystem corresponding to the connected component R^k. Therefore the corresponding subsystem has only a trivial solution.

Thus, we have decomposed the solution x of system (2.2.3) into linear combination of characteristic vectors. A constructive method of representation of the vector x as a linear combination of characteristic vectors is completely described. □

2.3. Network support criterion

Let's define a support of the network $S = (I,U)$ for system (2.1.1).

Definition 2.3.1. *Let's call an aggregate of sets $R = \{U_R, I_R^*\}$, $U_R \subseteq U$, and $I_R^* \subseteq I^*$ the support of the graph G for the system* (2.1.1) *if for $\tilde{R} = \{\tilde{U}, \tilde{I}^*\}$, $\tilde{U} = U_R$, $\tilde{I}^* = I_R^*$ the system*

$$\sum_{j \in I_i^+(\tilde{U})} x_{ij} - \sum_{j \in I_i^-(\tilde{U})} x_{ji} = \begin{cases} 0, & i \in I \setminus \tilde{I}^* \\ x_i \cdot sign[i], & i \in \tilde{I}^* \end{cases} \tag{2.3.1}$$

has only a trivial solution, but has a nontrivial solution for any of the following set aggregations:

$$\begin{cases} \tilde{R} = \{\tilde{U}, \tilde{I}^*\}, \quad \tilde{U} = U_R \cup (i_0, j_0), \text{for } (i_0, j_0) \in U \setminus U_R, \tilde{I}^* = I_R^*; \\ \tilde{R} = \{\tilde{U}, \tilde{I}^*\}, \quad \tilde{U} = U_R, \quad \tilde{I}^* = I_R^* \cup \{i_0\}, \quad \text{for } i_0 \in I^* \setminus I_R^*. \end{cases}$$

For some subset of arcs $U_1 \subseteq U$ we introduce the set of nodes

$$I(U_1) = \{i \in I : (i,j) \in U_1 \vee (j,i) \in U_1\}.$$

We construct a forest from s the trees $T^k = \left(I(U_T^k), U_T^k\right)$, $s \leq |I^*|$, such as every tree has exactly one node $u_k \in I_R^*$, for $k = \overline{1,s}$ and $\cup_{k=1}^s I(U_T^k) = I$. We form the sets

$$U_R = \bigcup_{k=1}^{s} U_T^k, \quad I_R^* = \bigcup_{k=1}^{s} \{u_k\}$$

Theorem 2.3.1 (Network Support Criterion). *An aggregate of sets $R = \{U_R, I_R^*\}, U_R \subseteq U$ and $I_R^* \subseteq I^*$ is the support of the graph S for the system (2.1.1) if and only if the following conditions are carried out:*

- *Each connected component* $T^k = \left(I(U_T^k), U_T^k\right), k = \overline{1,s}$ *is a tree;*
- $I(\cup_{k=1}^s U_T^k) = \cup_{k=1}^s I(U_T^k) = I$*;*
- $|I_k^*| = 1$*, where* $I_k^* = I_R^* \cap I(U_T^k), k = \overline{1,s}$.

Proof. Follows directly from combinatorial properties of the support of the graph S for the system (2.1.1) [32, 33]. □

In Figure 2.2 shown an example of a support $R = \{U_R, I_R^*\}$ to the graph which is presented in Figure 2.1 for the sparse system (2.3.2), where $U_R = \{(1,3), (2,1)\}$, $I_R^* = \{2,4,6\}$. Let us recall the following necessary and sufficient condition of consistency for system (2.3.2) implied by Kronecker-Capelli theorem: $b_1 + x_2 + b_2 + b_3 + x_4 + b_4 + x_6 + b_6 = 0$.

$$\begin{aligned} x_{1,2} + x_{1,3} - x_{2,1} &= b_1, \\ x_{2,4} + x_{2,6} + x_{2,1} - x_{1,2} &= x_2 + b_2, \\ -x_{1,3} &= b_3, \\ -x_{2,4} &= x_4 + b_4, \\ -x_{2,6} &= x_6 + b_6. \end{aligned} \tag{2.3.2}$$

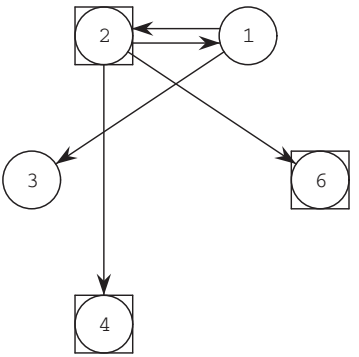

Fig. 2.1. Graph S

After the support $R = \{U_R, I_R^*\}$ of the graph S for the system (2.1.1) is chosen, let's consider what structures will be obtained after adding one nonsupporting element to the support R.

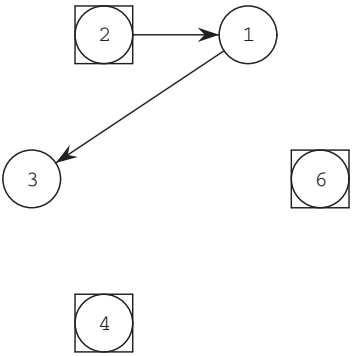

Fig. 2.2. Support of the graph S (Fig.2.1) for the system (2.3.2)

2.4. Basis of solution space

Definition 2.4.1. The **characteristic vector entailed by an arc** $(\tau,\rho) \in U \setminus U_R$ is the vector $\delta(\tau,\rho) = (\delta_{ij}^{\tau\rho}, (i,j) \in U; \delta_i^{\tau\rho}, i \in I^*)$ constructed according to the following rules:

- If the set $U_R \cup \{(\tau,\rho)\}$ has a cycle $L = \{I_L,U_L\}$, then the entailed characteristic vector $\delta(\tau,\rho)$ is the characteristic vector δ of that cycle, and the arc (τ,ρ) is chosen to define the detour direction of the cycle.
- If the set $U_R \cup \{(\tau,\rho)\}$ has a chain $C = \{I_C,U_C\}$ that connects nodes $u, v \in I_R^*$, then the entailed characteristic vector $\delta(\tau,\rho)$ is the characteristic vector δ of that chain, and the arc that defines the detour direction is chosen to be (τ,ρ).

Definition 2.4.2. The **haracteristic vector entailed by a node** $\gamma \in I^* \setminus I_R^*$ is the characteristic vector $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*)$ of the chain that connects nodes γ and $v \in I_R^*$, with node γ being chosen as the beginning of the chain.

Theorem 2.4.1. *Any solution of the homogeneous system (2.2.1) may be uniquely represented as a linear combination of characteristic vectors entailed by the non-supporting elements* $(\tau,\rho) \in U\setminus U_R$, *and* $\gamma \in I^*\setminus I_R^*$ *for the system (2.2.1) of the graph* $G = (I,U)$.

Proof. We have to prof that the aggregate of entailed characteristic vectors make up the basis of the space of solutions of the system (2.2.1).

The fact that each characteristic vector satisfies system (2.2.1) comes from Lemma 2.2.1. Let a support of the graph G for the system (2.1.1) be consisting of the s coherence component, then the number of nonsupporting arches equals $|U \backslash U_R| = |U| - (|I| - s)$ and the number of nonsupporting nodes equals $|I^* \setminus I^*_R| = (|I^*| - s)$. We have

$$|U \setminus U_R| + |I^* \setminus I^*_R| = |U| + |I^*| - s = |U| - |I| + |I^*|.$$

Each entailed characteristic vector always has one and only one nonsupporting component that equals one. It corresponds to the arc or the node that has entailed this vector. All other components non supporting component of the entailed characteristic vector are equal to zero. This means that any two different entailed characteristic vectors are linearly independent.

Thus, the aggregate of entailed characteristic vectors is a basis of the space of solutions of the homogeneous system (2.2.1). Therefore, any solution of the system (2.2.1) may be uniquely represented as their linear combination. □

We choose a support $R = \{U_R, I^*_R\}$ of the graph G for the system (2.1.1). It consists from the forest of s trees $T^k = \{I(U^k_T), U^k_T\}, s \leq |I^*|$, such as every tree has exactly one node $u_k \in I^*_R \cap I(U^k_T)$, for $k = \overline{1,s}$ and $\cup_{k=1}^{s} I(U^k_T) = I$.

We find characteristic vectors-columns $\delta(\tau,\rho) = (\delta^{\tau\rho}_{ij},\ (i,j) \in U;\ \delta^{\tau\rho}_i,\ i \in I^*)$ entailed by an non supporting arcs $(\tau,\rho) \in U \setminus U_R$ and $\delta(\gamma)$ where $\delta(\gamma) = (\delta^{\gamma}_{ij}, (i,j) \in U;\ \delta^{\gamma}_i,\ i \in I^*)$, entailed by an non supporting nodes $\gamma \in I^* \setminus I^*_R$.

Theorem 2.4.2. *The general solution of the system (2.1.1) may be uniquely represented using the following form:*

$$x_{ij} = \sum_{(\tau,\rho)\in U\backslash U_R} x_{\tau\rho}\delta^{\tau\rho}_{ij} + \sum_{\gamma\in I^*\backslash I^*_R} x_\gamma \delta^{\gamma}_{ij} + \tilde{x}_{ij}, \quad (i,j) \in U_R, \tag{2.4.1}$$

$$x_i = \sum_{(\tau,\rho)\in U\backslash U_R} x_{\tau\rho}\delta^{\tau\rho}_{i} + \sum_{\gamma\in I^*\backslash I^*_R} x_\gamma \delta^{\gamma}_{i} + \tilde{x}_{i}, \tag{2.4.2}$$

$$i \in I^*_R \cap I(U^k_T), \quad k = \overline{1,s},$$

$$x_{\tau\rho},\ x_\gamma \in \mathbf{R}, \quad (\tau,\rho) \in U\backslash U_R,\ \gamma \in I^*\backslash I^*_R,$$

where $\tilde{x} = (\tilde{x}_{ij}, (i,j) \in U, \tilde{x}_i, i \in I^*)$ is a partial solution of the inhomogeneous system (2.1.1).

Proof. We choose a supporting set $R = \{U_R, I_R^*\}$ of the graph G for the system and find the general solution of system (2.1.1). We consider it to be the sum of the general solution of the homogeneous system, entailed by system (2.1.1) and a any partial solution of the inhomogeneous system (2.1.1).

Let $y = (y_{ij}, (i,j) \in U; \quad y_i, i \in I^*)$ be any of the solutions of the homogeneous system, entailed by system (2.1.1). We consider vector

$$y' = y - \sum_{(\tau,\rho)\in U\setminus U_R} y_{\tau\rho}\delta(\tau,\rho) - \sum_{\gamma\in I^*\setminus I_R^*} y_\gamma \delta(\gamma)$$

According to Lemma 2.2.1 the characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node and the characteristic vector of a chain with the direction according to an arc satisfy of the homogeneous system (2.2.1). Moreover, y is also a solutions of the homogeneous system and therefore their linear combination also satisties system (2.2.1).And, furthermore, vector y' is constructed in such a way that $y'_{ij} = 0$, for $(i,j) \in U \setminus U_R$, and $y'_k = 0$, for $k \in I^* \setminus I_R^*$. In other words, all non supporting components of y' are equal to zero and therefore y' satisfies the following system:

$$\sum_{j\in I_i^+(U_R)} x_{ij} - \sum_{j\in I_i^-(U_R)} x_{ji} = \begin{cases} 0, & i \in I \setminus I_R^* \\ x_i \cdot sign[i], & i \in I_R^* \end{cases}$$

However, according support defination such system has only a trivial solution. Consequently, $y' = 0$ and the general solution of the homogeneous system (2.2.1) has the following look:

$$y = \sum_{(\tau,\rho)\in U\setminus U_R} y_{\tau\rho}\delta(\tau,\rho) + \sum_{\gamma\in I^*\setminus I_R^*} y_\gamma \delta(\gamma).$$

We have found the general solution of the homogeneous system, entailed by system (2.1.1). We write down the general solution of the homogeneous system, entailed by system (2.1.1) in network form:

$$y_{ij} = \sum_{(\tau,\rho)\in U\setminus U_R} y_{\tau\rho}\delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} y_\gamma \delta_{ij}^{\gamma}, \quad (i,j) \in U_R, \tag{2.4.3}$$

$$y_i = \sum_{(\tau,\rho)\in U\setminus U_R} y_{\tau\rho}\delta_{i}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} y_\gamma \delta_{i}^{\gamma}, i \in I_R^* \tag{2.4.4}$$

The general solution of the inhomogeneous system (2.1.1) is the sum of the general solution of the homogeneous system, entailed by system (2.1.1) and a any partial solution of the inhomogeneous system (2.1.1).

R e m a r k 2.4.1. The formulas (2.4.1) – (2.4.2) are correct, if the private solution $\tilde{x} = (\tilde{x}_{ij},(i,j) \in U, \tilde{x}_i, i \in I^*)$ is constructed according to rules: $\tilde{x}_{\tau\rho} = 0$, $(\tau,\rho) \in U \backslash U_R$, $\tilde{x}_\gamma = 0$, $\gamma \in I^* \backslash I_R^*$ and solve the system (2.1.1).

Further, we shall use the formula (2.4.1) – (2.4.2) where the private solution $\tilde{x}$ of the system (2.1.1) is constructed by the marked rules 2.4.1.

If the partial solution $\tilde{x} = (\tilde{x}_{ij},(i,j) \in U, \tilde{x}_i, i \in I^*)$ of the system (2.1.1) is constructed arbitrarily, then the general solution of the system (2.1.1) may be uniquely represented using the formulas in remark 2.4.2.

R e m a r k 2.4.2. The general solution of the system (2.1.1) may be uniquely represented using the following form:

$$x_{ij} = \sum_{(\tau,\rho)\in U\backslash U_R} x_{\tau\rho}\delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\backslash I_R^*} x_\gamma \delta_{ij}^{\gamma} +$$

$$+ \left(\tilde{x}_{ij} - \sum_{(\tau,\rho)\in U\backslash U_R} \tilde{x}_{\tau\rho}\delta_{ij}^{\tau\rho} - \sum_{\gamma\in I^*\backslash I_R^*} \tilde{x}_\gamma \delta_{ij}^{\gamma} \right), (i,j) \in U_R;$$

$$x_i = \sum_{(\tau,\rho)\in U\backslash U_R} x_{\tau\rho}\delta_{i}^{\tau\rho} + \sum_{\gamma\in I^*\backslash I_R^*} x_\gamma \delta_{i}^{\gamma} +$$

$$+ \left(\tilde{x}_{i} - \sum_{(\tau,\rho)\in U\backslash U_R} \tilde{x}_{\tau\rho}\delta_{i}^{\tau\rho} - \sum_{\gamma\in I^*\backslash I_R^*} \tilde{x}_\gamma \delta_{i}^{\gamma} \right), i \in I_R^*;$$

where $\tilde{x} = (\tilde{x}_{ij},(i,j) \in U, \tilde{x}_i, i \in I^*)$ is any partial solution of the inhomogeneous system (2.1.1), $x_{\tau\rho}$, $(\tau,\rho) \in U \setminus U_R$, x_γ, $\gamma \in I^* \backslash I_R^*$ are independent variables corresponding to arcs $(\tau,\rho) \in U \setminus U_R$ and to the nodes $\gamma \in I^* \backslash I_R^*$ respectively, $x_{\tau\rho}$, $x_\gamma \in \mathbf{R}$.

2.5. Decomposition of system

Let $R = \{U_R, I_R^*\}$ be a support of the network $S = \{I, U\}$ for the system (2.1.1). We define a set $W = \{U_W, I_W^*\}, |W| = q, U_W \subseteq U \setminus U_R$, where $I_W^* \subseteq I^* \setminus I_R^*$ by selecting q arbitrary arcs and nodes from the sets: $U \backslash U_R$, and $I^* \backslash I_R^*$.

Let's substitute the general solution (2.4.1) – (2.4.2) of the system (2.1.1) into (2.1.2):

$$\sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} = \sum_{(i,j)\in U_R} \lambda_{ij}^p x_{ij} + \sum_{(i,j)\in U\setminus U_R} \lambda_{ij}^p x_{ij} =$$

$$= \sum_{(i,j)\in U_R} \lambda_{ij}^p \left[x_{ij} = \sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho} \delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \delta_{ij}^\gamma + \tilde{x}_{ij} \right] +$$

$$+ \sum_{(\tau,\rho)\in U\setminus U_R} \lambda_{\tau\rho}^p x_{\tau\rho} = \alpha_p, \quad p = \overline{1,q} \tag{2.5.1}$$

We change the summing order in (2.5.1):

$$\sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho} \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma + \sum_{(i,j)\in U_R} \lambda_{ij}^p \tilde{x}_{ij} +$$

$$+ \sum_{(\tau,\rho)\in U\setminus U_R} \lambda_{\tau\rho}^p x_{\tau\rho} = \alpha_p, \quad p = \overline{1,q}\,. \tag{2.5.2}$$

In equations (2.5.2) we group the variables, corresponding to the sets $U \backslash U_R$ and $I^* \backslash I_R^*$:

$$\sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho} \left[\lambda_{\tau\rho}^p + \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} \right] + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \left[\sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma \right] =$$

$$= \alpha_p - \sum_{(i,j)\in U_R} \lambda_{ij}^p \tilde{x}_{ij}\,, p = \overline{1,q}\,. \tag{2.5.3}$$

Definition 2.5.1. *The number*

$$\Lambda_{\tau\rho}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^p \tag{2.5.4}$$

is the ***determinant of the structure, entailed by the arc*** $(\tau,\rho) \in U \setminus U_R$, *with respect to the equation with the number* p

of the system (2.1.2).

The number

$$\Lambda^p_\gamma = \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^\gamma_{ij} \tag{2.5.5}$$

is the ***determinant of the structure, entailed by the node*** $\gamma \in I^* \setminus I^*_R$, *with respect to the equation with the number* p *of the system (2.1.2).*

Let's denote

$$A^p = \alpha_p - \sum_{(i,j)\in U_R} \lambda^p_{ij}\,\tilde{x}_{ij}\,, p = \overline{1,q}\,. \tag{2.5.6}$$

The system (2.5.3), according to formulas (2.5.4) – (2.5.6), gets to the form:

$$\sum_{(\tau,\rho)\in U\setminus U_R} \Lambda^p_{\tau\rho}x_{\tau\rho} + \sum_{\gamma\in I^*\setminus I^*_R} \Lambda^p_\gamma x_\gamma = A^p, \quad p = \overline{1,q} \tag{2.5.7}$$

In (2.5.7) we group the variables that correspond to the set W and then we obtain:

$$\sum_{(\tau,\rho)\in U_W} \Lambda^p_{\tau\rho}x_{\tau\rho} + \sum_{\gamma\in I^*_W} \Lambda^p_\gamma x_\gamma =$$

$$= A^p - \sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)} \Lambda^p_{\tau\rho}x_{\tau\rho} - \sum_{\gamma\in I^*\setminus(I^*_W\cup I^*_R)} \Lambda^p_\gamma x_\gamma, p = \overline{1,q}. \tag{2.5.8}$$

Finally, let us rewrite equations (2.5.8) in the matrix form. For this purpose, we introduce arbitrary numberings of arcs and nodes within the set $W = \{U_W, I^*_W\}$. Thus, $t = t(\tau,\rho)$ is a number of a arc $(\tau,\rho) \in U_W$, and a $t = t(\gamma)$ is a number of a node $\gamma \in I^*_W$, $t \in \{1,2,\ldots,|U_W| + |I^*_W|\}$. In other words, we enumerate the elements from the set W. Note, the numbering of the elements from the set W is equivalent to the numbering of the structures, entailed by the elements from the set W with respect to trees $T^k = \{I(U^k_T),U^k_T\}, k = \overline{1,s}$ where every tree has exactly one node $u_k \in I^*_R$, for $k = \overline{1,s}$ and $\cup_{k=1}^s I(U^k_T) = I$.

Now equations (2.5.8) can be represented as following:

$$Dx_W = \beta, \tag{2.5.9}$$

where

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, D_1 = \left[\Lambda^p_{t(\tau,\rho)}, p = \overline{1,q};\ t(\tau,\rho) = \overline{1,|U_W|}\right],$$

$$D_2 = \left[\Lambda^p_{t(\gamma)}, p = \overline{1,q};\ t(\gamma) = \overline{1,|I^*_W|}\right],$$

D_1 is a submatrix of the size $q \times |U_W|$ and D_2 is a submatrix of the size $q \times |I^*_W|$, $x_W = (x_{\tau\rho},(\tau,\rho) \in U_W;\ x_\gamma,\gamma \in I^*_W)$ is the vector of unknowns with components ordered according to the numbering of arcs $t = t(\tau,\rho)$, $(\tau,\rho) \in U_W$, and nodes $t = t(\gamma)$, $\gamma \in I^*_W$.

The right-hand side of (2.5.9) has the form: $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_p \end{bmatrix}$,

$$\beta_p = A^p - \sum_{(\tau,\rho)\in(U_W\cup U_R)} \Lambda^p_{\tau\rho} x_{\tau\rho} - \sum_{\gamma\in I^*\setminus(I^*_W\cup I^*_R)} \Lambda^p_\gamma x_\gamma, \quad p = \overline{1,q}. \quad (2.5.10)$$

From (2.5.9), in case of non-singularity of the matrix D, we find the unknown variables x_W, corresponding to the set W:

$$x_W = D^{-1}\beta\,. \quad (2.5.11)$$

Remark 2.5.1. In general, because of an arbitrary selection of elements for the set W, non-singularity matrix D is not guaranteed. In the case when $det\ D = 0$ one should re-select arcs into the set U_W and nodes into the set I^*_W and re-compute D, β for the system (2.5.9).

The other unknowns $x_R = (x_{ij},(i,j) \in U_R, x_i, i \in I^*_R)$ that correspond to $R = \{U_R,I^*_R\}$ we compute from sparse system of linear algebraic equations of type (2.1.1) in linear time in worse case, using the combinatorial properties of support $R = \{U_R,I^*_R\}$ [10, 11, 32–34].

2.6. Properties of support

Definition 2.6.1. *Let's call the support of the graph G for the system (2.1.1) – (2.1.2) such an aggregate of sets $K = \{U_K,I^*_K\}$, $\tilde{U} = U_K$, $\tilde{I}^* = I^*_K$ that the system*

$$\begin{cases} \sum\limits_{j\in I_i^+(\tilde{U})} x_{ij} - \sum\limits_{j\in I_i^-(\tilde{U})} x_{ji} = \begin{cases} 0, & i \in I \setminus \tilde{I}^* \\ x_i \cdot sign[i], & i \in \tilde{I}^* \end{cases} \\ \sum\limits_{(i,j)\in\tilde{U}} \lambda^p_{ij} x_{ij} = 0, \quad p = \overline{1,q} \end{cases} \quad (2.6.1)$$

has only a trivial solution, but it has a nontrivial solution for any of the following aggregations of sets:

$$\begin{cases} \tilde{K} = \{\tilde{U},\tilde{I}^*\}, & \tilde{U} = U_K \cup \{(i_0,j_0)\}, \quad (i_0,j_0) \in U \setminus U_K, \tilde{I}^* = I_K^* \\ \tilde{K} = \{\tilde{U},\tilde{I}^*\}, & \tilde{U} = U_K, \quad \tilde{I}^* = I_K^* \cup \{i_0\}, \qquad i_0 \in I^* \setminus I_K^* \end{cases}$$

Theorem 2.6.1 (Network support criterion). *The aggregation of sets $K = \{U_K,I_K^*\}$ is a support of the network $G = \{I,U\}$ for the system (2.1.1) – (2.1.2) if and only if*

- *the aggregation of sets $K = \{U_K,I_K^*\}$ may be divided into two aggregations: $R = \{U_R,I_R^*\}$ and $W = \{U_W,I_W^*\}$, such as $U_R \cup U_W = U_K$, $U_R \cap U_W = \varnothing$, $I_R^* \cup I_W^* = I_K^*$, $I_R \cap I_W^* = \varnothing$, and the set R is a support of the network $G = \{I,U\}$ for the system (2.1.1);*
- *$|W| = r$, where r is the rank of matrix of the system (2.1.2);*
- *matrix D of the system (2.5.9) is nonsingular matrix.*

Remark 2.6.1. The matrix D of the system (2.5.9) is nonsingular matrix if the aggregation of sets $K = \{U_K,I_K^*\}$ is a support of the network G for the system (2.1.1) – (2.1.2).

Let's investigate graph theoretical properties of the structure of the support of the network $G = \{I,U\}$ for the system (2.1.1) – (2.1.2). According to the theorem 2.6.1 the supporting aggregate $K = \{U_K,I_K^*\}$ includes the support $R = \{U_R,I_R^*\}$ of the network G for the system (2.1.1). Supporting elements that correspond to the aggregate R make up a forest of trees that covers all the nodes of the set I, and each tree of the forest has exactly one node from the set I_R^*. Adding each additional element from $W = \{U_W,I_W^*\}$ to the support $R = \{U_R,I_R^*\}$ we will obtain a cycle or a chain that connects the nodes in the set I^*.

2.7. Sparse systems with extended additional part

We consider decomposition algorithms for solving sparse linear systems with embedded network structure and extended additional part. We investigate this problem on the base of graph theoretical specificity of the support and on the base of properties of the basis of the solution space.

Consider the following sparse linear underdetermined system

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} a_i, & i \in I\backslash I^*; \\ x_i \cdot sign[i], & i \in I^*; \end{cases} \tag{2.7.1}$$

$$\sum_{(ij)\in U} \lambda_{ij}^p x_{ij} + \sum_{i\in I^*} \lambda_i^p x_i = \beta_p, \quad \text{for} \quad p = \overline{1,q}, \tag{2.7.2}$$

where $G = (I,U)$ is a finite oriented connected graph without multiple arcs and loops with set of nodes I and set of arcs U , $|U| \gg| I |$, $I_i^+(U) = \{j : (i,j) \in U\}$, $I_i^-(U) = \{j : (j,i) \in U\}$, x_{ij} – a flow along the arc (i,j), I^* – a subset of I, $I^* \neq \oslash$. Nodes $i \in I^*$ are called nodes with variable intensity x_i, $sign(i) = 1$, if $i \in I_+^*$, $sign(i) = -1$, if $i \in I_-^*, I_+^*, I_-^* \subseteq I^*, I_+^* \cap I_-^* = \oslash$, a_i, λ_{ij}^p, λ_i^p β_p – rational numbers.

The matrix of system (2.7.1) – (2.7.2) can be written as a block matrix in the following form:

$$A = \begin{bmatrix} M & R \\ Q & T \end{bmatrix}.$$

Here M is a sparse incidence matrix of the graph $G = (I,U)$ of size $|I|\times|U|$. Matrix R has size $|I| \times |I^*|$ and has one nonzero element equal to $-\,sign[i]$ located at the intersection of column i and row i, where $i \in I^*$, with other elements being zeroes. Q is a matrix of size $q \times |U|$ and consists from the elements λ_{ij}^p for $(i,j) \in U$, $p = \overline{1,q}$. T is the matrix of size $q \times |I^*|$ and consists of elements λ_i^p for $i \in I^*$, $p = \overline{1,q}$.

Recall that the rank of the matrix of system (2.7.1) for a connected graph $G = \{I,U\}$, $I^* \neq \emptyset$, is equal to $|I|$.

The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc are constructed according to the rules in Section 2.4.

Theorem 2.7.1. *The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc satisfy the system (2.7.3) [32].*

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} 0, & i \in I\backslash I^* \\ x_i \cdot sign[i], & i \in I^* \end{cases} \tag{2.7.3}$$

Theorem 2.7.2. *Any solution of system (2.7.3) is a linear combination of characteristic vectors.*

The proof of the Theorem 2.7.2 is given in [53].

Definition 2.7.1. We call an aggregate of sets $R = \{U_R, I_R^*\}$, where $U_R \subseteq U$ and $I_R^* \subseteq I^*$ a support of graph G for system (2.7.1) if for the aggregate of sets $\widetilde{R} = \{\widetilde{U}, \tilde{I}^*\}, \widetilde{U} = U_R, \tilde{I}^* = I_R^*$ the system

$$\sum_{j \in I_i^+(\widetilde{U})} x_{ij} - \sum_{j \in I_i^-(\widetilde{U})} x_{ji} = \begin{cases} 0, & i \in I \backslash \widetilde{I^*} \\ x_i \cdot sign[i], & i \in \tilde{I}^* \end{cases} \tag{2.7.4}$$

has only a trivial solution, but has a nontrivial solution for any of the following set aggregations:

$$\begin{cases} \widetilde{R} = \{\widetilde{U}, \tilde{I}^*\}, \quad \widetilde{U} = U_R \cup (i_0 j_0), \quad \text{for} \quad (i_0 j_0) \in U \backslash U_R \quad \text{and,} \quad \tilde{I}^* = I_R^* \\ \widetilde{R} = \{\widetilde{U}, \tilde{I}^*\}, \quad \widetilde{U} = U_R, \quad \tilde{I}^* = I_R^* \cup \{i_0\}, \qquad \text{for} \quad i_0 \in I^* \backslash I_R^* \end{cases}$$

For a subset of arcs $U_1 \subseteq U$, we introduce the set of nodes $I(U_1)$ where $I(U_1) = \{i \in I : (i,j) \in U_1 \vee (j,i) \in U_1\}$. We construct a forest from s trees $T^k = \{I(U_T^k), U_T^k\}, s \leqslant |I^*|$, so that every tree contains exactly one node $u_k \in I_R^*$, for $k = \overline{1,s}$, and $\cup_{k=1}^s I(U_T^k) = I$. We form the sets

$$U_R = \bigcup_{k=1}^{s} U_T^k, \quad I_R^* = \bigcup_{k=1}^{s} \{u_k\}$$

Theorem 2.7.3. *An aggregate of sets $R = \{U_R, I_R^*\}, U_R \subseteq U$, and $I_R^* \subseteq I^*$ is a support of the graph G for system (2.7.1) if and only if the following conditions are carried out:*

- *Each connected component $T^k = \{I(U_T^k), U_T^k\}$, for $k = \overline{1,s}$ is a tree;*
- $I(\cup_{k=1}^s U_T^k) = \cup_{k=1}^s I(U_T^k) = I$;
- $|I_k^*| = 1$, *where* $I_k^* = I_R^* \cap I(U_T^k)$, *for* $k = \overline{1,s}$.

The proof of Theorem 2.7.3 follows from the network properties of the support.

After we choose the support $R = \{U_R, I_R^*\}$ of system (2.7.1), we determine what structures can be obtained after adding one non-supporting element to the support.

Definition 2.7.2. The **characteristic vector entailed by an arc** $(\tau,\rho) \in U \backslash U_R$ is the vector constructed according to the following rules:

- If the set $U_R \cup \{(\tau,\rho)\}$ has a cycle $L = \{I_L, U_L\}$, then the entailed characteristic vector is the characteristic vector of that cycle, and the arc (τ,ρ) defines the detour direction of the cycle.
- If the set $U_R \cup \{(\tau,\rho)\}$ has a chain $C = \{I_C, U_C\}$ that connects nodes $u,v \in I_R^*$, then the entailed characteristic vector is the characteristic vector of that chain, and the arc that defines the detour direction is (τ,ρ).

Definition 2.7.3. The **characteristic vector entailed by a node** $\gamma \in I^* \backslash I_R^*$ is the characteristic vector of the chain that connects nodes γ and $v \in I_R^*$ with node γ being chosen as the beginning of the chain.

We find characteristic vectors $\delta(\tau,\rho) = (\delta_{ij}^{\tau\rho}, (i,j) \in U;\ \delta_i^{\tau\rho}, i \in I^*)$ entailed by an non supporting arcs $(\tau,\rho) \in U \backslash U_R$ and characteristic vectors $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U;\ \delta_i^{\gamma}, i \in I^*)$, entailed by an non supporting nodes $\gamma \in I^* \setminus I_R^*$.

Theorem 2.7.4. *The general solution system (2.7.1) may be uniquely represented in the following way:*

$$x_{ij} = \sum_{(\tau,\rho) \in U \backslash U_R} x_{\tau\rho} \delta_{ij}^{\tau\rho} + \sum_{\gamma \in I^* \backslash I_R^*} x_\gamma \delta_{ij}^{\gamma} + \widetilde{x}_{ij}, \quad \textit{for} \quad (i,j) \in U_R; \quad (2.7.5)$$

$$x_i = \sum_{(\tau,\rho) \in U \backslash U_R} x_{\tau\rho} \delta_i^{\tau\rho} + \sum_{\gamma \in I^* \backslash I_R^*} x_\gamma \delta_i^{\gamma} + \widetilde{x}_i, \textit{for} \quad i \in I_R^* \qquad (2.7.6)$$

where $\widetilde{x} = (\widetilde{x}_{ij}, (i,j) \in U, \widetilde{x}_i, i \in I^*)$ is a partial solution of the nonhomogeneous system (2.7.1).

The proof of the Theorem 2.7.4 is given in [45] , where the general solution of the nonhomogeneous system (2.7.1) is the sum of the general solution of the homogeneous system, generated by the system (2.7.1) and a partial solution of the nonhomogeneous system (2.7.1).

Remark 2.7.1. The formulas (2.7.5) and (2.7.6) are correct, if the partial solution $\widetilde{x} = (\widetilde{x}_{ij}, (i,j) \in U, \widetilde{x}_i, i \in I^*)$ is constructed according to the rules:

$$\widetilde{x}_{\tau\rho} = 0, (\tau,\rho) \in U \setminus U_R, \widetilde{x}_\gamma = 0, \gamma \in I^* \setminus I_R^*$$

where the supporting components of the partial solution $\widetilde{x}$ are obtained from solving the system (2.7.1).

Later on, we shall use formulas (2.7.5) and (2.7.6) where the partial solution $\widetilde{x}$ is constructed according to the rules in Remark 2.7.1.

2.8. Matrix of determinants

Let $R = \{U_R, I_R^*\}$ be a support of the graph $G = \{I, U\}$ of system (2.7.1). In arbitrary order, we choose sets $W = \{U_W, I_W^*\}, |W| = q$ where $U_W \subseteq U \setminus U_R$, and $I_W^* \subseteq I^* \setminus I_R^*$. After substituting the general solution of system (2.7.1), which has the form (2.7.5) – (2.7.6), into (2.7.2), the system (2.7.2) takes the form:

$$\sum_{(\tau,\rho)\in U\setminus U_R} \Lambda^p_{\tau\rho} x_{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} \Lambda^p_\gamma x_\gamma = A_p \quad \text{and}\, p = \overline{1,q}, \tag{2.8.1}$$

$$\Lambda^p_{\tau\rho} = \lambda^p_{\tau\rho} + \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\tau\rho}_{ij} + \sum_{i\in I_R^*} \lambda^p_i \delta^{\tau\rho}_i,$$

$$\Lambda^p_\gamma = \lambda^p_\gamma + \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\gamma}_{ij} + \sum_{i\in I_R^*} \lambda^p_i \delta^{\gamma}_i,$$

$$A_p = \alpha_p - \sum_{(i,j)\in U_R} \lambda^p_{ij}\widetilde{x}_{ij} - \sum_{i\in I_R^*} \lambda^p_i \widetilde{x}_i,$$

where $\delta(\tau,\rho)$, $\delta(\gamma)$ are constructed according to Section 2.7.

Definition 2.8.1. *The number*

$$\Lambda^p_{\tau\rho} = \lambda^p_{\tau\rho} + \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\tau\rho}_{ij} + \sum_{i\in I_R^*} \lambda^p_i \delta^{\tau\rho}_i \tag{2.8.2}$$

is the ***determinant of the structure, entailed by the arc*** $(\tau,\rho) \in U \setminus U_R$, *with respect to the equation with the number* p *of the system (2.7.2).*

The number

$$\Lambda^p_\gamma = \lambda^p_\gamma + \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\gamma}_{ij} + \sum_{i\in I_R^*} \lambda^p_i \delta^{\gamma}_i \tag{2.8.3}$$

is the ***determinant of the structure, entailed by the node*** $\gamma \in I^* \setminus I_R^*$, *with respect to the equation with the number* p *of the system (2.7.2).*

Let's denote

$$A^p = \alpha_p - \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} - \sum_{i\in I_R^*} \lambda_i^p \widetilde{x}_i. \tag{2.8.4}$$

The system (2.8.1), according to formulas (2.8.2) – (2.8.4), gets to the form:

$$\sum_{(\tau,\rho)\in U\setminus U_R} \Lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} \Lambda_\gamma^p x_\gamma = A^p, \quad p = \overline{1,q} \tag{2.8.5}$$

In system (2.8.5), we separate variables that correspond to the set W and then we obtain (2.8.6).

$$\sum_{(\tau,\rho)\in U_W} \Lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{\gamma\in I_W^*} \Lambda_\gamma^p x_\gamma =$$

$$= A^p - \sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)} \Lambda_{\tau\rho}^p x_{\tau\rho} - \sum_{\gamma\in I^*\setminus(I_W^*\cup I_R^*)} \Lambda_\gamma^p x_\gamma \tag{2.8.6}$$

for $p = \overline{1,q}$.

Finally, let us rewrite equations (2.8.6) in the matrix form. For this purpose, we introduce arbitrary numberings of arcs and nodes within the set $W = \{U_W, I_W^*\}$. Thus, $t = t(\tau,\rho)$ is a number of a arc $(\tau,\rho) \in U_W$, and a $t = t(\gamma)$ is a number of a node $\gamma \in I_W^*$, $t \in \{1,2,\ldots,|U_W| + |I_W^*|\}$. In other words, we enumerate the elements from the set W. Note, the numbering of the elements from the set W is equivalent to the numbering of the structures, entailed by the elements from the set W with respect to trees $T^k = \{I(U_T^k), U_T^k\}, k = \overline{1,s}$ where every tree has exactly one node $u_k \in I_R^*$, for $k = \overline{1,s}$ and $\cup_{k=1}^s I(U_T^k) = I$.

Now equations (2.8.6) can be regarded as following:

$$Dx_W = \beta, \tag{2.8.7}$$

where

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, D_1 = \left[\Lambda_{t(\tau,\rho)}^p, p = \overline{1,q};\ t(\tau,\rho) = \overline{1,|U_W|}\right],$$

$$D_2 = \left[\Lambda^p_{t(\gamma)}, p = \overline{1,q};\ t(\gamma) = \overline{1,|I^*_W|}\right],$$

and D_1 – the submatrix of the size $q \times |U_W|$ and D_2 – the submatrix of the size $q \times |I^*_W|$, $x_W = (x_{\tau\rho},(\tau,\rho) \in U_W;\ x_\gamma,\gamma \in I^*_W)$ – vector of unknowns with components ordered according to the numbering arcs $t = t(\tau,\rho)$ for $(\tau,\rho) \in U_W$, and nodes $t = t(\gamma)$, $\gamma \in I^*_W$.

The right-hand side of (2.8.7) has the form: $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_p \end{bmatrix}$,

$$\beta_p = A^p - \sum_{(\tau,\rho)\in(U_W \cup U_R)} \Lambda^p_{\tau\rho} x_{\tau\rho} - \sum_{\gamma \in I^* \setminus (I^*_W \cup I^*_R)} \Lambda^p_\gamma x_\gamma, \quad p = \overline{1,q}. \quad (2.8.8)$$

From (2.8.7), in case of non-singularity of the matrix D, we find the unknown variables x_W, corresponding to the set W:

$$x_W = D^{-1}\beta\,. \quad (2.8.9)$$

Remark 2.8.1. In general, because of an arbitrary selection of elements for the set W, non-singularity matrix D is not guaranteed. In the case when $det\ D = 0$ one should re-select arcs into the set U_W and nodes into the set I^*_W and re-compute D, β for the system (2.8.7).

The other unknowns $x_R = (x_{ij},(i,j) \in U_R, x_i, i \in I^*_R)$ that correspond to $R = \{U_R,I^*_R\}$ we compute from sparse system of linear algebraic equations of type (2.7.1) in linear time in worse case, using the combinatorial properties of support $R = \{U_R,I^*_R\}$.

2.9. Graph theoretical properties

We call a support of graph G for system (2.7.1) – (2.7.2) such an aggregate of sets $K = \{U_K,I^*_K\}$, that for given $\widetilde{K} = \{\widetilde{U},\widetilde{I^*}\}, \widetilde{U} = U_K$, and $\widetilde{I^*} = I^*_K$, the system

$$\begin{cases} \sum\limits_{j\in I_i^+(\widetilde{U})} x_{ij} \quad - \quad \sum\limits_{j\in I_i^-(\widetilde{U})} x_{ji} = \begin{cases} 0, & i \in I\setminus\widetilde{I^*} \\ x_i \cdot sign[i], & i \in \widetilde{I^*} \end{cases} \\ \sum\limits_{(ij)\in\widetilde{U}} \lambda^p_{ij} x_{ij} + \sum\limits_{i\in\widetilde{I^*}} \lambda^p_i x_i = 0, \text{for} \quad p = \overline{1,q} \end{cases} \quad (2.9.1)$$

has only a trivial solution. But it has a nontrivial solution for any of the following aggregations of sets:

$$\begin{cases} \widetilde{K} = \{\widetilde{U},\widetilde{I^*}\}, \widetilde{U} = U_K \cup (i_0 j_0), & \text{for} \quad (i_0 j_0) \in U \backslash U_K, \quad \text{and} \quad \tilde{I}^* = I_K^* \\ \widetilde{K} = \{\widetilde{U},\widetilde{I^*}\}, \widetilde{U} = U_K, \tilde{I}^* = I_K^* \cup \{i_0\}, & \text{for} \quad i_0 \in I^* \backslash I_K^* \end{cases}$$

The following theorem holds:

Theorem 2.9.1. *The aggregation of sets $K = \{U_K, I_K^*\}$ is a support of network $G = \{I,U\}$ for system (2.7.1) – (2.7.2) if and only if*

- *the aggregation of sets $K = \{U_K, I_K^*\}$ may be divided into two aggregations: $R = \{U_R, I_R^*\}$ and $W = \{U_W, I_W^*\}$, such that $U_R \cup U_W = U_K$, $U_R \cap U_W = \emptyset$, $I_R^* \cup I_W^* = I_K^*$, $I_R \cap I_W^* = \emptyset$, and the aggregation of sets R is a support of the network $G = \{I,U\}$ for system (2.7.1);*
- *$|W| = q$, where q is the number of independent equations in system (2.7.2);*
- *matrix D of the system (2.8.7), which consists of determinants $\Lambda_{\tau\rho}^p$, Λ_γ^p of the structures entailed by the arcs and nodes of the aggregation W, is nonsingular.*

We now investigate graph theoretical properties of the structure of the support of network $G = \{I,U\}$ for system (2.7.1) – (2.7.2). According to Theorem 2.9.1, the aggregate of sets $K = \{U_K, I_K^*\}$ includes the support $R = \{U_R, I_R^*\}$ of network G for system (2.7.1). Supporting elements that correspond to the aggregate R make up a forest of trees that covers all the nodes of the set I, and each tree of the forest has exactly one node from the set I_R^*. Adding each additional element from $W = \{U_W, I_W^*\}$ to the set $R = \{U_R, I_R^*\}$, we obtain a cycle or a chain in the set $K = \{U_K, I_K^*\}$.

2.10. Implementation of decomposition algorithms

Let us consider the example of the system (2.10.1) – (2.10.2) for the network $G = (I,U)$ (see Figure 2.3), where

$$I = \{1,2,3,4,5,6,7,8\},$$

$$I^* = \{1,4,5,7,8\},$$

$$U = \{(1,2),(1,8),(2,8),(3,1),(3,7),(4,3),(4,6),(6,5),(6,7),(7,4),(7,8),(8,5)\}.$$

$$x_{1,2} + x_{1,8} - x_{3,1} = x_1$$

$$x_{2,8} - x_{1,2} = 5$$

$$x_{3,1} + x_{3,7} - x_{4,3} = 10$$

$$x_{4,3} + x_{4,6} - x_{7,4} = x_4$$

$$-x_{6,5} - x_{8,5} = x_5$$

$$x_{6,5} + x_{6,7} - x_{4,6} = -15 \tag{2.10.1}$$

$$x_{7,4} + x_{7,8} - x_{3,7} - x_{6,7} = -x_7$$

$$x_{8,5} - x_{1,8} - x_{2,8} - x_{7,8} = -x_8$$

$$x_{1,2} + 10x_{1,8} + 2x_{2,8} + 2x_{3,1} + x_{3,7} + x_{4,3} + 4x_{4,6} + x_{6,5} +$$

$$+3x_{6,7} + 13x_{7,4} + 2x_{7,8} + x_{8,5} = 37$$

$$2x_{1,2} + 4x_{1,8} + x_{3,1} + 3x_{3,7} + 7x_{4,3} + 2x_{4,6} + 5x_{6,5} + \tag{2.10.2}$$

$$+8x_{6,7} + x_{7,4} + 2x_{7,8} + 10x_{8,5} = 53$$

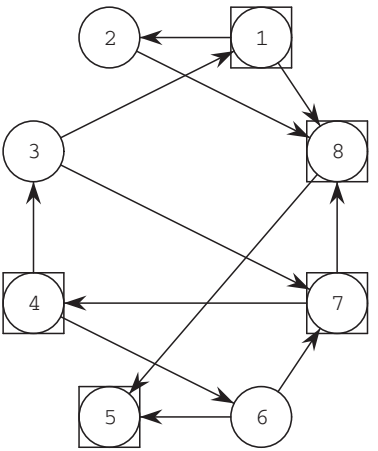

Fig. 2.3. Finite connected directed graph G

We choose a support $R = \{U_R, I_R^*\}$ of the network $G = (I,U)$ for the system (2.10.1). By Theorem 2.3.1, forest from $|K| = 3$ trees U_R where $U_R = \{U_T^k, k \in K\}$:

$$U_T^1 = \{(1,2),(1,8)\}, U_T^2 = \{(3,7)\}, U_T^3 = \{(4,6),(6,5)\},$$

$$I_R^* = \{1, 5, 7\},$$

which presented in Figure 2.4, is a support network (see Figure 2.3) for the system (2.10.1).

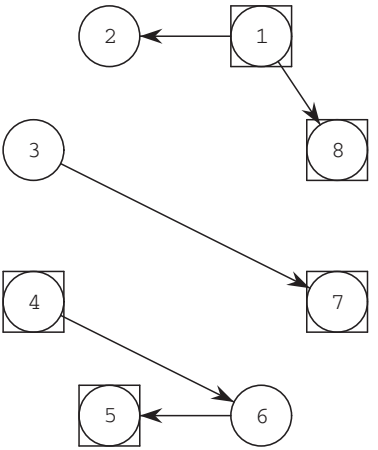

Fig. 2.4. Graph support for system (2.10.1)

Each tree forest $\{U_T^k, k \in K\}$, $K = \{1,2,3\}$ has single node from the set I_R^*: $|I(U_T^k) \cap I_R^*| = 1$,

$$I(U_T^1) \bigcap I_R^* = \{1\}, I(U_T^2) \bigcap I_R^* = \{7\}, I(U_T^3) \bigcap I_R^* = \{5\}.$$

We form $\{\delta(\tau,\rho), (\tau,\rho) \in U \setminus U_R\}$ and $\{\delta(\gamma), \gamma \in I^* \setminus I_R^*\}$ – the characteristic vectors, entailed by an non-supporting elements.

The dimension of each characteristic vector $\delta(\tau,\rho)$ or $\delta(\gamma)$ is $|U| + |I^*|$. In this example, the dimension of each characteristic vector is $|U| + |I^*| = 17$. The arcs of set $U \setminus U_R$ generate characteristic vectors:

$$\delta(2,8), \delta(3,1), \delta(4,3), \delta(6,7), \delta(7,4), \delta(7,8), \delta(8,5).$$

The nodes of the set $I^* \setminus I_R^*$ generate characteristic vectors:

$$\delta(4), \delta(8).$$

In the Table 2.10.1 are the data structures used for compute non-zero

Table 2.10.1

The collection of root trees

i	1	2	3	4	5	6	7	8
$p[i]$	0	1	7	6	0	5	0	1
$d[i]$	0	1	-1	-1	0	-1	0	1
$t[i]$	4	6	5	3	7	8	2	1
$dp[i]$	0	1	1	2	0	1	0	1
$sign[i]$	1	0	0	1	1	0	-1	-1

components of each characteristic vector

$$\delta(\tau,\rho),(\tau,\rho) \in U \setminus U_R, \delta(\gamma),\gamma \in I^* \setminus I_R^*$$

by number of operations $O(n)$ in the worst case, where $n = |I|$. Since each tree $\{U_T^k, k \in K\}$ contains a single node from the set I_R^*, $|I(U_T^k) \cap I_R^*| = 1$ the root of each rooted tree is the corresponding unique supporting node from set I_R^*.

The list $\{p[i], i = \overline{1,|I|}\}$ for each node $i \in I$ is the value of $p[i]$, who is the father of node i in root structure.

The list $\{t[i], i = \overline{1,|I|}\}$ is an inverted list index thread each root of the tree collection of rooted trees [2, 32, 53] with roots at the nodes of the set I_R^*.

The list $d[i], i = \overline{1,|I|}$ define for each node $i \in I$ direction of arc in root structure: if the node i is root, then $d[i] = 0$; if there is an arc $(p[i], i) \in U$, then $d[i] = 1$; if there is an arc $(i, p[i]) \in U$, then $d[i] = -1$. Data structures for store a collection of rooted trees are shown in Table 2.10.1.

Table 2.10.2

The nonzero components of vector $\delta(2,8)$

(i,j)	(2,8)	(1,8)	(1,2)
$\delta(2,8)$	1	-1	1

Table 2.10.3

Nonzero components of a characteristic vector $\delta(3,1)$

$(i,j), i$	(3,1)	(3,7)	1	7
$\delta(3,1)$	1	-1	-1	-1

Nonzero components of a characteristic vectors, entailed by arcs $U \setminus U_R$ where $U \setminus U_R = \{(2,8), (3,1), (4,3), (6,7), (7,4), (7,8), (8,5)\}$, presented in Tables 2.10.2 – 2.10.8. Nonzero components of a characteristic vectors

Table 2.10.4

Nonzero components of a characteristic vector $\delta(4,3)$

$(i,j), i$	(4,3)	(4,6)	(6,5)	(3,7)	7	5
$\delta(4,3)$	1	-1	-1	1	1	1

$\delta(4), \delta(8)$, entailed by nodes $I^* \setminus I^*_R = \{4, 8\}$, are shown in Tables 2.10.9 – 2.10.10.

Using the information storage support, which presented in Table 2.10.1, the construction of non-zero components each the characteristic vector and compute its determinant is the time $O(m)$ in the worst case, where $m = |I|$.

In accordance with Remark 2.4.1 we construct a particular solution for system (2.10.1). The nonzero components of a particular solution $\tilde{x}$ of system (2.10.1) are:

$$\tilde{x}_{6,5} = -15, \tilde{x}_5 = 15, \tilde{x}_{3,7} = 10, \tilde{x}_7 = 10, \tilde{x}_{1,2} = -5, \tilde{x}_1 = -5.$$

Table 2.10.5

Nonzero components of a characteristic vector $\delta(6,7)$

$(i,j), i$	(6,7)	(6,5)	7	5
$\delta(6,7)$	1	-1	1	1

Table 2.10.6

Nonzero components of a characteristic vector $\delta(7,4)$

$(i,j), i$	(7,4)	(4,6)	(6,5)	7	5
$\delta(7,4)$	1	1	1	-1	-1

Form the collection of sets $W = \{U_W, I^*_W\}$:

$$U_W = \{(3,1)\},\ I^*_W = \{4\}.$$

Using the formulas (2.8.2) and (2.8.3), we compute determinants of structures generated by the elements of the collection of sets W with respect to the equations (2.10.2) with the numbers $p = 1,2$.

To form the matrix D for system of type (2.8.7) must be arbitrarily numbered elements of the collection of sets W:

$$t(4) = 1,\ t(3,1) = 2.$$

From determinant of the structures, generated by elements of the collection of sets W, we form the matrix D of system type (2.8.7).

The matrix D has the form:

$$D = \begin{pmatrix} \Lambda^1_4 & \Lambda^1_{3,1} \\ \Lambda^2_4 & \Lambda^2_{3,1} \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 7 & -2 \end{pmatrix},\ \det D \neq 0.$$

We calculate the determinant of the structures generated by the arcs of the set $U_N = U \setminus (U_R \cup U_W)$ and nodes $\gamma \in I^*_N = I^* \setminus (I^*_R \cup I^*_W)$.

Using the formulas (2.8.4) we calculate the number of $A^1,\ A^2$:

$$A^1 = 47,\ \ A^2 = 108.$$

Table 2.10.7

Nonzero components of a characteristic vector $\delta(7,8)$

$(i,j), i$	(7,8)	(1,8)	7	1
$\delta(7,8)$	1	-1	-1	-1

Table 2.10.8

Nonzero components of a characteristic vector $\delta(8,5)$

$(i,j), i$	(8,5)	(1,8)	5	1
$\delta(8,5)$	1	1	-1	1

Components of the vector β — the right side of the system (2.8.7), calculate according to the (2.8.8).

Thus, the vector β has the form:

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$$\beta_1 = A^1 - \Lambda^1_8 x_8 - \Lambda^1_{2,8} x_{2,8} - \Lambda^1_{4,3} x_{4,3} - \Lambda^1_{6,7} x_{6,7} - \Lambda^1_{7,4} x_{7,4} - \Lambda^1_{7,8} x_{7,8} - \Lambda^1_{8,5} x_{8,5} =$$
$$= 47 - 10x_8 + 7x_{2,8} + 3x_{4,3} - 2x_{6,7} - 18x_{7,4} + 8x_{7,8} - 11x_{8,5};$$

$$\beta_2 = A^2 - \Lambda^2_8 x_8 - \Lambda^2_{2,8} x_{2,8} - \Lambda^2_{4,3} x_{4,3} - \Lambda^2_{6,7} x_{6,7} - \Lambda^2_{7,4} x_{7,4} - \Lambda^2_{7,8} x_{7,8} - \Lambda^2_{8,5} x_{8,5} =$$
$$= 108 - 4x_8 + 2x_{2,8} - 3x_{4,3} - 3x_{6,7} - 8x_{7,4} + 2x_{7,8} - 14x_{8,5}.$$

Thus, we calculated a right-hand side of system (2.8.7):

$$\beta = \begin{pmatrix} 47 - 10x_8 + 7x_{2,8} + 3x_{4,3} - 2x_{6,7} - 18x_{7,4} + 8x_{7,8} - 11x_{8,5} \\ \\ 108 - 4x_8 + 2x_{2,8} - 3x_{4,3} - 3x_{6,7} - 8x_{7,4} + 2x_{7,8} - 14x_{8,5} \end{pmatrix}.$$

Since the matrix D is non-singularity, then apply the formula (2.8.9) to find the components of the vector x_W — solution of the system (2.8.7):

Table 2.10.9

The nonzero components of vector $\delta(4)$

$(i,j), i$	4	(4,6)	(6,5)	5
$\delta(4)$	1	1	1	-1

Table 2.10.10

The nonzero components of vector $\delta(8)$

$(i,j), i$	8	(1,8)	1
$\delta(8)$	1	1	1

$$x_W = (x_{\tau,\rho},(\tau,\rho) \in U_W;\ x_\gamma, \gamma \in I_W^*) = (x_4,\ x_{3,1}),$$

where

$$x_4 = \frac{1}{17}(202 - 24x_8 + 16x_{2,8} + 3x_{4,3} -$$
$$-7x_{6,7} - 44x_{7,4} + 18x_{7,8} - 36x_{8,5}),$$

$$x_{3,1} = \frac{1}{17}(-211 - 50x_8 + 39x_{2,8} + 36x_{4,3} +$$
$$+x_{6,7} - 86x_{7,4} + 46x_{7,8} - 7x_{8,5}).$$

Finally, using the (2.4.1) – (2.4.2), we obtain the general solution of the system (2.10.1) – (2.10.2) for the elements of the aggregate $R = \{U_R, I_R^*\}$ with independent variables $x_{\tau,\rho}, (\tau,\rho) \in U_N$ и $x_\gamma, \gamma \in I_N^*$. Thus, the general solution of the system (2.10.1) – (2.10.2) for the elements of the aggregate $R = \{U_R, I_R^*\}$ has the form

$$x_{1,2} = -5 + x_{2,8},$$

$$x_{1,8} = x_8 - x_{2,8} - x_{7,8} + x_{8,5},$$

$$x_1 = \frac{1}{17}(126 + 67x_8 - 39x_{2,8} - 36x_{4,3} - x_{6,7} + \\ +86x_{7,4} - 63x_{7,8} + 24x_{8,5}),$$

$$x_{3,7} = \frac{1}{17}(381 + 50x_8 - 39x_{2,8} - 19x_{4,3} - x_{6,7} + \\ +86x_{7,4} - 46x_{7,8} + 7x_{8,5}),$$

$$x_7 = \frac{1}{17}(381 + 50x_8 - 39x_{2,8} - 19x_{4,3} + 16x_{6,7} + \\ +69x_{7,4} - 63x_{7,8} + 7x_{8,5}),$$

$$x_{4,6} = \frac{1}{17}(202 - 24x_8 + 16x_{2,8} - 14x_{4,3} - 7x_{6,7} - \\ -27x_{7,4} + 18x_{7,8} - 36x_{8,5}),$$

$$x_{6,5} = \frac{1}{17}(-53 - 24x_8 + 16x_{2,8} - 14x_{4,3} - 24x_{6,7} - \\ -27x_{7,4} + 18x_{7,8} - 36x_{8,5}),$$

$$x_5 = \frac{1}{17}(53 + 24x_8 - 16x_{2,8} + 14x_{4,3} + 24x_{6,7} + \\ +27x_{7,4} - 18x_{7,8} + 19x_{8,5}),$$

where $x_8, x_{2,8}, x_{4,3}, x_{6,7}, x_{7,4}, x_{7,8}, x_{8,5} \in \mathbf{R}$ are independent variables of underdetermined system (2.10.1) – (2.10.2).

3. SENSOR LOCATION PROBLEM

One of applications of the given approach for solution of systems of linear algebraic equations in the form (2.7.1) – (2.7.2) is the Sensor Location Problem (SLP) [6, 46, 50, 52]. That is the problem of location of the minimum number of sensors in the nodes of the graph, in order to determine the arc flows and node intensities. Let's apply the decomposition algorithms for systems of linear algebraic equations of type (2.7.1) – (2.7.2) to the Sensor Location Problem for the graph

3.1. Sensor Location Problem for graphs

Let's introduce the finite connected directed graph $G = (I, U)$. The set U is defined on $I \times I$ ($|I| < \infty, |U| < \infty$). We assume, that the graph G is symmetric: that is: if $(i,j) \in U$, then $(j,i) \in U$. We note that the graph G is not undirected: the flow on arc (i,j), in general, will not be the same as the flow on arc (j,i). To designate this distinction, we refer to the graph $G = (I, U)$ as a two way directed graph.

We represent the traffic flow by a network flow function $x : U \to \mathbb{R}$ that satisfies the following system:

$$\sum_{j \in I_i^+(U)} x_{i,j} - \sum_{j \in I_i^-(U)} x_{j,i} = \begin{cases} x_i, & i \in I^*, \\ 0, & i \in I \setminus I^* \end{cases} \tag{3.1.1}$$

where I^* is the set of nodes with variable intensities, x_i is the variable intensity of node $i \in I^*$, $I_i^+(U) = \{j \in I : (i,j) \in U\}$ and $I_i^-(U) = \{j \in I : (j,i) \in U\}$. If the variable intensity x_i of node i is positive, the node i is a source; if it is negative, this node i is a sink. For system (3.1.1) is true the following condition: $\sum\limits_{i \in I} x_i = 0$. According to Theorem 2.2.1 if $I^* \neq \emptyset$, then the rank of the matrix of system (3.1.1) for a connected graph $G = (I, U)$ is equal to $|I|$.

In order to obtain information about the network flow function x and variables x_i of nodes $i \in I^*$, sensors are placed at the nodes of the graph $G = (I, U)$. The nodes in the graph $G = (I, U)$ with sensors we call monitored ones and denote the set of monitored nodes by M, $M \subseteq I$. We assume that if a node i is monitored, we know the values of flows on all outgoing and all incoming arcs for the node $i \in M$:

$$x_{ij} = f_{ij}, j \in I_i^+(U), \;\; x_{ji} = f_{ji}, j \in I_i^-(U), \quad i \in M.$$

If the set M includes the nodes from the set I^*, then we also know the values $x_i = f_i, i \in M \cap I^*$. So, we have

$$x_{ij} = f_{ij}, j \in I_i^+(U), \quad x_{ji} = f_{ji}, j \in I_i^-(U), \quad i \in M; \\ x_i = f_i, i \in M \cap I^*. \tag{3.1.2}$$

Consider any node i of the network. For every outgoing arc $(i,j) \in U$ for this node i determine a real number $p_{ij} \in (0,1]$ which denotes the part of the total outgoing flow $\sum\limits_{j \in I_i^+(U)} x_{ij}$ from node i corresponding to the arc (i,j). That is,

$$x_{ij} = p_{ij} \sum_{j \in I_i^+(U)} x_{ij}.$$

If $|I_i^+(U)| \geqslant 2$ for the node $i \in I$ then we can write the arc flow along all outgoing arcs from node i (except any selected arc) in terms of arc flow a single outgoing arc, for example, $(i,v_i), v_i \in I_i^+(U)$:

$$x_{i,j} = \frac{p_{i,j}}{p_{i,v_i}} x_{i,v_i}, \quad j \in I_i^+(U) \setminus v_i. \tag{3.1.3}$$

We continue this process for each node $i \in I$, if $|I_i^+(U)| \geqslant 2$.

Let $|I_i^+(U)| \geqslant 2$ for any node $i \in I$ and x_{i,v_i} is known for the arc (i,v_i) and equal to f_{i,v_i}. Then we can write the unknown arc flow along all outgoing arcs from node i (except any selected arc (i,v_i)) in terms of arc flow for a single outgoing arc (i,v_i), where x_{i,v_i} is known and equal to f_{i,v_i}.

So we shall formulate the Sensor Location Problem: **what is the minimum number of monitored nodes $|M|$ such that system (3.1.1) has an unique solution?**

Let's substitute the calculated arc flows according to (3.1.2) and (3.1.3) in the equations of system (3.1.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flow are known. Let's delete from graph G the set of the nodes $i \in M$. Then we have a new graph $\overline{G} = (\overline{I}, \overline{U})$. A new set of nodes with variable intensity for a new graph $\overline{G}$ is $\overline{I}^*$, where $\overline{I}^* = I^* \setminus (M \cap I^*)$. The new graph $\overline{G}$ can be non-connected. The graph $\overline{G}$ consists of connected components. Some connected components may contain no nodes of the set $\overline{I}^*$. The system (3.1.1) for graph $\overline{G} = (\overline{I}, \overline{U})$ will be the following one:

$$\sum_{j \in I_i^+(\overline{U})} x_{i,j} - \sum_{j \in I_i^-(\overline{U})} x_{j,i} = \begin{cases} x_i + b_i, & i \in \overline{I}^*, \\ a_i, & i \in I \setminus \overline{I}^* \end{cases} \tag{3.1.4}$$

$$\sum_{(i,j)\in\overline{U}} \lambda_{ij}^{p} x_{ij} = 0, p = \overline{1,q}, \tag{3.1.5}$$

where a_i, b_i, λ_{ij}^{p} – are constants.

Let's state the steps of the algorithm for modeling the new graph $\overline{G}$ for the given set M and the formation of set $I \setminus M^*$.

Step 1. Construct a cut $CC(M)$ [17, 29, 30, 53] with respect to the set of monitored nodes M.

Step 2. Find the nodes of the set $I(CC(M))$.

Step 3. Construct the set $M^+ = I(CC(M)) \setminus M$.

Step 4. Form sets $M^* = M \cup M^+$ and $I \setminus M^*$.

The unknowns of the system (3.1.4) – (3.1.5) are the flows for outgoing arcs from the nodes of the set $I \setminus M^*$. Also the unknowns in the system (3.1.4) – (3.1.5) are the variable intensities x_i, where $i \in \overline{I}^*$ for the new graph $\overline{G} = (\overline{I}, \overline{U})$. The system (3.1.4) – (3.1.5) is a particular case of the system (2.7.1) – (2.7.2) if a connected component includes at least one node from the set $\overline{I}^*$. The system (3.1.4) – (3.1.5) has an unique solution for the given set M if and only if the rank of the matrix of system (3.1.4) – (3.1.5) is equal to the number of unknowns of the system (3.1.4) – (3.1.5). According to Theorem 2.2.1, Theorem 2.7.3 computes the rank of the matrix of system (3.1.4) for the given connected component for graph $\overline{G} = (\overline{I}, \overline{U})$ if the connected component contains nodes of the set $\overline{I}^*$. We use the theory of decomposition [40, 45, 53] if the connected components of the graph $\overline{G}$ don't include the nodes of the set $\overline{I}^*$. For computing the rank of the matrix of system (3.1.4) – (3.1.5) we use graph theoretical properties of the support according to Theorem 2.9.1.

Consider the examples to the Sensor Location Problem in the graph G for the case when the set M of monitored nodes contains a single node: $|M| = 1$.

3.2. Example 1 (Underdetermined system)

In Figure 3.1 we represent a finite connected directed symmetric graph $G = (I, U)$, where

$$I = \{1, 2, 3, 4, 5, 6\},$$

$$U = \{(1,2),(1,3),(2,1),(2,4),(2,6),(3,1),(3,5),(4,2),(4,6),(4,5),$$

$$(5,3),(5,4),(5,6),(6,2),(6,4),(6,5)\},$$

The set I^* consists of nodes:

$$I^* = \{2, 4, 5, 6\}.$$

For the graph $G = (I, U)$ (see Figure 3.1) we write the system of linear algebraic equations of type (3.1.1).

$$\begin{aligned}
x_{1,2} + x_{1,3} - x_{2,1} - x_{3,1} &= 0 \\
x_{2,4} + x_{2,6} + x_{2,1} - x_{4,2} - x_{1,2} - x_{6,2} &= x_2 \\
x_{3,1} + x_{3,5} - x_{1,3} - x_{5,3} &= 0 \\
x_{4,2} + x_{4,5} + x_{4,6} - x_{2,4} - x_{5,4} - x_{6,4} &= x_4 \\
x_{5,4} + x_{5,3} + x_{5,6} - x_{3,5} - x_{4,5} - x_{6,5} &= x_5 \\
x_{6,2} + x_{6,4} + x_{6,5} - x_{2,6} - x_{4,6} - x_{5,6} &= x_6
\end{aligned} \tag{3.2.1}$$

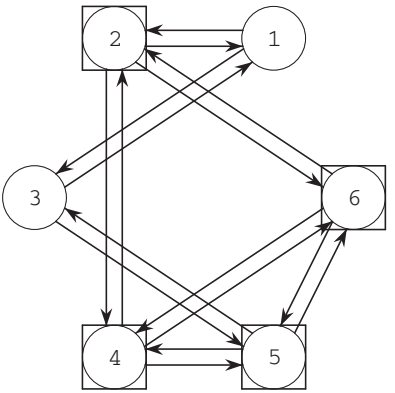

Fig. 3.1. Finite connected directed symmetric graph G

Suppose that the set M of monitored nodes is $M = \{1\}$ for the graph shown in Figure 3.1.

We form the sets:

$$M^+ = I(CC(M)) \setminus M = \{2,3\};$$

$$M^* = M \bigcup M^+ = \{1\} \bigcup \{2,3\} = \{1,2,3\}.$$

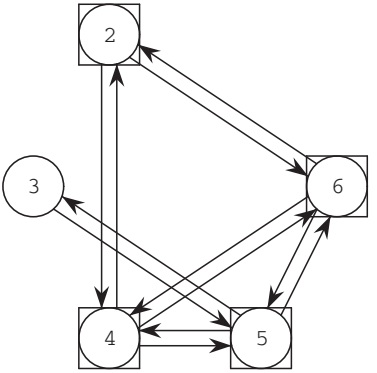

Fig. 3.2. Graph G'

Thus $M^* = \{1,2,3\}$ and $I \setminus M^* = \{4,5,6\}$.

In the Sensor Location Problem the values of flows on all incoming and outgoing arcs for the each node i of the set M (monitored nodes) are known and also we known the values $x_i = f_i$, $i \in M \cap I^*$:

$$x_{1,2} = f_{1,2},\ x_{2,1} = f_{2,1},$$
$$x_{1,3} = f_{1,3},\ \ x_{3,1} = f_{3,1}. \tag{3.2.2}$$

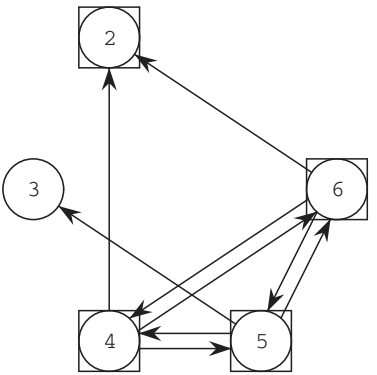

Fig. 3.3. Graph $\overline{G}$

We substitute the known values of the variables (3.2.2) to the system of equations (3.2.1) and delete the corresponding arcs from the graph G. We also delete the nodes $i \in M$ from the graph G. The graph G' obtained after substitution (3.2.2) is shown in Figure 3.2.

The rest of the flows for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{2,3\}$, can be expressed from the flows of the outgoing arcs for M^+ by the following equations:

$$x_{2,4} = \frac{p_{2,4}}{p_{21}} f_{2,1};\ x_{2,6} = \frac{p_{2,6}}{p_{21}} f_{2,1};\ x_{3,5} = \frac{p_{3,5}}{p_{3,1}} f_{3,1}. \tag{3.2.3}$$

We substitute (3.2.3) into the system (3.2.1). We obtain the graph $\overline{G}$ where $\overline{G} = (\overline{I}, \overline{U})$, by deleting from the graph G' the arcs corresponding to variables (3.2.3). The graph $\overline{G}$ is shown in Figure 3.3.

The system (3.2.1) for the graph $\overline{G}$ (see Fig.3.3) after substitution of known arc flows (3.2.2) and (3.2.3) transforms to the form (3.2.4):

$$\begin{gathered}
f_{1,2} + f_{1,3} - f_{2,1} - f_{3,1} = 0 \\
\frac{p_{2,4}}{p_{2,1}} f_{21} + \frac{p_{2,6}}{p_{2,1}} f_{2,1} + f_{2,1} - x_{4,2} - f_{1,2} - x_{6,2} = x_2 \\
f_{3,1} + \frac{p_{3,5}}{p_{3,1}} f_{3,1} - f_{1,3} - x_{5,3} = 0 \\
x_{4,2} + x_{4,5} + x_{4,6} - \frac{p_{2,4}}{p_{2,1}} f_{2,1} - x_{5,4} - x_{6,4} = x_4 \\
x_{5,4} + x_{5,3} + x_{5,6} - \frac{p_{3,5}}{p_{3,1}} f_{3,1} - x_{4,5} - x_{6,5} = x_5 \\
x_{6,2} + x_{6,4} + x_{6,5} - \frac{p_{2,6}}{p_{2,1}} f_{2,1} - x_{4,6} - x_{5,6} = x_6
\end{gathered} \tag{3.2.4}$$

We form additional equations of type (2.7.2) for the system (3.2.4) for the graph $\overline{G}$ (see Figure 3.3), where $\lambda_i^p = 0$, $i \in \overline{I}^*$, $p = \overline{1,q}$. Part of the unknowns of the system (3.2.4) constitute the arc flows for arcs, outgoing from nodes of the set $I \setminus M^* = \{4,5,6\}$. For these arc flows we perform the following steps.

- Choose arbitrary outgoing arc that starts from a node of set $I \setminus M^*$ where $I \setminus M^* = \{4,5,6\}$, for example, the arc (4,5) is outgoing arc

from node 4. Let us express the arc flows to all other arcs outgoing from the node $i = 4$ through the arc flow of $x_{4,5}$ of selected arc (4,5).

$$x_{4,2} = \frac{p_{4,2}}{p_{4,5}} x_{4,5}, \;\; x_{4,6} = \frac{p_{4,6}}{p_{4,5}} x_{4,5}.$$

- Choose any outgoing arc from the node $i = 5$, for example, (5,6). Let us express the arc flows to all other outgoing arcs from node $i = 5$ through the arc flow of $x_{5,6}$ outgoing arc (5,6).

$$x_{5,3} = \frac{p_{5,3}}{p_{5,6}} x_{5,6}, \;\; x_{5,4} = \frac{p_{5,4}}{p_{56}} x_{5,6}.$$

- Choose any outgoing arc from the node $i = 6$, for example, (6,2). Let us express the arc flows to all other outgoing arcs from node $i = 6$ through the arc flow of $x_{6,2}$.

$$x_{64} = \frac{p_{64}}{p_{62}} x_{62},$$

$$x_{65} = \frac{p_{65}}{p_{62}} x_{62}.$$

So, we have:

$$\begin{aligned} x_{4,2} &= \frac{p_{4,2}}{p_{4,5}} x_{4,5}, \;\; x_{4,6} = \frac{p_{4,6}}{p_{4,5}} x_{4,5}, \\ x_{5,3} &= \frac{p_{5,3}}{p_{5,6}} x_{5,6}, \;\; x_{5,4} = \frac{p_{5,4}}{p_{5,6}} x_{5,6}, \\ x_{6,4} &= \frac{p_{6,4}}{p_{6,2}} x_{6,2}, \;\; x_{6,5} = \frac{p_{6,5}}{p_{6,2}} x_{6,2}. \end{aligned} \tag{3.2.5}$$

In this example, the additional equations of type (2.7.2) for the graph $\overline{G}$ are as follows:

$$\begin{aligned} x_{4,2} - \frac{p_{4,2}}{p_{4,5}} x_{4,5} &= 0, \;\; x_{4,6} - \frac{p_{4,6}}{p_{4,5}} x_{4,5} = 0, \\ x_{5,3} - \frac{p_{5,3}}{p_{5,6}} x_{5,6} &= 0, \;\; x_{5,4} - \frac{p_{5,4}}{p_{5,6}} x_{5,6} = 0, \\ x_{6,4} - \frac{p_{6,4}}{p_{6,2}} x_{6,2} &= 0, \;\; x_{6,5} - \frac{p_{6,5}}{p_{6,2}} x_{6,2} = 0. \end{aligned} \tag{3.2.6}$$

We define the number of unknowns of the system (3.2.4), (3.2.6). Part of the unknowns of the system (3.2.4), (3.2.6) makes up outgoing arc flows for arcs outgoing from the set of nodes $I \setminus M^* = \{4,5,6\}$:

$$x_{4,2},\ x_{4,5},\ x_{4,6},\ x_{5,3},\ x_{5,4},\ x_{5,6},\ x_{6,2},\ x_{6,4},\ x_{6,5}.$$

The remaining part of the unknowns of the system (3.2.4), (3.2.6) defines the values of the variables x_i, $i \in \overline{I}^*$, where $\overline{I}^* = \{2,4,5,6\}$:

$$x_2,\ x_4,\ x_5,\ x_6.$$

Thus, the number of unknowns of the system (3.2.4), (3.2.6) is equal to 13. Number of equations in the system (3.2.4), (3.2.6) is equal to 11. Hence the rank of the matrix of system (3.2.4), (3.2.6) is not greater than the number 11. System (3.2.4), (3.2.6) is a underdetermined system of linear algebraic equations. System (3.2.4), (3.2.6) doesn't have unique solution for given set of monitored nodes $M = \{1\}$. Consequently the system (3.2.1) for the set of monitored nodes $M = \{1\}$ also has no unique solution. So, when we locate single sensor into the node 1, we cannot get unique solution for system (3.2.1).

3.3. Example 2 (Unique solution)

In Figure 3.4 we show a finite connected directed symmetric graph G with the set of nodes I and the set of arcs U where

$$I = \{1, 2, 3, 4, 5, 6\},$$

$$U = \{(1,2),(1,3),(2,1),(2,4),(2,6),(3,1),(3,5),(4,2),(4,6),(4,5),$$
$$(5,3),(5,4),(5,6),(6,2),(6,4),(6,5)\},$$
$$I^* = \{2, 4, 5, 6\}.$$

For the graph $G = (I, U)$ (see Figure 3.4) we write the system of linear algebraic equations in the form:

$$\begin{aligned}
x_{1,2} + x_{1,3} - x_{2,1} - x_{3,1} &= 0 \\
x_{2,4} + x_{2,6} + x_{2,1} - x_{4,2} - x_{1,2} - x_{6,2} &= x_2 \\
x_{3,1} + x_{3,5} - x_{1,3} - x_{5,3} &= 0 \\
x_{4,2} + x_{4,5} + x_{4,6} - x_{2,4} - x_{5,4} - x_{6,4} &= x_4 \\
x_{5,4} + x_{5,3} + x_{5,6} - x_{3,5} - x_{4,5} - x_{6,5} &= x_5 \\
x_{6,2} + x_{6,4} + x_{6,5} - x_{2,6} - x_{4,6} - x_{5,6} &= x_6
\end{aligned} \tag{3.3.1}$$

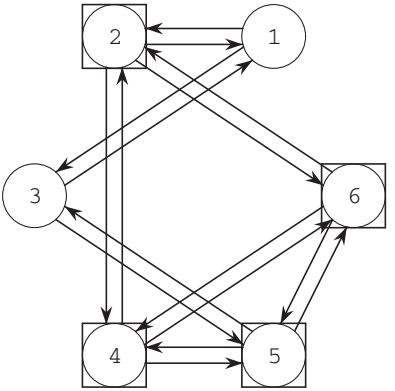

Fig. 3.4. Finite connected directed symmetric graph G

Suppose that the set of monitoring nodes is $M = \{2\}$ for the graph shown in Figure 3.4. Construct the cut $CC(M)$ with respect to the set M. We form the sets

$$M^+ = I(CC(M)) \setminus M = \{1,4,6\}; \quad M^* = M \bigcup M^+ = \{1,2,4,6\};$$

$$M^* = \{1,2,4,6\}, \ I \setminus M^* = \{3,5\}.$$

In the Sensor Location Problem (SLP) the values of flows on all incoming and outgoing arcs for the each node i of the set M (monitored nodes) are known and we also know the values $x_i = f_i$, $i \in M \bigcap I^*$:

$$\begin{gathered} x_{1,2} = f_{1,2}, \ x_{2,1} = f_{2,1}, \ x_{2,4} = f_{2,4}, \\ x_{4,2} = f_{4,2}, \ x_{2,6} = f_{2,6}, \ x_{6,2} = f_{6,2}, \ x_2 = f_2. \end{gathered} \tag{3.3.2}$$

We substitute the known values of the variables (3.3.2) to the system of equations (3.3.1) and delete the corresponding arcs from the graph G. Also, we delete the nodes $i \in M$ from the graph G. The graph G' obtained after deleting the arcs corresponding to the variables (3.3.2) and nodes $i \in M$ from graph G is shown in Figure 3.5. The rest of the flows for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{1,4,6\}$, can be

expressed from the flows of the outgoing arcs for $M^+ = \{1,4,6\}$ by the following equations:

$$
\begin{aligned}
x_{1,3} &= \frac{p_{1,3}}{p_{1,2}} f_{1,2}, \\
x_{4,5} &= \frac{p_{4,5}}{p_{4,2}} f_{4,2}, \\
x_{4,6} &= \frac{p_{4,6}}{p_{4,2}} f_{4,2}, \\
x_{6,4} &= \frac{p_{6,4}}{p_{6,2}} f_{6,2}, \\
x_{6,5} &= \frac{p_{6,5}}{p_{6,2}} f_{6,2}.
\end{aligned}
\tag{3.3.3}
$$

Let us substitute (3.3.3) to the system of linear equations (3.3.1). We delete from the graph G arcs which correspond to the known values of the arc flows (3.3.2) and (3.3.3). The graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the arcs corresponding to variables (3.3.3) from the graph G' is shown in Figure 3.6. The system (3.3.1) for the graph $\overline{G} = (\overline{I}, \overline{U})$ (see Figure 3.6) transforms to the form (3.3.4).

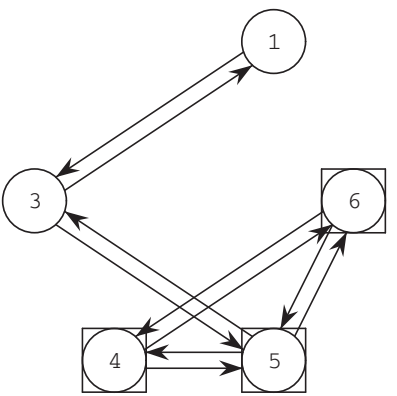

Fig. 3.5. Graph G'

$$
\begin{aligned}
& f_{1,2} + \frac{p_{1,3}}{p_{1,2}} f_{1,2} - f_{2,1} - x_{3,1} = 0 \\
& f_{2,1} + f_{2,4} + f_{2,6} - f_{1,2} - f_{4,2} - f_{6,2} = f_2 \\
& x_{3,1} + x_{3,5} - \frac{p_{1,3}}{p_{1,2}} f_{1,2} - x_{5,3} = 0 \\
& f_{4,2} + \frac{p_{4,5}}{p_{4,2}} f_{4,2} + \frac{p_{4,6}}{p_{4,2}} f_{4,2} - f_{2,4} - x_{5,4} - \frac{p_{6,4}}{p_{6,2}} f_{6,2} = x_4 \\
& x_{5,4} + x_{5,3} + x_{5,6} - x_{3,5} - \frac{p_{4,5}}{p_{4,2}} f_{4,2} - \frac{p_{6,5}}{p_{6,2}} f_{6,2} = x_5 \\
& f_{6,2} + \frac{p_{6,4}}{p_{6,2}} f_{6,2} + \frac{p_{6,5}}{p_{6,2}} f_{6,2} - f_{2,6} - \frac{p_{4,6}}{p_{4,2}} f_{4,2} - x_{5,6} = x_6
\end{aligned}
\tag{3.3.4}
$$

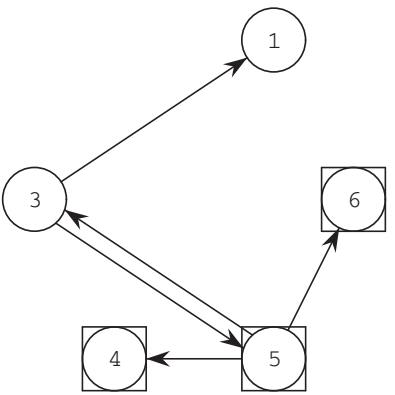

Fig. 3.6. Graph $\overline{G} = (\overline{I}, \overline{U})$

Arc flows $x_{i,j}$,$(i,j) \in \overline{U}$, corresponding to the arcs outgoing from node set $I \setminus M^* = \{3,5\}$ are unknown. For these unknown flows $x_{i,j}$, $(i,j) \in \overline{U}$ we form the additional equation of the type (2.7.2).

- Choose arbitrary outgoing arc that starts from a node set i of set $I \setminus M^* = \{3,5\}$, for example, for the node $i = 3$ we choose the arc (3,5). Let us express the arc flows to all other arcs outgoing from the

node $i = 3$ through the arc flow of $x_{3,5}$ for the chosen outgoing arc (3,5).

- Choose any outgoing arc from the node $i = 5$, for example (5,6). Let us express the arc flows to all other arcs outgoing from the node $i = 5$ through the arc flow of $x_{5,6}$.

$$x_{3,1} = \frac{p_{3,1}}{p_{3,5}} x_{3,5},$$

$$x_{5,3} = \frac{p_{5,3}}{p_{5,6}} x_{5,6}, \; x_{5,4} = \frac{p_{5,4}}{p_{5,6}} x_{5,6}.$$

Additional equations have the form:

$$\begin{aligned} x_{3,1} - \frac{p_{3,1}}{p_{3,5}} x_{3,5} &= 0, \\ x_{5,3} - \frac{p_{5,3}}{p_{5,6}} x_{5,6} &= 0, \\ x_{5,4} - \frac{p_{5,4}}{p_{5,6}} x_{5,6} &= 0. \end{aligned} \tag{3.3.5}$$

Part of the unknowns of the system (3.3.4), (3.3.5) makes up outgoing arc flows for arcs from node sets $I \setminus M^* = \{3,5\}$ for the graph $\overline{G}$:

$$x_{3,1}, \; x_{3,5}, \; x_{5,3}, \; x_{5,4}, \; x_{5,6}.$$

The remaining part of the unknowns of the system (3.3.4), (3.3.5) defines the variables x_i, $i \in \overline{I}^* = \{4,5,6\}$:

$$x_4, \; x_5, \; x_6.$$

Thus, the number of unknowns of the system (3.3.4), (3.3.5) is equal to 8. Number of equations in the system (3.3.4), (3.3.5) coincides with the number of unknowns and is equal to 8.

We compute the rank of the matrix of the system (3.3.4), (3.3.5). If the system (3.3.4), (3.3.5) is a system of full rank, then it has a unique solution.

For that we choose any support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.3.4) (see Figure 3.7), $U_R = \{(3,1),(5,3)\}$, $I_R^* = \{4,5,6\}$ [32, 53].

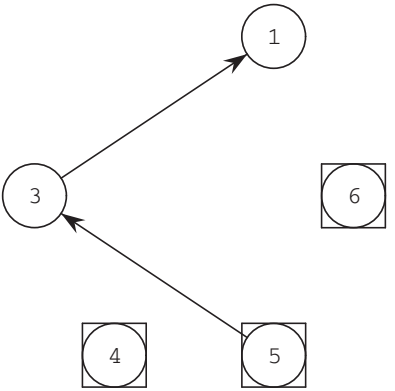

Fig. 3.7. Support of the graph $\overline{G}$ for the system (3.3.4)

Support $R = \{U_R, I_R^*\}$ (see. Figure 3.7) of the graph $\overline{G}$ for the system (3.3.4) consists of three trees. After selecting the support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.3.4) we determine what structures can be obtained after adding one non supporting element from the sets $\overline{U} \backslash U_R$, $\overline{I}^* \backslash I_R^*$ to the support R.

We construct a system of characteristic vectors (basis of the solution space) of the homogeneous system generated by the system (3.3.4). They are the vectors $\delta(3,5)$,$\delta(5,4)$,$\delta(5,6)$, entailed by the arcs (3,5),(5,4),(5,6). We compute the characteristic vector $\delta(3,5) = (\delta_{ij}^{3,5}, (i,j) \in \overline{U};\ \delta_i^{3,5}, i \in \overline{I}^*)$, entailed by the arc (3,5):

$$\delta(3,5) = (\delta_{3,5}^{3,5} \to 1, \delta_{3,1}^{3,5} \to 0, \delta_{5,3}^{3,5} \to 1, \delta_{5,4}^{3,5} \to 0,$$

$$\delta_{5,6}^{3,5} \to 0, \delta_4^{3,5} \to 0, \delta_5^{3,5} \to 0, \delta_6^{3,5} \to 0).$$

Characteristic vectors $\delta(5,4)$, $\delta(5,6)$, entailed by the arcs (5,4) и (5,6) respectively, are:

$$\delta(5,4) = (\delta_{5,4}^{5,4} \to 1, \delta_{3,1}^{5,4} \to 0, \delta_{3,5}^{5,4} \to 0, \delta_{5,3}^{5,4} \to 0,$$

$$\delta_{5,6}^{5,4} \to 0, \delta_4^{5,4} \to -1, \delta_5^{5,4} \to 1, \delta_6^{5,4} \to 0);$$

$$\delta(5,6) = (\delta_{5,6}^{5,6} \to 1, \delta_{3,1}^{5,6} \to 0, \delta_{3,5}^{5,6} \to 0, \delta_{5,3}^{5,6} \to 0,$$

$$\delta_{5,4}^{5,6} \to 0, \delta_4^{5,6} \to 0, \delta_5^{5,6} \to 1, \delta_6^{5,6} \to -1).$$

We enumerate the equations (3.3.5) of the system (3.3.4) − (3.3.5). We compute the determinants $\Lambda_{\tau\rho}^p$, $(\tau,\rho) \in \overline{U} \backslash U_R = \{(3,5),(5,4),(5,6)\}$ with respect to the equations (3.3.5), where p is a number of equation from the system (3.3.5), $p = 1, 2, 3$ using the formulas:

$$\Lambda_{\tau\rho}^p = \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^p, p = \overline{1,q}.$$

We form a matrix of determinants:

$$D = \begin{pmatrix} \Lambda_{3,5}^1 & \Lambda_{5,4}^1 & \Lambda_{5,6}^1 \\ \Lambda_{3,5}^2 & \Lambda_{5,4}^2 & \Lambda_{5,6}^2 \\ \Lambda_{3,5}^3 & \Lambda_{5,4}^3 & \Lambda_{5,6}^3 \end{pmatrix}.$$

Thus, the matrix of determinants for this example has the form:

$$D = \begin{pmatrix} -\dfrac{p_{3,1}}{p_{3,5}} & 0 & 0 \\ 1 & 0 & -\dfrac{p_{5,3}}{p_{5,6}} \\ 0 & 1 & -\dfrac{p_{5,4}}{p_{5,6}} \end{pmatrix}.$$

Considering $0 < p_{i,j} \leqslant 1$, $(i,j) \in U$, the determinant of the matrix D is not zero:

$$det\ D = \frac{p_{3,1}\, p_{5,3}}{p_{3,5}\, p_{5,6}} \neq 0.$$

Thus, the system (3.3.4), (3.3.5) is a system of full rank. The number of unknowns of the system (3.3.4), (3.3.5) equal to the rank of the matrix and is equal to 8. The system (3.3.4), (3.3.5) has the unique solution for given set of monitored nodes $M = \{2\}$. The required unique solution (3.3.4), (3.3.5) has the form:

$$x_{3,5} = -\frac{(f_{2,1} + f_{1,2}(-1 - \dfrac{p_{1,3}}{p_{1,2}}))p_{3,5}}{p_{3,1}},$$

$$x_{5,4} = \frac{(-f_{2,1}p_{1,2}(p_{3,1}+p_{3,5}) + f_{1,2}(p_{1,3}p_{3,5} + p_{1,2}(p_{3,1}+p_{3,5})))p_{5,4}}{p_{1,2}p_{3,1}p_{5,3}},$$

$$x_{5,6} = \frac{(-f_{2,1}p_{1,2}(p_{3,1}+p_{3,5}) + f_{1,2}(p_{1,3}p_{3,5} + p_{1,2}(p_{3,1}+p_{3,5})))p_{5,6}}{p_{1,2}p_{3,1}p_{5,3}},$$

$$x_{3,1} = -f_{2,1} + f_{1,2}\left(1 + \frac{p_{1,3}}{p_{1,2}}\right),$$

$$x_{5,3} = f_{1,2} - f_{2,1} - \frac{\left(f_{2,1} + f_{1,2}(-1 - \frac{p_{1,3}}{p_{1,2}})\right)p_{3,5}}{p_{3,1}},$$

$$x_4 = -f_{2,4} + f_{4,2} + \frac{f_{4,2}p_{4,5}}{p_{4,2}} + \frac{f_{4,2}p_{4,6}}{p_{4,2}} +$$

$$+\frac{(-f_{1,2}+f_{2,1})p_{5,4}}{p_{5,3}} + \frac{(f_{2,1} + f_{1,2}(-1 - \frac{p_{1,3}}{p_{1,2}}))p_{3,5}p_{5,4}}{p_{3,1}p_{5,3}} - \frac{f_{6,2}p_{6,4}}{p_{6,2}},$$

$$x_5 = \frac{1}{p_{1,2}p_{3,1}p_{4,2}p_{5,3}p_{6,2}}\Bigg(f_{1,2}p_{4,2}(p_{1,3}p_{3,5}(p_{5,4}+p_{5,6}) +$$

$$+p_{1,2}(p_{3,5}(p_{5,4}+p_{5,6}) +$$

$$+p_{3,1}(p_{5,3}+p_{5,4}+p_{5,6})))p_{6,2} - p_{1,2}(f_{2,1}p_{4,2}(p_{3,5}(p_{5,4}+p_{5,6}) +$$

$$+p_{3,1}(p_{5,3}+p_{5,4}+p_{5,6}))p_{6,2} + p_{3,1}p_{5,3}(f_{4,2}p_{4,5}p_{6,2} + f_{6,2}p_{4,2}p_{6,5}))\Bigg),$$

$$x_6 = -f_{2,6} + f_{6,2} - \frac{f_{4,2}p_{4,6}}{p_{4,2}} + \frac{(-f_{1,2}+f_{2,1})p_{5,6}}{p_{5,3}} +$$

$$+\frac{(f_{2,1} + f_{1,2}(-1 - \frac{p_{1,3}}{p_{1,2}}))p_{3,5}p_{5,6}}{p_{3,1}p_{5,3}} + \frac{f_{6,2}p_{6,4}}{p_{6,2}} + \frac{f_{6,2}p_{6,5}}{p_{6,2}}.$$

As a result of the localization of a single sensor to the node 2 in the graph G which is represented in Figure 3.4 we have a unique solution of system (3.3.1).

3.4. Example 3 (Underdetermined system)

For the finite connected directed symmetric graph $G = (I, U)$ which is shown in Figure 3.8, $I = \{1, 2, 3, 4, 5, 6\}$, $U = \{(1,2), (1,3), (2,1), (2,4), (2,6), (3,1), (3,5), (4,2), (4,6), (4,5), (5,3), (5,4), (5,6),(6,2),(6,4),(6,5)\}$, $I^* = \{2, 4, 5, 6\}$ consider a system of linear algebraic equations (3.4.1).

$$
\begin{aligned}
x_{1,2} + x_{1,3} - x_{2,1} - x_{3,1} &= 0 \\
x_{2,4} + x_{2,6} + x_{2,1} - x_{4,2} - x_{1,2} - x_{6,2} &= x_2 \\
x_{3,1} + x_{3,5} - x_{1,3} - x_{5,3} &= 0 \\
x_{4,2} + x_{4,5} + x_{4,6} - x_{2,4} - x_{5,4} - x_{6,4} &= x_4 \\
x_{5,4} + x_{5,3} + x_{5,6} - x_{3,5} - x_{4,5} - x_{6,5} &= x_5 \\
x_{6,2} + x_{6,4} + x_{6,5} - x_{2,6} - x_{4,6} - x_{5,6} &= x_6
\end{aligned}
\tag{3.4.1}
$$

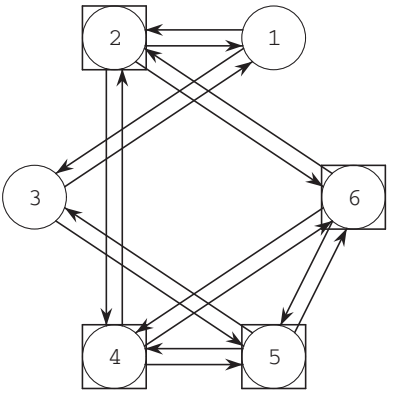

Fig. 3.8. Connected symmetric directed graph G

Suppose that the set M of monitoring nodes for the graph shown in Figure 3.8 is a single node: $M = \{3\}$. Let us form the sets:

$$M^+ = I(CC(M)) \setminus M = \{1,5\}; \quad M^* = M \bigcup M^+ = \{1,3,5\};$$

$$I \setminus M^* = \{2,4,6\}.$$

In the Sensor Location Problem (SLP) the values of flows on all incoming and outgoing arcs for the each node i of the set M (monitored nodes) are known and also we known the values $x_i = f_i$, $i \in M \cap I^*$ of variable intensities of the nodes for the set $M \cap I^*$. Considering $M \cap I^* = \emptyset$ we have:

$$\begin{aligned} x_{3,1} &= f_{3,1}, \\ x_{3,5} &= f_{3,5}, \\ x_{1,3} &= f_{13}, \\ x_{5,3} &= f_{5,3}. \end{aligned} \tag{3.4.2}$$

Let us substitute (3.4.2) to the system of linear equations (3.4.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flow are known according to (3.4.2). Also, we delete from graph G the set of the nodes $i \in M$, $M = \{3\}$.

We obtain a graph G' which is shown in Figure 3.9.

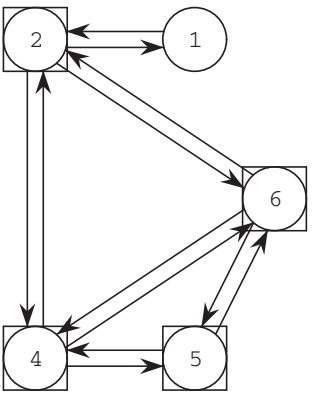

Fig. 3.9. Graph G'

The values for the arc flows for arcs outgoing from the node of sets $M^+ = I(CC(M)) \setminus M = \{1,5\}$, we express in terms of known arc flows of outgoing arcs from the nodes of the set M^+ as follows:

$$\begin{aligned} x_{1,2} &= \frac{p_{1,2}}{p_{1,3}} f_{1,3}, \\ x_{5,4} &= \frac{p_{5,4}}{p_{5,3}} f_{5,3}, \\ x_{5,6} &= \frac{p_{5,6}}{p_{5,3}} f_{5,3}. \end{aligned} \tag{3.4.3}$$

Let's delete from graph G' (see Figure 3.9) the set of the arcs on which the arc flow are known in accordance with (3.4.3). We obtain a new graph $\overline{G} = (\overline{I}, \overline{U})$ which is shown in Figure 3.10.

System (3.4.1) for the graph $\overline{G}$ transforms to the form (3.4.4).

$$\begin{aligned} f_{1,3} + \frac{p_{1,2}}{p_{1,3}} f_{1,3} - x_{21} - f_{3,1} &= 0 \\ x_{2,1} + x_{2,4} + x_{2,6} - \frac{p_{1,2}}{p_{1,3}} f_{1,3} - x_{4,2} - x_{6,2} &= x_2 \\ f_{3,1} + f_{3,5} - f_{1,3} - f_{5,3} &= 0 \\ x_{4,2} + x_{4,5} + x_{4,6} - x_{2,4} - \frac{p_{5,4}}{p_{5,3}} f_{5,3} - x_{6,4} &= x_4 \\ f_{5,3} + \frac{p_{5,4}}{p_{5,3}} f_{5,3} + \frac{p_{5,6}}{p_{5,3}} f_{5,3} - f_{3,5} - x_{4,5} - x_{6,5} &= x_5 \\ x_{6,2} + x_{6,4} + x_{6,5} - x_{2,6} - x_{4,6} - \frac{p_{5,6}}{p_{5,3}} f_{5,3} &= x_6 \end{aligned} \tag{3.4.4}$$

Arcs flows corresponding to the arcs outgoing from node set $I \setminus M^*$ are unknown where $I \setminus M^* = \{2,4,6\}$. For these unknown flows $x_{i,j}$, $(i,j) \in \overline{U}$ we form the additional equation of the type (2.7.2), if possible.

- Choose arbitrary outgoing arc that starts from a node i of set $I \setminus M^*$, where $I \setminus M^* = \{2,4,6\}$, for example, for the node $i = 2$ we choose the arc (2,4). Let us express the arc flows to all other arcs outgoing from

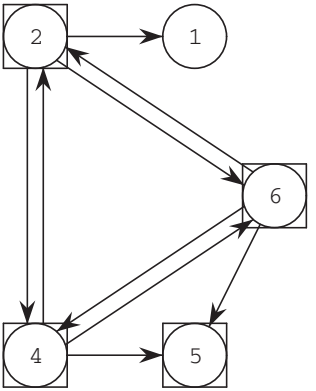

Fig. 3.10. Graph $\overline{G}$

the node $i = 2$ through the arc flow of $x_{2,4}$ for the chosen outgoing arc (2,4).

$$x_{2,1} = \frac{p_{2,1}}{p_{2,4}} x_{2,4},$$

$$x_{2,6} = \frac{p_{2,6}}{p_{24}} x_{2,4}.$$

- Choose any outgoing arc from node $i = 4$, for example, (4,5). Then, for all outgoing arcs from a node $i = 4$ except (4,5) the equalities are true:

$$x_{4,2} = \frac{p_{4,2}}{p_{4,5}} x_{4,5},$$

$$x_{4,6} = \frac{p_{4,6}}{p_{4,5}} x_{4,5}.$$

- Choose any outgoing arc from node $i = 6$, for example, (6,2). Then, for all outgoing arcs from a node $i = 6$ except (6,2) the equalities are true:

$$x_{6,4} = \frac{p_{6,4}}{p_{6,2}} x_{6,2},$$

$$x_{6,5} = \frac{p_{6,5}}{p_{6,2}} x_{6,2}.$$

As a result, we obtained the following expressions for the unknown arc flows in graph $\overline{G}$:

$$\begin{aligned} x_{2,1} &= \frac{p_{2,1}}{p_{2,4}} x_{2,4}, \\ x_{2,6} &= \frac{p_{2,6}}{p_{2,4}} x_{2,4}, \\ x_{4,2} &= \frac{p_{4,2}}{p_{4,5}} x_{4,5}, \\ x_{4,6} &= \frac{p_{4,6}}{p_{4,5}} x_{4,5}, \\ x_{6,4} &= \frac{p_{6,4}}{p_{6,2}} x_{6,2}, \\ x_{6,5} &= \frac{p_{6,5}}{p_{6,2}} x_{6,2}. \end{aligned} \tag{3.4.5}$$

Thus, additional equations of type (2.7.2) for the graph $\overline{G}$ (see Figure 3.10) have the form:

$$\begin{aligned} x_{2,1} - \frac{p_{2,1}}{p_{24}} x_{2,4} = 0, \quad x_{2,6} - \frac{p_{2,6}}{p_{2,4}} x_{2,4} = 0, \\ x_{4,2} - \frac{p_{4,2}}{p_{4,5}} x_{4,5} = 0, \quad x_{4,6} - \frac{p_{4,6}}{p_{4,5}} x_{4,5} = 0, \\ x_{6,4} - \frac{p_{6,4}}{p_{6,2}} x_{6,2} = 0, \quad x_{6,5} - \frac{p_{6,5}}{p_{6,2}} x_{6,2} = 0. \end{aligned} \tag{3.4.6}$$

We define the number of unknowns of the system (3.4.4), (3.4.6). Part of the unknown of the system (3.4.4), (3.4.6) makes up outgoing arc flows for arcs from node sets $I \setminus M^* = \{2,4,6\}$ for graph $\overline{G}$. They are

$$x_{2,1},\ x_{2,4},\ x_{2,6},\ x_{4,2},\ x_{4,5},\ x_{4,6},\ x_{6,2},\ x_{6,4},\ x_{6,5}.$$

The remaining part of the unknowns of the system (3.4.4), (3.4.6) defines the variable intensities x_i, $i \in \overline{I}^*$ of the nodes $\overline{I}^*$, where $\overline{I}^* = \{2,4,5,6\}$:

$$x_2,\ x_4,\ x_5,\ x_6.$$

Thus, the number of unknowns of the system (3.4.4), (3.4.6) is equal to 13. Number of equations in the system (3.4.4), (3.4.6) is equal to 11. System (3.4.4), (3.4.6) is a underdetermined system of linear algebraic equations. Consequently the system (3.4.4), (3.4.6) has no unique solution for given monitoring node $M = \{3\}$.

3.5. Example 4 (Unique solution)

We will build the analytical and numerical solutions sparse system of linear algebraic equations of the type (3.1.1) for the case when the sensor is installed in the node $M = \{4\}$ for the graph G shown in Figure 3.11.

3.5.1. Analytical solution

In Figure 3.11 represented by a finite connected directed symmetric graph $G = (I, U)$, where

$$I = \{1, 2, 3, 4, 5, 6\},$$

$$U = \{(1,2), (1,3), (2,1), (2,4), (2,6), (3,1), (3,5), (4,2), (4,6), (4,5), (5,3), (5,4), (5,6), (6,2), (6,4), (6,5)\},$$

$$I^* = \{2, 4, 5, 6\}.$$

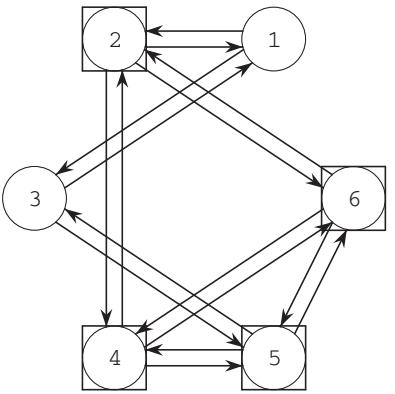

Fig. 3.11. Finite connected directed symmetric graph G

For the graph (see Figure 3.11) we consider the system of linear algebraic equations in the form (3.5.1).

$$\begin{gathered} x_{1,2} + x_{1,3} - x_{2,1} - x_{3,1} = 0 \\ x_{2,4} + x_{2,6} + x_{2,1} - x_{4,2} - x_{1,2} - x_{6,2} = x_2 \\ x_{3,1} + x_{3,5} - x_{1,3} - x_{5,3} = 0 \\ x_{4,2} + x_{4,5} + x_{4,6} - x_{2,4} - x_{5,4} - x_{6,4} = x_4 \\ x_{5,4} + x_{5,3} + x_{5,6} - x_{3,5} - x_{4,5} - x_{6,5} = x_5 \\ x_{6,2} + x_{6,4} + x_{6,5} - x_{2,6} - x_{4,6} - x_{5,6} = x_6 \end{gathered} \tag{3.5.1}$$

Suppose that the set of monitored nodes is $M = \{4\}$ for the graph G shown in Figure 3.11. We denote a cut of the graph G with respect to a given set M of nodes with $CC(M)$. We construct the cut $CC(M)$ for the graph G respect to the set $M = \{4\}$.

We form the sets:

$$\begin{gathered} M^+ = I(CC(M)) \setminus M = \{2,5,6\}; \\ M^* = M \bigcup M^+ = \{2,4,5,6\}; \\ I \setminus M^* = \{1,3\}. \end{gathered}$$

The values of arc flows on all incoming and outgoing arcs for the each node i of the set $M = \{4\}$ (monitored node) are known and also we know the values $x_i = f_i$, $i \in M \bigcap I^*$:

$$\begin{gathered} x_{4,2} = f_{4,2},\ x_{4,5} = f_{4,5}, \\ x_{4,6} = f_{4,6},\ x_{2,4} = f_{2,4}, \\ x_{5,4} = f_{5,4},\ x_{6,4} = f_{6,4},\ x_4 = f_4. \end{gathered} \tag{3.5.2}$$

We substitute the known values of the variables (3.5.2) to the system of equations (3.5.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flows are known according to (3.5.2). Also, delete from graph G the set of the nodes $i \in M$, $M = \{4\}$. We obtain a graph G' which is shown in Figure 3.12.

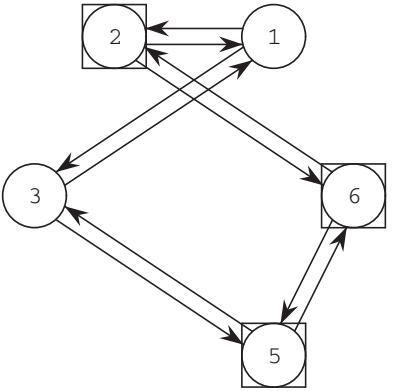

Fig. 3.12. Graph G'

The rest of the arcs flow for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{2,5,6\}$, can be expressed from the arcs flow of the outgoing arcs for M^+ by the following equalities:

$$x_{2,1} = \frac{p_{2,1}}{p_{2,4}} f_{2,4},$$

$$x_{2,6} = \frac{p_{2,6}}{p_{2,4}} f_{2,4};$$

$$x_{5,6} = \frac{p_{5,6}}{p_{5,4}} f_{5,4}, \quad x_{5,3} = \frac{p_{5,3}}{p_{5,4}} f_{5,4}; \tag{3.5.3}$$

$$x_{6,2} = \frac{p_{6,2}}{p_{6,4}} f_{6,4}; \quad x_{6,5} = \frac{p_{6,5}}{p_{6,4}} f_{6,4}.$$

Let's delete from graph G' (see Figure 3.12) the set of the arcs on which the arc flow are known according to (3.5.3).

In Figure 3.13 represented graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set of the arcs on which the arc flow are known according to (3.5.3) from graph G' (see Figure 3.12).

The system (3.5.1) for the graph $\overline{G}$ (see. Figure 3.13) transforms to the form (3.5.4).

$$
\begin{aligned}
x_{1,2} + x_{1,3} - \frac{p_{2,1}}{p_{2,4}} f_{2,4} - x_{3,1} &= 0 \\
\frac{p_{2,1}}{p_{2,4}} f_{2,4} + f_{2,4} + \frac{p_{2,6}}{p_{2,4}} f_{2,4} - x_{1,2} - f_{4,2} - \frac{p_{6,2}}{p_{6,4}} f_{6,4} &= x_2 \\
x_{3,1} + x_{3,5} - x_{1,3} - \frac{p_{5,3}}{p_{5,4}} f_{5,4} &= 0 \\
f_{4,2} + f_{4,5} + f_{4,6} - f_{2,4} - f_{5,4} - f_{6,4} &= f_4 \\
\frac{p_{5,3}}{p_{5,4}} f_{5,4} + f_{5,4} + \frac{p_{5,6}}{p_{5,4}} f_{5,4} - x_{3,5} - f_{4,5} - \frac{p_{6,5}}{p_{6,4}} f_{6,4} &= x_5 \\
\frac{p_{6,2}}{p_{6,4}} f_{6,4} + f_{6,4} + \frac{p_{6,5}}{p_{6,4}} f_{6,4} - \frac{p_{2,6}}{p_{2,4}} f_{2,4} - f_{4,6} - \frac{p_{5,6}}{p_{5,4}} f_{5,4} &= x_6
\end{aligned}
\tag{3.5.4}
$$

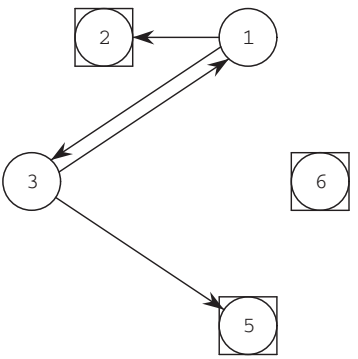

Fig. 3.13. Disconnected graph $\overline{G}$

Arc flows $x_{i,j}$,$(i,j) \in \overline{U}$ corresponding to the arcs outgoing from node set $I \setminus M^* = \{1,3\}$ are unknown. For these unknown flows $x_{i,j}$, $(i,j) \in \overline{U}$ we form the additional equations of the type (2.7.2).

- Choose arbitrary outgoing arc that starts from a node i of set $I \setminus M^*$, where $I \setminus M^* = \{1,3\}$ for example, for the node $i = 1$ we choose the

arc (1,2). Then, for all outgoing arcs from a node $i = 1$ except (1,2) the equality is true:

$$x_{1,3} = \frac{p_{1,3}}{p_{1,2}} x_{1,2}.$$

- Choose arbitrary outgoing arc that starts from a node $i = 3$, for example, (3,5). Then, for the arc (3,1) outgoing from the node $i = 3$ and the arc (3,5) the equality is true:

$$x_{3,1} = \frac{p_{3,1}}{p_{3,5}} x_{3,5}.$$

As a result, we obtained the expressions for unknown arc flows for the graph $\overline{G}$ (3.5.5):

$$\begin{aligned} x_{1,3} &= \frac{p_{1,3}}{p_{1,2}} x_{1,2}, \\ x_{3,1} &= \frac{p_{3,1}}{p_{3,5}} x_{3,5}. \end{aligned} \tag{3.5.5}$$

Thus, additional equations of type (2.7.2) for the graph $\overline{G}$ (see Figure 3.13) are as follows:

$$\begin{aligned} x_{1,3} - \frac{p_{1,3}}{p_{1,2}} x_{1,2} &= 0, \\ x_{3,1} - \frac{p_{3,1}}{p_{3,5}} x_{3,5} &= 0. \end{aligned} \tag{3.5.6}$$

Part of the unknowns of the system (3.5.4), (3.5.6) makes up outgoing arc flows for arcs from nodes of the set $I \setminus M^* = \{1,3\}$:

$$x_{12},\ x_{13},\ x_{31},\ x_{35}.$$

The remaining part of the unknowns of the system (3.5.4), (3.5.6) defines the variables intensities x_i, $i \in \overline{I}^*$, $\overline{I}^* = \{2,5,6\}$:

$$x_2,\ x_5,\ x_6.$$

Number of unknowns of the system (3.5.4), (3.5.6) is equal to 7. Number of equations in the system (3.5.4), (3.5.6) is equal to 7. We find the rank

of the matrix of the system (3.5.4), (3.5.6). If the system (3.5.4), (3.5.6) is a system of full rank, then it has a unique solution.

We construct any support $R = \{U_R, I_R^*\}$ (see Figure 3.14) of the graph $\overline{G}$ for the system (3.5.4), where $U_R = \{(1,2),(1,3)\}$, $I_R^* = \{2,5,6\}$ [32, 53].

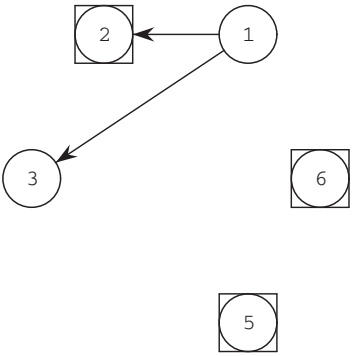

Fig. 3.14. Support R of the graph $\overline{G}$ for the system (3.5.4)

After the support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.5.4) is chosen (see Figure 3.14)), we determine what structures can be obtained after adding one no supporting element from $\overline{U} \backslash U_R$ or $\overline{I}^* \backslash I_R^*$ to the support R. We construct a system of characteristic vectors (basis of the solution space) of the homogeneous system generated by the system (3.5.4). The vectors $\delta(3,1)$, $\delta(3,5)$, entailed by arcs (3,1),(3,5) respectively constitute the system of characteristic vectors. We compute the characteristic vectors $\delta(3,1) = (\delta_{ij}^{3,1}, (i,j) \in \overline{U};\ \delta_i^{3,1}, i \in \overline{I}^*)$ and $\delta(3,5) = (\delta_{ij}^{3,5}, (i,j) \in \overline{U};\ \delta_i^{3,5}, i \in \overline{I}^*)$:

$$\delta(3,1) = (\delta_{31}^{31} \to 1, \delta_{12}^{31} \to 0, \delta_{35}^{31} \to 0, \delta_{13}^{31} \to 1,$$
$$\delta_2^{31} \to 0, \delta_5^{31} \to 0, \delta_6^{31} \to 0);$$

$$\delta(3,5) = (\delta_{35}^{35} \to 1, \delta_{12}^{35} \to -1, \delta_{13}^{35} \to 1, \delta_{31}^{35} \to 0,$$
$$\delta_2^{35} \to 1, \delta_5^{35} \to -1, \delta_6^{35} \to 0).$$

We enumerate the additional equations (3.5.6). For each additional equation with number p we compute the determinants $\Lambda_{\tau\rho}^p$,

$(\tau,\rho) \in \overline{U} \backslash U_R = \{(3,1),(3,5)\}, p = 1,2$:

$$\Lambda^p_{\tau\rho} = \sum_{(i,j)\in U_R} \lambda^p_{ij}\delta^{\tau\rho}_{ij} + \lambda^p_{\tau\rho}, p = \overline{1,q}.$$

We form the matrix D. Matrix D consists of determinants of the structures, entailed by no supporting elements:

$$D = \begin{pmatrix} \Lambda^1_{3,1} & \Lambda^1_{3,5} \\ \Lambda^2_{3,1} & \Lambda^2_{3,5} \end{pmatrix}.$$

So, matrix of determinants D takes the form:

$$D = \begin{pmatrix} 1 & 1 + \dfrac{p_{1,3}}{p_{1,2}} \\ 1 & -\dfrac{p_{3,1}}{p_{3,5}} \end{pmatrix}.$$

Since the following relations are true: $0 < p_{i,j} \leqslant 1$, $(i,j) \in U$, then the determinant of the matrix D is not equal to zero:

$$det\ D = -1 - \frac{p_{1,3}}{p_{1,2}} - \frac{p_{3,1}}{p_{3,5}} \neq 0.$$

Therefore, the system (3.5.4), (3.5.6) is system of full rank. The number of the unknown of system (3.5.4), (3.5.6) is equal to the rank of the matrix. Also, the number of the equations in the system (3.5.4), (3.5.6) is equal to the rank of the matrix. The system (3.5.4), (3.5.6) has the unique solution for given set $M = \{4\}$. The required unique solution of the system (3.5.4), (3.5.6) has the form:

$$x_{3,1} = \frac{p_{3,1}(f_{5,4}(p_{1,2} + p_{1,3})p_{2,4}p_{5,3} + f_{2,4}p_{1,3}p_{2,1}p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5} + p_{1,2}(p_{3,1} + p_{3,5}))p_{5,4}},$$

$$x_{3,5} = \frac{p_{3,5}(f_{5,4}(p_{1,2} + p_{1,3})p_{2,4}p_{5,3} + f_{2,4}p_{1,3}p_{2,1}p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5} + p_{1,2}(p_{3,1} + p_{3,5}))p_{5,4}},$$

$$x_{1,2} = \frac{p_{1,2}(f_{5,4}p_{2,4}p_{3,1}p_{5,3} + f_{2,4}p_{2,1}(p_{3,1} + p_{3,5})p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5} + p_{1,2}(p_{3,1} + p_{3,5}))p_{5,4}},$$

$$x_2 = f_{2,4} - f_{4,2} + \frac{f_{2,4}p_{2,6}}{p_{2,4}} - \frac{f_{5,4}p_{5,3}}{p_{5,4}} +$$

$$+\frac{p_{3,5}(f_{5,4}(p_{1,2}+p_{1,3})p_{2,4}p_{5,3}+f_{2,4}p_{1,3}p_{2,1}p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5}+p_{1,2}(p_{3,1}+p_{3,5}))p_{5,4}}-\frac{f_{6,4}p_{6,2}}{p_{6,4}},$$

$$x_{1,3}=\frac{p_{1,3}(f_{5,4}p_{2,4}p_{3,1}p_{5,3}+f_{2,4}p_{2,1}(p_{3,1}+p_{3,5})p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5}+p_{1,2}(p_{3,1}+p_{3,5}))p_{5,4}},$$

$$x_5=-f_{4,5}+f_{5,4}+\frac{f_{5,4}p_{5,3}}{p_{5,4}}-$$

$$-\frac{p_{3,5}(f_{5,4}(p_{1,2}+p_{1,3})p_{2,4}p_{5,3}+f_{2,4}p_{1,3}p_{2,1}p_{5,4})}{p_{2,4}(p_{1,3}p_{3,5}+p_{1,2}(p_{3,1}+p_{3,5}))p_{5,4}}+\frac{f_{5,4}p_{5,6}}{p_{5,4}}-\frac{f_{6,4}p_{6,5}}{p_{6,4}},$$

$$x_6=-f_{4,6}-\frac{f_{2,4}p_{2,6}}{p_{2,4}}-\frac{f_{5,4}p_{5,6}}{p_{5,4}}+\frac{f_{6,4}(p_{6,2}+p_{6,4}+p_{6,5})}{p_{6,4}}.$$

3.5.2. Numerical solution

We locate a single sensor into the node 4. The values of arc flows on all incoming and outgoing arcs for the each node i of the set $M=\{4\}$ (monitored node) are known and also we know the values $x_i=f_i$, where $i\in M\cap I^*$:

$$\begin{gathered} x_{4,2}=f_{4,2},\ x_{4,5}=f_{4,5},\\ x_{4,6}=f_{4,6},\ x_{2,4}=f_{2,4},\\ x_{5,4}=f_{5,4},\ x_{6,4}=f_{6,4},\ x_4=f_4. \end{gathered} \tag{3.5.7}$$

We substitute the known values of the variables

$$f_{4,2}=7,\ f_{4,5}=3,\ f_{4,6}=8,$$

$$f_{2,4}=6,\ f_{5,4}=5,\ f_{6,4}=9,\ f_4=-2$$

to the system of equations (3.5.1). Let's delete from graph $G=(I,U)$ the set of the arcs on which the arc flows are known:

(4,2), (4,5), (4,6), (2,4), (5,4), (6,4).

Also, we delete from graph G the set of the nodes $i\in M$, $M=\{4\}$. We obtain a graph G' which is shown in Figure 3.15.

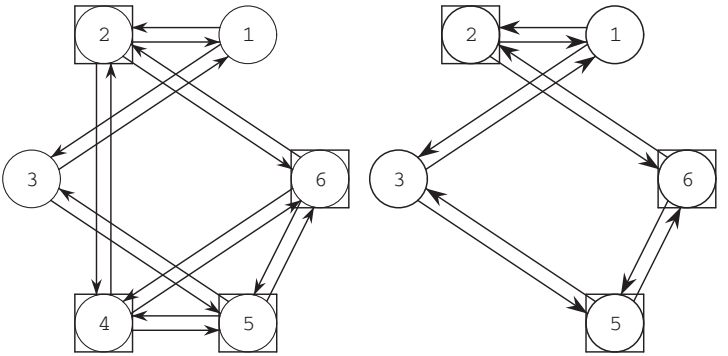

Fig. 3.15. Connected graphs G (on the left) and G' (on the right)

Table 3.5.1

Values of components of the vector $p = (p_{i,j}, (i,j) \in U)$

(i,j)	(1,2)	(1,3)	(2,1)	(2,4)	(2,6)	(3,1)	(3,5)	(4,2)
$p_{i,j}$	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{3}$
(i,j)	(4,5)	(4,6)	(5,3)	(5,4)	(5,6)	(6,2)	(6,4)	(6,5)
$p_{i,j}$	$\frac{7}{15}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{11}$	$\frac{7}{11}$	$\frac{1}{11}$

Numerical values of the components of vector $p = (p_{ij}), (i,j) \in U$ are provided in Table 3.5.1. The rest of the arcs flow for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{2,5,6\}$, can be expressed from the arcs flow of the outgoing arcs for M^+ by the following way:

$$x_{2,1} = \frac{p_{2,1}}{p_{2,4}} f_{2,4} = \frac{15}{2},$$

$$x_{2,6} = \frac{p_{2,6}}{p_{2,4}} f_{2,4} = \frac{9}{2};$$

$$x_{5,6} = \frac{p_{5,6}}{p_{5,4}} f_{5,4} = \frac{5}{2},$$

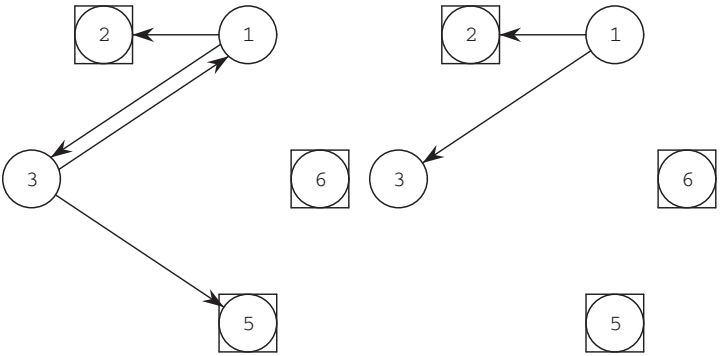

Fig. 3.16. Disconnected graph $\overline{G}$ (on the left) and support R of the graph $\overline{G}$ for the system (3.5.4) (on the right)

Table 3.5.2

Values of known arc flows and of variable intensity of nodes

(i,j)	(1,2)	(1,3)	(2,1)	(2,4)	(2,6)	(3,1)	(3,5)	(4,2)
$x_{i,j}$	$\frac{400}{57}$	$\frac{80}{19}$	$\frac{15}{2}$	6	$\frac{9}{2}$	$\frac{425}{114}$	$\frac{170}{57}$	7
(i,j)	(4,5)	(4,6)	(5,3)	(5,4)	(5,6)	(6,2)	(6,4)	(6,5)
$x_{i,j}$	3	8	$\frac{5}{2}$	5	$\frac{5}{2}$	$\frac{27}{7}$	9	$\frac{9}{7}$

$$x_{5,3} = \frac{p_{5,3}}{p_{5,4}} f_{5,4} = \frac{5}{2};$$

$$x_{6,2} = \frac{p_{6,2}}{p_{6,4}} f_{6,4} = \frac{27}{7};$$

$$x_{6,5} = \frac{p_{6,5}}{p_{6,4}} f_{6,4} = \frac{9}{7}.$$

Let's delete from graph G' (see Figure 3.12) the set of the arcs on which the arc flow are known according to (3.5.3). In Figure 3.16 represented graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set of the arcs on which the arc flow are known according to (3.5.3) from graph G' (see Figure 3.12). The system (3.5.1) for the graph $\overline{G}$ (see Figure 3.13) transforms to the form

Table 3.5.3

Values of variable intensities of nodes

i	2	4	5	6
x_i	$\frac{50}{399}$	-2	$\frac{1090}{399}$	$\frac{6}{7}$

(3.5.4). As a result of the localization of a single sensor to the node 4 in the graph G which is represented in Figure 3.15 we have a unique solution of system (3.5.1). The numerical values of arc flows of the unique solution of system (3.5.1) is presented in Table 3.5.2. The numerical values of variable intensities of the unique solution of system (3.5.1) for the nodes of set I^* is presented in Table 3.5.3.

In Examples 1–4, we locate in the graph G the single sensor to single node from set of nodes $\{1,2,3,4\} \subset I$. The result is that a single sensor should be installed to the node 2 or to node 4 to see all arc flows in the graph G, and also variable intensities of nodes from set I^*.

3.6. Example 5 (Underdetermined system)

For the finite connected directed symmetric graph $G = \{I,U\}$ where $I^* = \{5,6\}$ (see Figure 3.17) we consider the system of linear algebraic equations in the form (3.6.1).

$$\begin{gathered}
x_{1,2} + x_{1,4} - x_{2,1} - x_{4,1} = 0 \\
x_{2,1} + x_{2,3} - x_{1,2} - x_{3,2} = 0 \\
x_{3,2} + x_{3,4} - x_{2,3} - x_{4,3} = 0 \\
x_{4,1} + x_{4,3} + x_{4,5} + x_{4,6} - x_{1,4} - x_{3,4} - x_{5,4} - x_{6,4} = 0 \\
x_{5,4} - x_{4,5} = x_5 \\
x_{6,4} - x_{4,6} = x_6
\end{gathered} \tag{3.6.1}$$

Choose the set $M = \{1\}$. For the set of monitored nodes M we construct the cut $CC(M) = \{(1,2),(2,1),(1,4),(4,1)\}$ and the sets: $I(CC(M)) = \{1,2,4\}$, $M^+ = I(CC(M)) \setminus M = \{2,4\}$, $M^* = M \cup M^+ = \{1,2,4\}$, $I \setminus M^* = \{3,5,6\}$.

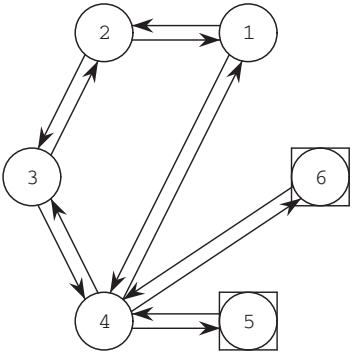

Fig. 3.17. Graph G

In the Sensor Location Problem we shall assume, that the values of flows on all incoming and outgoing arcs for the each node i of the set M (monitored nodes) are known and the values $x_i = f_i$, $i \in M \cap I^*$ are known.

$$
\begin{aligned}
x_{12} &= f_{12}, \\
x_{14} &= f_{14}, \\
x_{21} &= f_{21}, \\
x_{41} &= f_{41}.
\end{aligned}
\tag{3.6.2}
$$

We substitute the known values of the variables (3.6.2) to the system of equations (3.6.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flows are known according to (3.6.2). Also, delete from graph G the set of the nodes $i \in M$, $M = \{1\}$.

We obtain a graph G' which is shown in Figure 3.18.

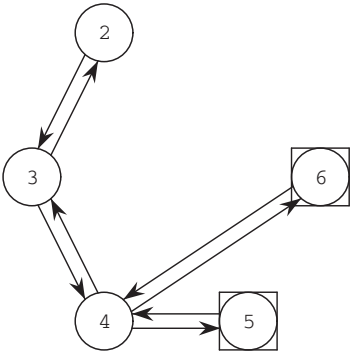

Fig. 3.18. Graph G'

The rest of the flows for the outgoing arcs from the nodes of the set M^+, where

$$M^+ = \{2,4\},$$

can be expressed from the flows of the outgoing arcs for M^+ by (3.6.3).

$$\begin{aligned} x_{23} &= \frac{p_{23}}{p_{21}} f_{21}, \\ x_{43} &= \frac{p_{43}}{p_{41}} f_{41}, \\ x_{45} &= \frac{p_{45}}{p_{41}} f_{41}, \\ x_{46} &= \frac{p_{46}}{p_{41}} f_{41} \end{aligned} \tag{3.6.3}$$

where f_{21}, f_{41} are constants. Let's delete from graph G' (see Figure 3.18) the set of the arcs on which the arc flows are known according to (3.6.3). In Figure 3.19 represented graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set

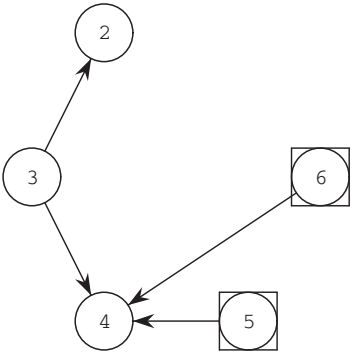

Fig. 3.19. Graph $\overline{G}$

of the arcs on which the arc flow are known according to (3.6.3) from graph G' (see Figure 3.18).

The system of the type (2.7.1) for the graph $\overline{G}$ will be following:

$$
\begin{gathered}
-x_{32} = a_2,\ a_2 = -f_{21} - \frac{p_{23}}{p_{21}} f_{21} + f_{12}; \\
x_{32} + x_{34} = a_3,\ a_3 = \frac{p_{23}}{p_{21}} f_{21} + \frac{p_{43}}{p_{41}} f_{41}; \\
-x_{34} - x_{54} - x_{64} = a_4, \\
a_4 = f_{14} - f_{41} - \frac{p_{43}}{p_{41}} f_{41} - \frac{p_{45}}{p_{41}} f_{41} - \frac{p_{46}}{p_{41}} f_{41}; \\
x_{54} = x_5 + b_5,\ b_5 = \frac{p_{45}}{p_{41}} f_{41}; \\
x_{64} = x_6 + b_6,\ b_6 = \frac{p_{46}}{p_{41}} f_{41}.
\end{gathered}
\tag{3.6.4}
$$

Arcs flows corresponding to the arcs outgoing from nodes of set $I \setminus M^*$ where $I \setminus M^* = \{3,5,6\}$ are unknown. For these unknown arc flows $x_{i,j}, (i,j) \in \overline{U}$ we form the additional equations of the type (2.7.2), if possible.

The additional equations of the type (2.7.2) will be following:

$$x_{32} - \frac{p_{32}}{p_{34}} x_{34} = 0. \tag{3.6.5}$$

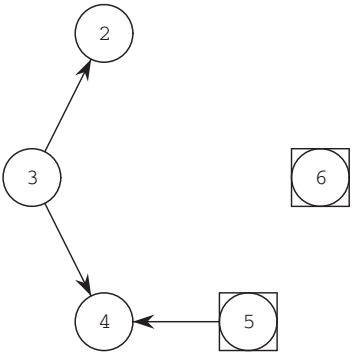

Fig. 3.20. Support R of the graph $\overline{G}$ for the system (3.6.4)

The unknown flows x_{32}, x_{34}, x_{54}, x_{64} for the system (3.6.4) – (3.6.5) determine the arcs outgoing from the nodes of the set $I \setminus M^* = \{3,5,6\}$ for the graph $\overline{G}$. Also the unknown variables of the system (3.6.4) – (3.6.5) determine variable intensities x_i, $i \in \overline{I}^*$. There are the following unknown variable intensities:

$$x_5, x_6.$$

Construct a support [53] $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.6.4), $U_R = \{(3,2),(3,4),(5,4)\}$, $I_R^* = \{5,6\}$ (see Figure 3.20).

Form matrix D. Matrix D consists of one element: $D = [\Lambda_{64}^1 = 0]$. The determinant of a matrix is equal to zero. Therefore, the rank of the matrix of the system (3.6.4) – (3.6.5) of the graph $\overline{G}$ is not full rank. The graph $\overline{G}$ is connected and contains the nodes from the set $\overline{I}^*$. The rank of the matrix of the system (3.6.4) – (3.6.5) of the graph $\overline{G}$ is equal to 5. The number of the unknowns of system (3.6.4) – (3.6.5) is equal to 6. As the rank of a matrix of system (3.6.4) – (3.6.5) is less the number of the unknown, then the system (3.6.4) – (3.6.5) is underdetermined. The system (3.6.4) – (3.6.5) has no unique solution for given set $M = \{1\}$.

In Example 5, to determine the fact that can not be localized in a single sensor node 1, we perform the decomposition of the system (3.6.4) – (3.6.5). In Examples 1 and 3 is no need to perform the decomposition of the corresponding system to determine the fact that can not be localized a sensor to the single node 1 or 3.

3.7. Example 6 (Unique solution)

We will build the analytical and numerical solutions sparse system of linear algebraic equations of the type (3.1.1) for the case when the sensor is installed in the node $M = \{6\}$ for the graph G shown in Figure 3.21.

3.7.1. Analytical solution

In Figure 3.21 represented by a finite connected directed symmetric graph $G = (I, U)$, where

$$\begin{gathered} I^* = \{2,3,6,7\}, \\ I = \{1,2,3,4,5,6,7,8\}, \\ U = \{(1,2),(1,5),(2,1),(2,6),(3,4), \\ (3,6),(4,3),(4,7),(4,8),(5,1), \\ (6,2),(6,3),(6,7),(7,4),(7,6),(8,4)\}. \end{gathered}$$

For the graph G (see Figure 3.21) we consider the system of linear algebraic equations of the type (3.5.1):

$$\begin{gathered} x_{1,2} + x_{1,5} - x_{2,1} - x_{5,1} = 0 \\ x_{2,1} + x_{2,6} - x_{1,2} - x_{6,2} = x_2 \\ x_{3,4} + x_{3,6} - x_{4,3} - x_{6,3} = x_3 \\ x_{4,3} + x_{4,7} + x_{4,8} - x_{3,4} - x_{7,4} - x_{8,4} = 0 \\ x_{5,1} - x_{1,5} = 0 \\ x_{6,2} + x_{6,3} + x_{6,7} - x_{2,6} - x_{3,6} - x_{7,6} = x_6 \\ x_{7,4} + x_{7,6} - x_{4,7} - x_{6,7} = x_7 \\ x_{8,4} - x_{4,8} = 0 \end{gathered} \tag{3.7.1}$$

Suppose that the set of monitored nodes is $M = \{6\}$ for the graph G shown in Figure 3.21. We denote a cut of the graph G with respect to a

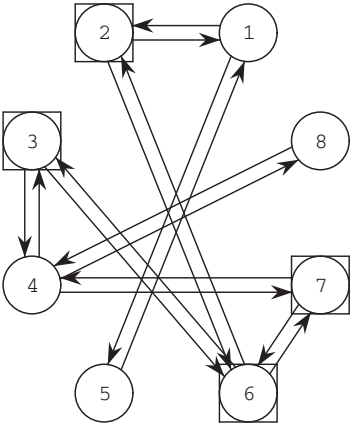

Fig. 3.21. Finite connected directed symmetric graph G

given set M of nodes with $CC(M)$. We construct the cut $CC(M)$ for the graph G respect to the set $M = \{6\}$.

We form the sets:

$$M^+ = I(CC(M)) \setminus M = \{2,3,7\};$$

$$M^* = M \cup M^+ = \{6\} \cup \{2,3,7\} = \{2,3,6,7\};$$

$$I \setminus M^* = \{1,4,5,8\}.$$

The values of arc flows on all incoming and outgoing arcs for the each node i of the set $M = \{6\}$ (monitored node) are known:

$$x_{6,2} = f_{6,2},$$

$$x_{6,3} = f_{6,3},$$

$$x_{6,7} = f_{6,7}, \quad x_{2,6} = f_{2,6},$$

$$x_{3,6} = f_{3,6}, \quad x_{7,6} = f_{7,6},$$

Also we know the values $x_i = f_i$, $i \in M \cap I^*$:

$$x_6 = f_6.$$

We have:

$$\begin{gathered} x_{6,2} = f_{6,2}, \quad x_{6,3} = f_{6,3}, \\ x_{6,7} = f_{6,7}, \quad x_{2,6} = f_{2,6}, \\ x_{3,6} = f_{3,6}, \quad x_{7,6} = f_{7,6}, \\ x_6 = f_6. \end{gathered} \tag{3.7.2}$$

Let us substitute (3.7.2) to the system of linear equations (3.7.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flow are known according to (3.7.2). Also, we delete from graph G the set of the nodes $i \in M$, $M = \{6\}$. We obtain a graph G' which is shown in Figure 3.22.

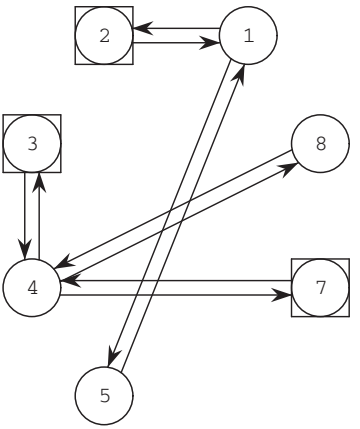

Fig. 3.22. Graph G'

The rest of the arcs flow for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{2,3,7\}$, can be expressed from the arcs flow of the outgoing arcs for M^+.

We obtain the following equalities:

$$
\begin{aligned}
x_{2,1} &= \frac{p_{2,1}}{p_{2,6}} x_{2,6}, \\
x_{2,6} &= f_{2,6}; \\
x_{3,4} &= \frac{p_{3,4}}{p_{3,6}} x_{3,6}, \\
x_{3,6} &= f_{3,6}; \\
x_{7,4} &= \frac{p_{7,4}}{p_{7,6}} x_{7,6}, \\
x_{7,6} &= f_{7,6}.
\end{aligned}
\tag{3.7.3}
$$

In Figure 3.23 represented graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set of the arcs on which the arc flow are known according to (3.7.3) from graph G' (see Figure 3.22). The system (3.7.1) for the graph $\overline{G}$ (see. Figure 3.23) transforms to the form (3.7.4).

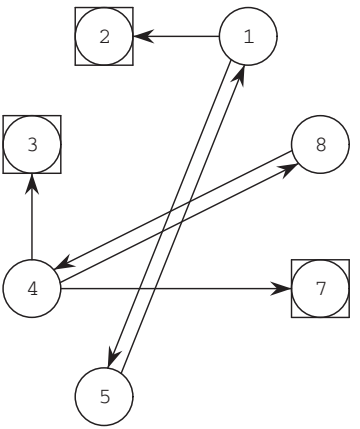

Fig. 3.23. Graph $\overline{G}$

$$
\begin{gathered}
x_{1,2} + x_{1,5} - \frac{p_{2,1}}{p_{2,6}} f_{2,6} - x_{5,1} = 0 \\
\frac{p_{2,1}}{p_{2,6}} f_{2,6} + f_{2,6} - x_{1,2} - f_{6,2} = x_2 \\
\frac{p_{3,4}}{p_{3,6}} f_{3,6} + f_{3,6} - x_{4,3} - f_{6,3} = x_3 \\
x_{4,3} + x_{4,7} + x_{4,8} - \frac{p_{3,4}}{p_{3,6}} f_{3,6} - \frac{p_{7,4}}{p_{7,6}} f_{7,6} - x_{8,4} = 0 \\
x_{5,1} - x_{1,5} = 0 \\
f_{6,2} + f_{6,3} + f_{6,7} - f_{2,6} - f_{3,6} - f_{7,6} = f_6 \\
\frac{p_{7,4}}{p_{7,6}} f_{7,6} + f_{7,6} - x_{4,7} - f_{6,7} = x_7 \\
x_{8,4} - x_{4,8} = 0
\end{gathered}
\tag{3.7.4}
$$

Represent the system (3.7.4) in the form (3.7.5).

$$
\begin{gathered}
x_{1,2} + x_{1,5} - x_{5,1} = b_1 \\
-x_{1,2} = x_2 + b_2 \\
-x_{4,3} = x_3 + b_3 \\
x_{4,3} + x_{4,7} + x_{4,8} - x_{8,4} = b_4 \\
x_{5,1} - x_{1,5} = 0 \\
-x_{4,7} = x_7 + b_7 \\
x_{8,4} - x_{4,8} = 0
\end{gathered}
\tag{3.7.5}
$$

where

$$
b_1 = \frac{p_{2,1}}{p_{2,6}} f_{2,6},\ b_2 = -\frac{p_{2,1}}{p_{2,6}} f_{2,6} - f_{2,6} + f_{6,2},
$$

$$
b_3 = -\frac{p_{3,4}}{p_{3,6}} f_{3,6} - f_{3,6} + f_{6,3},\ b_4 = \frac{p_{3,4}}{p_{3,6}} f_{36} + \frac{p_{7,4}}{p_{7,6}} f_{7,6},
$$

$$
b_5 = 0,\ b_7 = -\frac{p_{7,4}}{p_{7,6}} f_{7,6} - f_{7,6} + f_{6,7},\ b_8 = 0.
$$

Arc flows $x_{i,j}$,$(i,j) \in \overline{U}$ corresponding to the arcs outgoing from node set $I \setminus M^* = \{1,4,5,8\}$ are unknown. For these unknown flows $x_{i,j}$ we form the additional equations of the type (2.7.2), $(i,j) \in \overline{U}$.

- Choose arbitrary outgoing arc that starts from a node i of set $I \setminus M^*$ where $I \setminus M^* = \{1,4,5,8\}$, if it is possible, for example, for the node $i = 1$ we choose the arc (1,2). Then, for all outgoing arcs from a node $i = 1$ except (1,2) the equality is true:

$$x_{1,5} = \frac{p_{1,5}}{p_{1,2}} x_{1,2}. \tag{3.7.6}$$

- Choose arbitrary outgoing arc that starts from a node $i = 4$ of set $I \setminus M^* = \{1,4,5,8\}$, for example, (4,8). Then, for the arc (4,3) outgoing from the node $i = 4$ and the arc (4,7) the equalities are true:

$$x_{4,3} = \frac{p_{4,3}}{p_{4,8}} x_{4,8}, \quad x_{4,7} = \frac{p_{4,7}}{p_{4,8}} x_{4,8}. \tag{3.7.7}$$

As a result, we obtained the expressions (3.7.8) for unknown arc flows for the graph $\overline{G}$:

$$\begin{aligned} x_{1,5} - \frac{p_{1,5}}{p_{1,2}} x_{1,2} &= 0, \\ x_{4,3} - \frac{p_{4,3}}{p_{4,8}} x_{4,8} &= 0, \\ x_{4,7} - \frac{p_{4,7}}{p_{4,8}} x_{4,8} &= 0. \end{aligned} \tag{3.7.8}$$

Note that for the nodes $\{5,8\} \subset I \setminus M^*$ impossible to generate additional equations of type (2.7.2), because for each of these nodes has only one outgoing arc.

Join (3.7.6) and (3.7.7). Thus, the additional equations of type (2.7.2) for the graph $\overline{G}$ (see Figure 3.23) are (3.7.8).

Part of the unknowns of the system (3.7.5), (3.7.8) makes up outgoing arc flows for arcs from nodes of the set $I \setminus I^* = \{1,4,5,8\}$:

$$x_{1,2},\ x_{1,5},\ x_{4,3},\ x_{4,7},\ x_{4,8},\ x_{5,1},\ x_{8,4}.$$

The remaining part of the unknowns of the system (3.7.5), (3.7.8) defines the variables intensities x_i, $i \in \overline{I}^*$, where $\overline{I}^* = \{2,3,7\} : x_2, x_3, x_7$.

New graph $\overline{G} = (\overline{I}, \overline{U})$ (see Figure 3.23) where $\overline{I} = \{1,2,3,4,5,7,8\}$, $\overline{U} = \{(1,2),(1,5),(4,3),(4,7),(4,8),(5,1),(8,4)\}$, $\overline{I}^* = \{2,3,7\}$ consists of two connected components: $\overline{G}_m = (\overline{I}_m, \overline{U}_m)$, $m = 1,2$.

The connected component $\overline{G}_1 = (\overline{I}_1, \overline{U}_1)$ consists of nodes $\overline{I}_1 = \{1,2,5\}$, of arcs $\overline{U}_1 = \{(1,2),(1,5),(5,1)\}$ and nodes with variable intensities $\overline{I}_1^* = \{2\}$. The connected component $\overline{G}_2 = (\overline{I}_2, \overline{U}_2)$ consists of nodes $\overline{I}_2 = \{3,4,7,8\}$, of arcs $\overline{U}_2 = \{(4,3),(4,7),(4,8),(8,4)\}$ and nodes with variable intensities $\overline{I}_2^*$ where $\overline{I}_2^* = \{3,7\}$.

Choose the support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.7.5) (see Figure 3.24): $U_R = \{(1,2),(1,5),(4,3),(4,8)\}$, $I_R^* = \{2,3,7\}$.

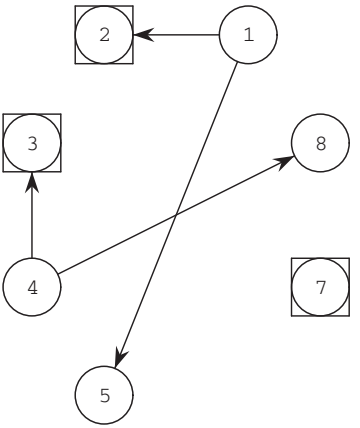

Fig. 3.24. Support of the graph $\overline{G}$ for the system (3.7.5)

After the support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.7.5) is chosen (see Figure (3.24)), we determine what structures can be obtained after adding one no supporting element from $\overline{U} \backslash U_R$ or $\overline{I}^* \backslash I_R^*$ to the support R. We construct a system of characteristic vectors (basis of the solution space) of the homogeneous system generated by the system (3.7.5). The vectors $\delta(4,7), \delta(5,1), \delta(8,4)$ entailed by arcs $(4,7),(5,1),(8,4)$ respectively constitute the system of characteristic vectors. We compute the characteristic vectors $\delta(4,7), \delta(5,1), \delta(8,4)$. We compute the characteristic vector $\delta(4,7) = (\delta_{i,j}^{4,7}, (i,j) \in \overline{U};\ \delta_i^{4,7}, i \in \overline{I}^*)$ entailed by arc $(4,7)$:

$$\delta(4,7) = (\delta_{1,2}^{4,7} \to 0, \delta_{1,5}^{4,7} \to 0, \delta_{4,3}^{4,7} \to -1, \delta_{4,7}^{4,7} \to 1,$$

$$\delta_{4,8}^{4,7} \to 0, \delta_{8,4}^{4,7} \to 0, \delta_{5,1}^{4,7} \to 0, \delta_{2}^{4,7} \to 0, \delta_{3}^{4,7} \to 1, \delta_{7}^{4,7} \to -1).$$

We compute the characteristic vector $\delta(5,1)$ entailed by arc $(5,1)$:

$$\delta(5,1) = (\delta_{1,2}^{5,1} \to 0, \delta_{1,5}^{5,1} \to 1, \delta_{4,3}^{5,1} \to 0, \delta_{4,7}^{5,1} \to 0,$$

$$\delta_{4,8}^{5,1} \to 0, \delta_{8,4}^{5,1} \to 0, \delta_{5,1}^{5,1} \to 1, \delta_{2}^{5,1} \to 0, \delta_{3}^{4,7} \to 0, \delta_{7}^{4,7} \to 0).$$

We compute the characteristic vector $\delta(8,4)$ entailed by arc $(8,4)$:

$$\delta(8,4) = (\delta_{1,2}^{8,4} \to 0, \delta_{1,5}^{8,4} \to 0, \delta_{4,3}^{8,4} \to 0, \delta_{4,7}^{8,4} \to 0,$$
$$\delta_{4,8}^{8,4} \to 1, \delta_{8,4}^{8,4} \to 1, \delta_{5,1}^{8,4} \to 0, \delta_{2}^{8,4} \to 0, \delta_{3}^{8,4} \to 0, \delta_{7}^{8,4} \to 0).$$

We enumerate the additional equations (3.7.8). For each additional equation with number p we compute the determinants $\Lambda_{\tau\rho}^{p}$, $(\tau,\rho) \in \overline{U} \backslash U_R = \{(4,7),(5,1),(8,4)\}, p = 1, 2, 3$:

$$\Lambda_{\tau\rho}^{p} = \sum_{(i,j)\in U_R} \lambda_{ij}^{p} \delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^{p}, \; p = \overline{1,q}.$$

We form the matrix D. Matrix D consists of determinants of the structures, entailed by no supporting elements:

$$\Lambda_{4,7}^{1} = 0, \Lambda_{5,1}^{1} = 1, \Lambda_{8,4}^{1} = 0,$$
$$\Lambda_{4,7}^{2} = -1, \Lambda_{5,1}^{2} = 0, \Lambda_{8,4}^{2} = -\frac{p_{4,3}}{p_{4,8}},$$
$$\Lambda_{4,7}^{3} = 1, \Lambda_{5,1}^{3} = 0, \Lambda_{8,4}^{3} = -\frac{p_{4,7}}{p_{4,8}}.$$

We form the set $W = \{U_W, I_W^*\}, |W| = q, \; U_W \subseteq \overline{U} \setminus U_R, \; I_W^* \subseteq \overline{I}^* \setminus I_R^*$. As $I_W^* = \emptyset$, then $W = U_W, \; U_W = \{(4,7),(5,1),(8,4)\}\}$. We enumerate the arcs of set U_W, where $t = t(\tau,\rho)$ – the number of arc $(\tau,\rho) \in U_W$ in the entered numbering:

$$t(4,7) = 1, \; t(5,1) = 2, \; t(8,4) = 3.$$

We form the matrix D:

$$D = \begin{pmatrix} \Lambda_{4,7}^{1} & \Lambda_{5,1}^{1} & \Lambda_{8,4}^{1} \\ \Lambda_{4,7}^{2} & \Lambda_{5,1}^{2} & \Lambda_{8,4}^{2} \\ \Lambda_{4,7}^{3} & \Lambda_{5,1}^{3} & \Lambda_{8,4}^{3} \end{pmatrix}.$$

So, the matrix D is:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\dfrac{p_{4,3}}{p_{4,8}} \\ 1 & 0 & -\dfrac{p_{4,7}}{p_{4,8}} \end{pmatrix}.$$

Since the following relations are true: $0 < p_{i,j} \leqslant 1$, $(i,j) \in U$, then the determinant of the matrix D is not equal to zero:

$$\det D = -\frac{p_{4,3}}{p_{4,8}} - \frac{p_{4,7}}{p_{4,8}} \neq 0.$$

Therefore, the system (3.7.5), (3.7.8) is system of full rank. The number of the unknown of system (3.7.5), (3.7.8) is equal to the rank of the matrix of system. Also, the number of the equations in the system (3.7.5), (3.7.8) is equal to the rank of the matrix. The system (3.7.5), (3.7.8) has the unique solution for given set $M = \{6\}$.

We use the theory of decomposition to build the unique solution of the system (3.7.5), (3.7.8) [53]. Apply the decomposition algorithms for solution of the system (3.7.5), (3.7.8). To construct a general solution of system (3.7.5) need build any particular solution $\tilde{x} = (\tilde{x}_{i,j}, (i,j) \in \overline{U}, \tilde{x}_i, i \in \overline{I}^*)$ of system (3.7.5). A particular solution $\tilde{x}$ of the system (3.7.5) construct in according with the rules of remark 2.4.1. Non-supporting components of a particular solution $\tilde{x}$ are equal to zeroes $\tilde{x}_{\tau,\rho} = 0$, $(\tau,\rho) \in \overline{U} \backslash U_R$, $\tilde{x}_\gamma = 0$, $\gamma \in \overline{I}^* \backslash I_R^*$: $\tilde{x}_{4,7} = 0$, $\tilde{x}_{5,1} = 0$, $\tilde{x}_{8,4} = 0$. Using the graph-theoretic properties of the support of the graph $\overline{G}$ for the system (3.7.5) support components of a particular solution of system (3.7.5) we find in the time $O(n)$, $n = |\overline{I}|$. Support components of particular solution $\tilde{x}$ are:

$$\tilde{x}_{1,2} = \frac{p_{2,1}}{p_{2,6}} f_{2,6}, \quad \tilde{x}_2 = f_{2,6} - f_{6,2},$$

$$\tilde{x}_{4,3} = \frac{p_{3,4}}{p_{3,6}} f_{3,6} + \frac{p_{7,4}}{p_{7,6}} f_{7,6},$$

$$\tilde{x}_3 = f_{3,6} - f_{6,3} - \frac{p_{7,4}}{p_{7,6}} f_{76}, \quad \tilde{x}_{1,5} = 0,$$

$$\tilde{x}_7 = \frac{p_{7,4}}{p_{7,6}} f_{7,6} + f_{7,6} - f_{6,7}, \quad \tilde{x}_{4,8} = 0.$$

Using (2.8.4), we compute the numbers $A^1,\ A^2,\ A^3$:

$$A^1 = \frac{f_{2,6}\, p_{1,5} p_{2,1}}{p_{1,2}\, p_{2,6}},$$

$$A^2 = -\frac{p_{3,4}}{p_{3,6}} f_{3,6} - \frac{p_{7,4}}{p_{7,6}} f_{7,6},$$

$$A^3 = 0.$$

Since the matrix D is nonsingular, then we apply the formula (2.8.9) to find the components of solution $x_W = (x_{\tau,\rho,(\tau,\rho)} \in U_W) = (x_{4,7},\ x_{5,1},\ x_{8,4})$ from the system in the form:

$$x_{5,1} = A^1,$$

$$-x_{4,7} - \frac{p_{4,3}}{p_{4,8}} x_{8,4} = A^2, \tag{3.7.9}$$

$$x_{4,7} - \frac{p_{4,7}}{p_{4,8}} x_{8,4} = A^3.$$

The components of the vector β — the right-hand side of the system of type (2.8.7) for the considered example are:

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

We calculate the components $x_W = (x_{4,7},\, x_{5,1},\, x_{8,4})$ of vector x_W from system (3.7.9):

$$x_{5,1} = \frac{f_{2,6}\, p_{1,5}\, p_{2,1}}{p_{1,2}\, p_{2,6}},$$

$$x_{4,7} = \frac{p_{4,7}\,(f_{7,6}\, p_{3,6}\, p_{7,4} + f_{3,6}\, p_{3,4}\, p_{7,6})}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}}, \tag{3.7.10}$$

$$x_{8,4} = \frac{p_{4,8}\,(f_{7,6}\, p_{3,6}\, p_{7,4} + f_{3,6}\, p_{3,4}\, p_{7,6})}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}}.$$

The remaining components of the desired unique solution of system (3.7.5), (3.7.8) we found from formulas (2.4.1) – (2.4.2), using network properties of support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for system (3.7.5). So, we have:

$$x_{1,2} = \frac{f_{2,6}\, p_{2,1}}{p_{2,6}},$$

$$x_2 = f_{2,6} - f_{6,2},$$

$$x_3 = f_{3,6} - f_{6,3} - \frac{f_{7,6}\, p_{7,4}}{p_{7,6}} + \frac{p_{4,7}\,(f_{7,6}\, p_{3,6}\, p_{7,4} + f_{3,6}\, p_{3,4}\, p_{7,6})}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}},$$

$$x_{4,3} = \frac{p_{4,3}\,(f_{7,6}\, p_{3,6}\, p_{7,4} + f_{3,6}\, p_{3,4}\, p_{7,6})}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}},$$

$$x_{1,5} = \frac{f_{2,6}\, p_{1,5}\, p_{2,1}}{p_{1,2}\, p_{2,6}},$$

$$x_7 = -f_{6,7} + \frac{-f_{3,6}\, p_{3,4}\, p_{4,7}\, p_{7,6} + f_{7,6}\, p_{3,6}(\, p_{4,7}\, p_{7,6} + p_{4,3}(p_{7,4} + p_{7,6}))}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}},$$

$$x_{4,8} = \frac{p_{4,8}\,(f_{7,6}\, p_{3,6}\, p_{7,4} + f_{3,6}\, p_{3,4}\, p_{7,6})}{p_{3,6}\,(p_{4,3} + p_{4,7})\, p_{7,6}}.$$

3.7.2. Numerical solution

We locate a single sensor to the node 6. The values of arc flows on all incoming and outgoing arcs for the each node i of the set $M = \{6\}$ (monitored node) are known and also we know the values of variable intensities $x_i = f_i, i \in M \cap I^*$.

We substitute the known values of the variables:

$$f_{2,6} = 8,\ f_{3,6} = 3,\ f_{6,3} = 3,$$

$$f_{6,2} = 6,\ f_{6,7} = 5,\ f_{7,6} = 2,$$

$$f_6 = 1$$

to the system of equations (3.7.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flows are known:

(2,6), (3,6), (6,3), (6,2), (6,7), (7,6).

Also, we delete from graph G the set of the nodes $i \in M$, $M = \{6\}$. We obtain a graph G' which is shown in Figure 3.25.

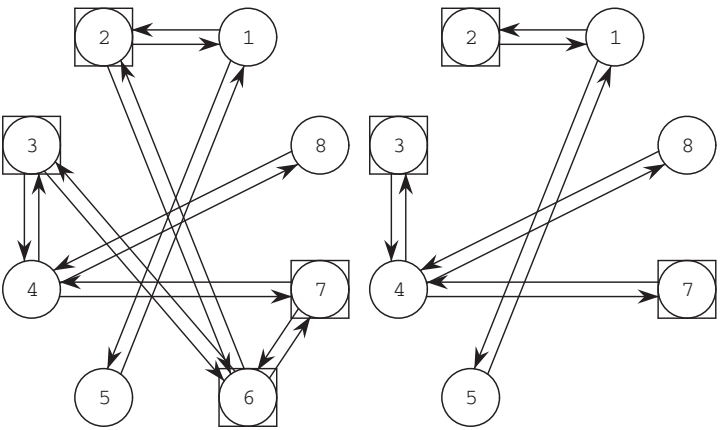

Fig. 3.25. Graph G (on the left) and disconnected graph G' (on the right)

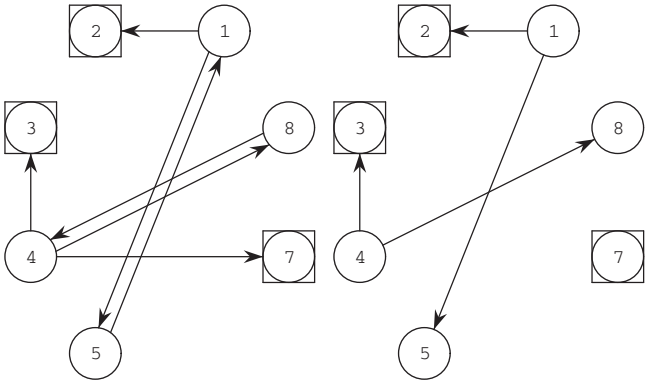

Fig. 3.26. Disconnected graph $\overline{G}$ (on the left) and support R of the graph $\overline{G}$ for the system (3.7.5) (on the right)

The rest of the arc flows for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{2,3,7\}$, can be expressed from the arc flows of the outgoing arcs for M^+ by the following way:

$$\begin{aligned} x_{2,1} &= \frac{p_{2,1}}{p_{2,6}} f_{2,6} = 8, \\ x_{3,4} &= \frac{p_{3,4}}{p_{3,6}} f_{3,6} = 9, \\ x_{7,4} &= \frac{p_{7,4}}{p_{7,6}} f_{7,6} = 1. \end{aligned} \tag{3.7.11}$$

Let's delete from graph G' (see Figure 3.25) the set of the arcs on which the arc flows are known according to (3.7.11).

In Figure 3.26 we represent graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set of the arcs on which the arc flows are known according to (3.7.11) from graph G' (see Figure 3.26).

The numerical values $p_{i,j}$, $(i,j) \in U$ are presented in Table 3.7.1.

Values of variable intensities of nodes from set $I^* = \{2,3,6,7\}$ are:

$$x_2 = 2, \quad x_3 = 1, \quad x_6 = 1, \quad x_7 = -4.$$

Table 3.7.1

Values of components of the vector $p = (p_{i,j},(i,j) \in U)$

(i,j)	(1,2)	(1,5)	(2,1)	(2,6)	(3,4)	(3,6)	(4,3)	(4,7)
$p_{i,j}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{5}{13}$	$\frac{4}{13}$
(i,j)	(4,8)	(5,1)	(6,2)	(6,3)	(6,7)	(7,4)	(7,6)	(8,4)
$p_{i,j}$	$\frac{4}{13}$	1	$\frac{5}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{3}$	$\frac{2}{3}$	1

Table 3.7.2

Values of known arc flows $x = (x_{i,j},(i,j) \in U)$

(i,j)	(1,2)	(1,5)	(2,1)	(2,6)	(3,4)	(3,6)	(4,3)	(4,7)
$x_{i,j}$	8	16	8	8	9	3	$\frac{50}{9}$	$\frac{40}{9}$
(i,j)	(4,8)	(5,1)	(6,2)	(6,3)	(6,7)	(7,4)	(7,6)	(8,4)
$x_{i,j}$	$\frac{40}{9}$	16	6	3	5	1	2	$\frac{40}{9}$

As a result of the localization of a single sensor to the node 6 in the graph G which is represented in Figure 3.25 we have a unique solution of system (3.7.1).

The arc flows of the unique solution of system (3.7.1) is presented in Table 3.7.2.

The numerical values of variable intensities of the unique solution of system (3.7.1) for the nodes of set I^* are:

$$x_2 = 2,$$

$$x_3 = 1,$$

$$x_6 = 1, \ x_7 = -4.$$

3.8. Example of multiple monitored nodes

For the graph $G = (I, U)$ where $I^* = \{2, 4, 5, 6, 7, 9\}$ (see Figure 3.27) we consider the system of linear algebraic equations of the type (3.1.1):

$$\begin{gathered}
x_{1,8} - x_{8,1} = 0 \\
x_{2,4} - x_{4,2} = x_2 \\
x_{3,4} + x_{3,8} - x_{4,3} - x_{8,3} = 0 \\
x_{4,2} + x_{4,3} + x_{4,5} + x_{4,7} + x_{4,8} - x_{2,4} - \\
-x_{3,4} - x_{5,4} - x_{7,4} - x_{8,4} = x_4 \\
x_{5,4} - x_{4,5} = x_5 \\
x_{6,8} + x_{6,9} - x_{8,6} - x_{9,6} = x_6 \\
x_{7,4} - x_{4,7} = x_7 \\
x_{8,1} + x_{8,3} + x_{8,4} + x_{8,6} - x_{1,8} - \\
-x_{3,8} - x_{4,8} - x_{6,8} = 0 \\
x_{9,6} - x_{6,9} = x_9
\end{gathered} \tag{3.8.1}$$

We consider the Sensor Location Problem for the graph (see Figure 3.27) for the case of multiple monitored nodes $M = \{2,5,7,9\}$. We construct the cut $CC(M)$ with respect to the set of nodes M (see Figure 3.28). We show that for the set $M = \{2,5,7,9\}$ of the monitored nodes the system (3.8.1) has a unique solution. Construct the sets:

$$CC(M) = \{(2,4), (4,2), (4,5), (5,4), (7,4), (4,7), (6,9), (9,6)\};$$

$$I(CC(M)) = \{2, 4, 5, 6, 7, 9\};$$

$$M^+ = I(CC(M)) \setminus M = \{4,6\};$$

$$M^* = M \bigcup M^+ = \{2,5,7,9,4,6\};$$

$$I \setminus M^* = \{1,3,8\}.$$

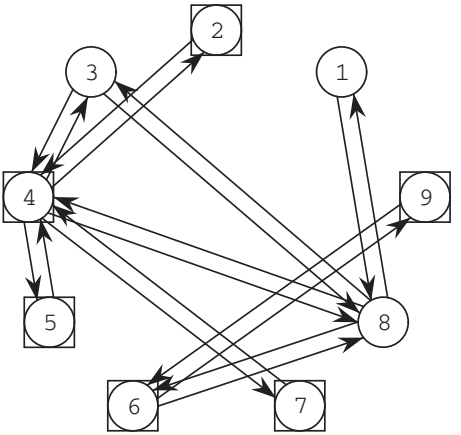

Fig. 3.27. Finite connected directed symmetric graph G

In the Sensor Location Problem the values of flows on all incoming and outgoing arcs for the each node i of the set M (monitored nodes) are known and also we know the values $x_i = f_i$, $i \in M \cap I^*$:

$$\begin{gathered} x_{2,4} = f_{2,4},\ x_{4,2} = f_{4,2}, \\ x_{4,5} = f_{4,5},\ x_{4,7} = f_{4,7}, \\ x_{5,4} = f_{5,4},\ x_{6,9} = f_{6,9}, \\ x_{7,4} = f_{7,4},\ x_{9,6} = f_{9,6}, \\ x_2 = f_2,\ x_5 = f_5, \\ x_7 = f_7,\ x_9 = f_9. \end{gathered} \tag{3.8.2}$$

We substitute the known values of the variables (3.8.2) to the system of linear equations (3.8.1). Let's delete from graph $G = (I, U)$ the set of the arcs on which the arc flow are known according to (3.8.2). Also, we delete from graph G the set of the nodes $i \in M = \{2,5,7,9\}$. We obtain a graph G' which is shown in Figure 3.29.

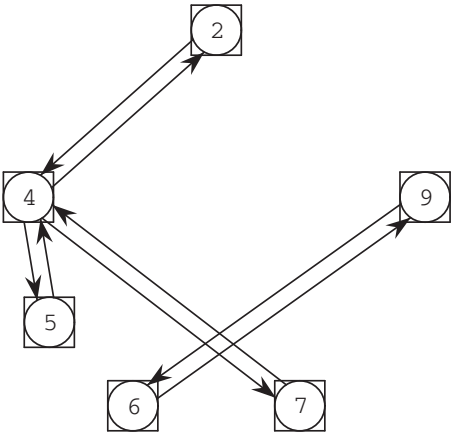

Fig. 3.28. Cut $CC(M)$ of the graph G

Values of arc flows for outgoing arcs from nodes of the set M^+ can be expressed through known arc flows for outgoing arcs from the nodes of the set M^+, $M^+ = \{4,6\}$ by the following equations:

$$x_{4,3} = \frac{p_{4,3}}{p_{4,2}} f_{4,2};$$

$$x_{4,8} = \frac{p_{4,8}}{p_{4,2}} f_{4,2}; \tag{3.8.3}$$

$$x_{6,8} = \frac{p_{6,8}}{p_{6,9}} f_{6,9}.$$

Let us substitute (3.8.3) to the system of linear algebraic equations (3.8.1).

Let's delete from graph G' (see Figure 3.29) the set of the arcs on which the arc flows are known according to (3.8.3).

In Figure 3.30 we show graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting from graph G' the set of the arcs on which the arc flows are known according to (3.8.3).

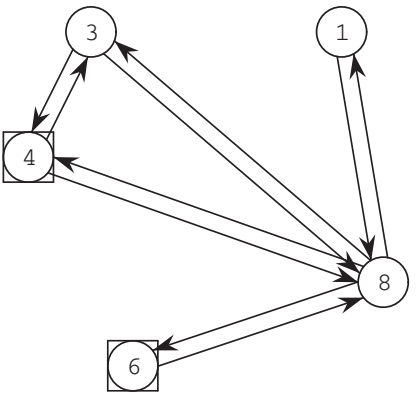

Fig. 3.29. Graph G'

The system (3.8.1) for the graph $\overline{G}$ (see Figure 3.30) transforms to the form (3.8.4).

$$
\begin{gathered}
x_{1,8} - x_{8,1} = 0, \quad f_{2,4} - f_{4,2} = f_2, \\
x_{3,4} + x_{3,8} - \frac{p_{4,3}}{p_{4,2}} f_{4,2} - x_{8,3} = 0, \\
f_{4,2} + \frac{p_{4,3}}{p_{4,2}} f_{4,2} + f_{4,5} + f_{4,7} + \frac{p_{4,8}}{p_{4,2}} f_{4,2} - \\
-f_{2,4} - x_{3,4} - f_{5,4} - f_{7,4} - x_{8,4} = x_4, \, f_{5,4} - f_{4,5} = f_5, \\
\frac{p_{6,8}}{p_{6,9}} f_{6,9} + f_{6,9} - x_{8,6} - f_{9,6} = x_6, \\
f_{7,4} - f_{4,7} = f_7, \\
x_{8,1} + x_{8,3} + x_{8,4} + x_{8,6} - \\
-x_{1,8} - x_{3,8} - \frac{p_{4,8}}{p_{4,2}} f_{4,2} - \frac{p_{6,8}}{p_{6,9}} f_{6,9} = 0, \quad f_{9,6} - f_{6,9} = f_9.
\end{gathered}
\tag{3.8.4}
$$

We represent the system (3.8.4) as (3.8.5).

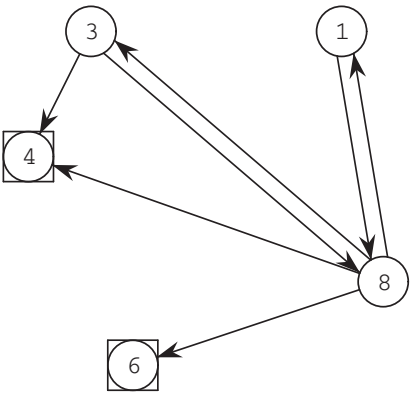

Fig. 3.30. Graph $\overline{G}$

$$
\begin{aligned}
x_{1,8} - x_{8,1} &= 0,\\
x_{3,4} + x_{3,8} - x_{8,3} &= b_3,\\
-x_{3,4} - x_{8,4} &= x_4 + b_4,\\
-x_{8,6} &= x_6 + b_6,\\
x_{8,1} + x_{8,3} + x_{8,4} + x_{8,6} - x_{1,8} - x_{3,8} &= b_8,
\end{aligned}
\tag{3.8.5}
$$

where

$$b_1 = 0,\ \ b_3 = \frac{p_{4,3}}{p_{4,2}} f_{4,2},$$

$$b_4 = -f_{4,2} - \frac{p_{4,3}}{p_{4,2}} f_{4,2} - f_{4,5} - f_{4,7} - \frac{p_{4,8}}{p_{4,2}} f_{4,2} + f_{2,4} + f_{5,4} + f_{7,4},$$

$$b_6 = -\frac{p_{6,8}}{p_{6,9}} f_{6,9} - f_{6,9} + f_{9,6},$$

$$b_8 = \frac{p_{4,8}}{p_{4,2}} f_{4,2} + \frac{p_{6,8}}{p_{6,9}} f_{6,9}.$$

Arc flows $x_{i,j}$,$(i,j) \in \overline{U}$, corresponding to the arcs outgoing from node set $I \setminus M^* = \{1,3,8\}$ are unknown. For these unknown flows $x_{i,j}$, $(i,j) \in \overline{U}$ we form the additional equations of the type (2.7.2) as follows:

- Choose arbitrary outgoing arc that starts from a node i of set $I \setminus M^*$ where $I \setminus M^* = \{1,3,8\}$. For example, for the node $i = 3$ we choose the arc (3,4). Then we can express the arc flows to all outgoing arcs from node $i = 3$, except for the arc (3,4), through the arc flow $x_{3,4}$ of selected outgoing arc (3,4).

$$x_{3,8} = \frac{p_{3,8}}{p_{3,4}} x_{3,4},$$

- Choose arbitrary outgoing arc from node $i = 8$, for example, the arc (8,1). Let us express the arc flows to all outgoing arcs from the node $i = 8$, except for the arc (8,1), through the arc flow $x_{8,1}$ of selected outgoing arc (8,1).

$$x_{8,3} = \frac{p_{8,3}}{p_{8,1}} x_{8,1}, \;\; x_{8,4} = \frac{p_{8,4}}{p_{8,1}} x_{8,1}, \;\; x_{8,6} = \frac{p_{8,6}}{p_{8,1}} x_{8,1}.$$

It is obvious that we form additional equations for each node where the number of outgoing arcs is equal or greater than 2. The additional equations of the type (2.7.2) for the graph $\overline{G}$ (see Figure 3.30) are as follows:

$$\begin{aligned}
x_{3,8} - \frac{p_{3,8}}{p_{3,4}} x_{3,4} &= 0, \\
x_{8,3} - \frac{p_{8,3}}{p_{8,1}} x_{8,1} &= 0, \\
x_{8,4} - \frac{p_{8,4}}{p_{8,1}} x_{8,1} &= 0, \\
x_{8,6} - \frac{p_{8,6}}{p_{8,1}} x_{8,1} &= 0.
\end{aligned} \tag{3.8.6}$$

Part of the unknowns of the system (3.8.5), (3.8.6) makes up outgoing arc flows for arcs from nodes of the set $I \setminus M^*$ of the graph $\overline{G}$, where $I \setminus M^* = \{1,3,8\}$:

$$x_{1,8}, \; x_{3,4}, \; x_{3,8}, \; x_{8,1}, \; x_{8,3}, \; x_{8,4}, \; x_{8,6}.$$

The remaining part of the unknowns of the system (3.8.5), (3.8.6) defines the variables intensities for the nodes of set $\overline{I}^* = \{4,6\}$:

$$x_4,\ x_6.$$

Number of unknowns of the system (3.8.5), (3.8.6) is equal to 9. Number of equations in the system (3.8.5), (3.8.6) also is equal to 9.

We will find the rank of the matrix of the system (3.8.5), (3.8.6). If the system (3.8.5), (3.8.6) is a system of full rank, then it has a unique solution.

We compute the rank of the sparse matrix of the system (3.8.5) using the following rules:

- If the connected component of the graph $\overline{G}$ contains at least one node from the set of nodes with variable intensities, then the system corresponding to this connected component is a system of full rank and the rank of the matrix of this system is equal to the number of nodes in this connected component.
- If the connected component of the graph $\overline{G}$ contains no nodes from the set of nodes with variable intensities, then the system corresponding to this connected component is not of full rank and the rank of the matrix of this system is equal to the number of nodes in this connected component minus one.
- As the matrix of the system (3.8.5) is a block-diagonal matrix, then its rank is equal to the sum of ranks of matrices of the systems corresponding to the connected components of the graph $\overline{G}$.

To calculate the rank of the matrix of the system (3.8.5), (3.8.6) we use the theory of decomposition (see Section 2.5).

Let's show the use of the decomposition algorithms for the system under consideration. For this purpose we construct any support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.8.5) (see Figure 3.31), where

$$U_R = \{(3,4), (3,8), (8,1)\},\ \ I_R^* = \{4, 6\}.$$

Support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.8.5) consists of two connected components. We determine what structures can be obtained after adding one no supporting element from sets $\overline{U} \backslash U_R$ or $\overline{I}^* \backslash I_R^*$ to the support R.

We construct a system of characteristic vectors (basis of the solution space) of the homogeneous system generated by the system (3.8.5).

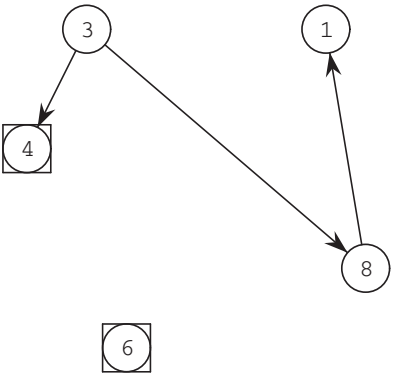

Fig. 3.31. Support of the graph $\overline{G}$ for the system (3.8.5)

The system of characteristic vectors $\delta(1,8)$,$\delta(8,3)$,$\delta(8,4)$,$\delta(8,6)$, entailed by arcs (1,8),(8,3),(8,4),(8,6) respectively is:

$$\delta(1,8) = (\delta_{1,8}^{1,8} \to 1,\ \delta_{3,4}^{1,8} \to 0,\ \delta_{3,8}^{1,8} \to 0,\ \delta_{8,1}^{1,8} \to 1,$$
$$\delta_{8,3}^{1,8} \to 0,\ \delta_{8,4}^{1,8} \to 0,\ \delta_{8,6}^{1,8} \to 0,\ \delta_{4}^{1,8} \to 0,\ \delta_{6}^{1,8} \to 0);$$

$$\delta(8,3) = (\delta_{1,8}^{8,3} \to 0,\ \delta_{3,4}^{8,3} \to 0,\ \delta_{3,8}^{8,3} \to 1,\ \delta_{8,1}^{8,3} \to 0,$$
$$\delta_{8,3}^{8,3} \to 1,\ \delta_{8,4}^{8,3} \to 0,\ \delta_{8,6}^{8,3} \to 0,\ \delta_{4}^{8,3} \to 0,\ \delta_{6}^{8,3} \to 0);$$

$$\delta(8,4) = (\delta_{1,8}^{8,4} \to 0,\ \delta_{3,4}^{8,4} \to -1,\ \delta_{3,8}^{8,4} \to 1,\ \delta_{8,1}^{8,4} \to 0,$$
$$\delta_{8,3}^{8,4} \to 1,\ \delta_{8,4}^{8,4} \to 1,\ \delta_{8,6}^{8,4} \to 0,\ \delta_{4}^{8,4} \to 0,\ \delta_{6}^{8,4} \to 0);$$

$$\delta(8,6) = (\delta_{1,8}^{8,6} \to 0,\ \delta_{3,4}^{8,6} \to -1,\ \delta_{3,8}^{8,6} \to 1,\ \delta_{8,1}^{8,6} \to 0,$$
$$\delta_{8,3}^{8,6} \to 0,\ \delta_{8,4}^{8,6} \to 0,\ \delta_{8,6}^{8,6} \to 1,\ \delta_{4}^{8,6} \to 1,\ \delta_{6}^{8,6} \to -1).$$

By using formulas $\Lambda_{\tau\rho}^{p} = \sum\limits_{(i,j)\in U_R} \lambda_{ij}^{p}\delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^{p}$, $p = \overline{1,q}$ we calculate the determinants $\Lambda_{\tau\rho}^{p}$, $(\tau,\rho) \in \overline{U}\backslash U_R$ of the structures, relatively to additional

equations (3.8.6), where p – number of equation from the system (3.8.6), $p = 1,2,3,4$ where $(\tau,\rho) \in \overline{U} \backslash U_R = \{(1,8),(8,3),(8,4),(8,6)\}$.

We form the matrix of determinants D. Matrix D consists from determinants of the structures, entailed by non-supporting elements:

$$D = \begin{pmatrix} \Lambda^1_{1,8} & \Lambda^1_{8,3} & \Lambda^1_{8,4} & \Lambda^1_{8,6} \\ \Lambda^2_{1,8} & \Lambda^2_{8,3} & \Lambda^2_{8,4} & \Lambda^1_{8,6} \\ \Lambda^3_{1,8} & \Lambda^3_{8,3} & \Lambda^3_{8,4} & \Lambda^1_{8,6} \\ \Lambda^4_{1,8} & \Lambda^3_{8,3} & \Lambda^3_{8,4} & \Lambda^1_{8,6} \end{pmatrix}.$$

So, matrix of determinants D takes the form:

$$D = \begin{pmatrix} 0 & 1 & 1 + \dfrac{p_{3,8}}{p_{3,4}} & 1 + \dfrac{p_{3,8}}{p_{3,4}} \\ -\dfrac{p_{8,3}}{p_{8,1}} & 1 & 0 & 0 \\ -\dfrac{p_{8,4}}{p_{8,1}} & 0 & 1 & 0 \\ -\dfrac{p_{8,6}}{p_{8,1}} & 0 & 0 & 1 \end{pmatrix}.$$

Since the following relations are true: $0 < p_{i,j} \leqslant 1$, $(i,j) \in U$, then the determinant of the matrix D is not equal to zero:

$$det\ D = \frac{p_{8,3}}{p_{8,1}} + \frac{p_{8,4}}{p_{8,1}} + \frac{p_{3,8}p_{8,4}}{p_{3,4}p_{8,1}} + \frac{p_{8,6}}{p_{8,1}} + \frac{p_{3,8}p_{8,6}}{p_{3,4}p_{8,1}} \neq 0.$$

Therefore, the system (3.8.5), (3.8.6) is system of full rank. The number of the unknowns of system (3.8.5), (3.8.6) is equal to the rank of its matrix and equal to 9. The system (3.8.5), (3.8.6) has the unique solution for given set $M = \{2,5,7,9\}$ of monitored nodes of the graph G.

The partial solution $\tilde{x} = (\tilde{x}_{i,j},(i,j) \in \overline{U}, \tilde{x}_i, i \in \overline{I}^*)$ of the system (3.8.5) we construct in the following way. The non-supporting components of a partial solution $\tilde{x}$ are equal to zero: $\tilde{x}_{\tau,\rho} = 0$, $(\tau,\rho) \in \overline{U} \backslash U_R$, $\tilde{x}_\gamma = 0$, $\gamma \in \overline{I}^* \backslash I^*_R$.

So, we have

$$\tilde{x}_{1,8} = 0,\ \tilde{x}_{8,3} = 0,\ \tilde{x}_{8,4} = 0,\ \tilde{x}_{8,6} = 0.$$

Using graph-theoretical properties of the support of graph $\overline{G}$ for system (3.8.5), we compute the supporting components of the partial solution of system (3.8.5) in the $O(n)$ time, $n = |\overline{I}|$ in the worst case. The supporting components of a partial solution $\tilde{x}$ are:

$$\tilde{x}_{3,4} = \frac{f_{4,2}p_{4,3}}{p_{4,2}} + \frac{f_{4,2}p_{4,8}}{p_{4,2}} + \frac{f_{6,9}p_{6,8}}{p_{6,9}},$$

$$\tilde{x}_{3,8} = -\frac{f_{4,2}p_{4,8}}{p_{4,2}} - \frac{f_{6,9}p_{6,8}}{p_{6,9}},$$

$$\tilde{x}_{8,1} = 0,$$

$$\tilde{x}_4 = -f_{2,4} + f_{4,2} + f_{4,5} + f_{4,7} - f_{5,4} - f_{7,4} - \frac{f_{6,9}p_{6,8}}{p_{6,9}},$$

$$\tilde{x}_6 = f_{6,9} - f_{9,6} + \frac{f_{6,9}p_{6,8}}{p_{6,9}}.$$

Using (2.8.4), we calculate $A^1,\ A^2,\ A^3,\ A^4$:

$$A^1 = \frac{f_{4,2}\,p_{4,8}}{p_{4,2}} + \frac{p_{3,8}\left(\dfrac{f_{4,2}\,p_{4,3}}{p_{4,2}} + \dfrac{f_{4,2}\,p_{4,8}}{p_{4,2}} + \dfrac{f_{6,9}\,p_{6,8}}{p_{6,9}}\right)}{p_{3,4}} + \frac{f_{6,9}\,p_{6,8}}{p_{6,9}},$$

$$A^2 = 0,\ \ A^3 = 0,\ \ A^4 = 0.$$

We compute the components of the vector $x_W = (x_{1,8},\ x_{8,3},\ x_{8,4},\ x_{8,6})$ from the system (3.8.7):

$$\begin{aligned} x_{8,3} + \left(1 + \frac{p_{3,8}}{p_{3,4}}\right) x_{8,4} + \left(1 + \frac{p_{3,8}}{p_{3,4}}\right) x_{8,6} &= A^1, \\ -\frac{p_{8,3}}{p_{8,1}} x_{1,8} + x_{8,3} &= 0, \\ -\frac{p_{8,4}}{p_{8,1}} x_{1,8} + x_{8,4} &= 0, \\ -\frac{p_{8,6}}{p_{8,1}} x_{1,8} + x_{8,6} &= 0. \end{aligned} \tag{3.8.7}$$

Thus, the components of the vector $x_W = (x_{1,8},\ x_{8,3},\ x_{8,4},\ x_{8,6})$ are equal to:

$$x_{1,8} = \frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,1}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_{8,3} = \frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,3}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_{8,4} = \frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,4}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_{8,6} = \frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,6}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))}.$$

The remaining components of the vector x, which correspond to elements of support of the graph $\overline{G}$ for the system (3.8.5), we compute using the formulas (2.7.5) – (2.7.6) and network properties of the support $R = \{U_R, I_R^*\}$ of the graph $\overline{G}$ for the system (3.8.5) (see Theorem 2.7.3).

So, we write the components of the vector x, which correspond to supporting elements of the graph $\overline{G}$ for the system (3.8.5):

$$x_{8,1} = \frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,1}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_{3,4} = \frac{p_{3,4}(f_{6,9}p_{4,2}p_{6,8}p_{8,3} + f_{4,2}p_{6,9}(p_{4,8}p_{8,3} + p_{4,3}(p_{8,3}+p_{8,4}+p_{8,6})))}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_4 = -f_{2,4} + f_{4,2} + f_{4,5} + f_{4,7} - f_{5,4} - f_{7,4} - \frac{f_{6,9}p_{6,8}}{p_{6,9}} +$$

$$+\frac{(f_{6,9}(p_{3,4}+p_{3,8})p_{4,2}p_{6,8} + f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{6,9})p_{8,6}}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6}) + p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))},$$

$$x_6 = (-p_{6,9}(f_{4,2}(p_{3,4}p_{4,8} + p_{3,8}(p_{4,3}+p_{4,8}))p_{8,6} +$$

$$+f_{9,6}p_{4,2}(p_{3,8}(p_{8,4}+p_{8,6})+p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6})))+$$

$$+f_{6,9}p_{4,2}(p_{3,8}(p_{6,8}p_{8,4}+p_{6,9}(p_{8,4}+p_{8,6}))+$$

$$+p_{3,4}(p_{6,8}(p_{8,3}+p_{8,4})+p_{6,9}(p_{8,3}+p_{8,4}+$$

$$+p_{8,6})))))/(p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6})+p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))),$$

$$x_{3,8}=\frac{p_{3,8}(f_{6,9}p_{4,2}p_{6,8}p_{8,3}+f_{4,2}p_{6,9}(p_{4,8}p_{8,3}+p_{4,3}(p_{8,3}+p_{8,4}+p_{8,6})))}{p_{4,2}p_{6,9}(p_{3,8}(p_{8,4}+p_{8,6})+p_{3,4}(p_{8,3}+p_{8,4}+p_{8,6}))}.$$

Now we construct a numerical solution of the system (3.8.1). We locate sensors in the nodes $i \in M = \{2,5,7,9\}$. The values of arc flows on all incoming and outgoing arcs for the each node i of the set M (monitored node) are known and also we know the values $x_i = f_i$, $i \in M \cap I^*$. We substitute the known values of the variables

$$x_{2,4}=3,\ x_{4,2}=2,\ x_{4,5}=2,\ x_{4,7}=1$$

$$x_{5,4}=4,\ x_{6,9}=1,\ x_{7,4}=5,\ x_{9,6}=4,$$
$$x_2=1,\ x_5=2,\ x_7=4,\ x_9=3$$

to the system of equations (3.8.1). Let's delete from graph $G=(I,U)$ the set of the arcs on which the arc flows are known:

$$(2,4),\ (4,2),\ (4,5),\ (4,7),\ (5,4),\ (6,9),\ (7,4),\ (9,6).$$

Also, delete from graph G the set of the nodes $i \in M$, $M=\{2,5,7,9\}$. We obtain a graph G' which is shown in Figure 3.33.

Consider any node i of the graph G. For every outgoing arc $(i,j) \in U$ for this node i determined a real number $p_{ij} \in (0,1]$ which denotes the part of the total outgoing flow $\sum\limits_{j \in I_i^+(U)} x_{ij}$ from node i corresponding to the arc (i,j). That is,

$$x_{ij}=p_{ij}\sum_{j \in I_i^+(U)} x_{ij},\ p_{ij} \in (0,1],\ \sum_{j \in I_i^+(U)} p_{ij}=1.$$

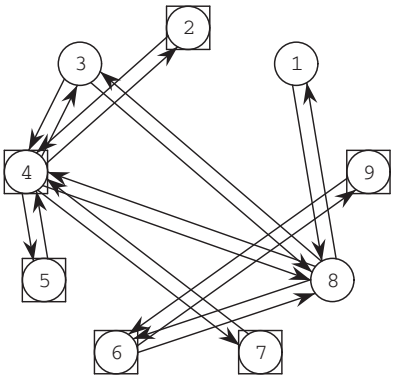

Fig. 3.32. Connected graphs G

The numerical values p_{ij}, $(i,j) \in U$ are shown in Table 3.8.1.

The rest of the arcs flow for the outgoing arcs from the nodes of the set $M^+ = I(CC(M)) \setminus M = \{4,6\}$, can be expressed from the arcs flow of the outgoing arcs for M^+ by (3.8.8).

$$x_{4,3} = \frac{p_{4,3}}{p_{4,2}} f_{4,2} = 3; \quad x_{4,8} = \frac{p_{4,8}}{p_{4,2}} f_{4,2} = 1; \qquad x_{6,8} = \frac{p_{6,8}}{p_{6,9}} f_{6,9} = 2. \tag{3.8.8}$$

Let's delete from graph G' (see Figure 3.33) the set of the arcs on which the arc flow are known according to (3.8.8). In Figure 3.34 we represent graph $\overline{G} = (\overline{I}, \overline{U})$ obtained by deleting the set of the arcs on which the arc flow are known according to (3.8.8) from graph G' (see Figure 3.33). As a result of the localization of multiple monitored nodes $M = \{2,5,7,9\}$ in the graph G which is represented in Figure 3.32 we have a unique solution of system (3.8.1). The numerical values of arc flows of the unique solution of system (3.8.1) is presented in Table 3.8.2. The numerical values of variable intensities of nodes from set $I^* = \{2, 4, 5, 6, 7, 9\}$ of the unique solution of system (3.8.1) for the nodes of set I^* are:

$$x_2 = 1, \quad x_4 = -\frac{87}{13}, \quad x_5 = 2, \quad x_6 = -\frac{43}{13}, \quad x_7 = 4, \quad x_9 = 3.$$

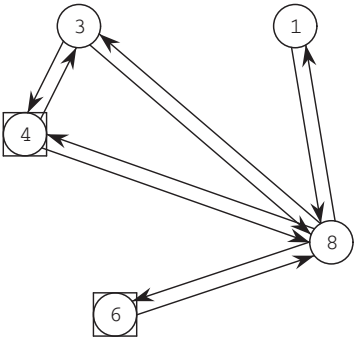

Fig. 3.33. Connected graph G'

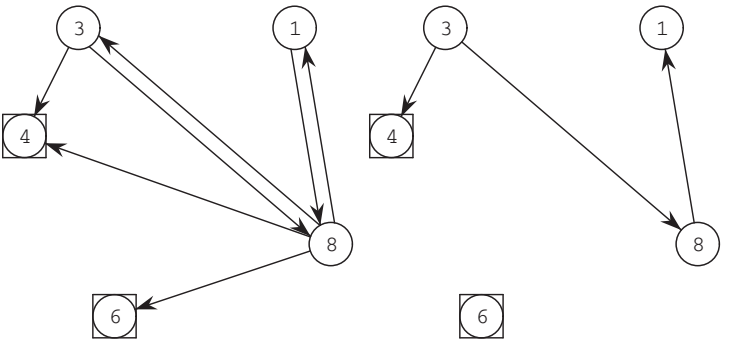

Fig. 3.34. Connected graph $\overline{G}$ (on the left) and support R of the graph $\overline{G}$ for the system (3.8.5) (on the right)

Table 3.8.1

Values of components of the vector $p = (p_{i,j},(i,j) \in U)$

(i,j)	(1,8)	(2,4)	(3,4)	(3,8)	(4,2)	(4,3)	(4,5)	(4,7)	(4,8)
$p_{i,j}$	1	1	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{5}{13}$	$\frac{2}{13}$	$\frac{1}{13}$
(i,j)	(5,4)	(6,8)	(6,9)	(7,4)	(8,1)	(8,3)	(8,4)	(8,6)	(9,6)
$p_{i,j}$	1	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$	1

Table 3.8.2

Values of known arc flows $x = (x_{i,j},(i,j) \in U)$

(i,j)	(1,8)	(2,4)	(3,4)	(3,8)	(4,2)	(4,3)	(4,5)	(4,7)	(4,8)
$x_{i,j}$	$\frac{15}{26}$	3	$\frac{81}{26}$	$\frac{27}{26}$	2	3	2	1	1
(i,j)	(5,4)	(6,8)	(6,9)	(7,4)	(8,1)	(8,3)	(8,4)	(8,6)	(9,6)
$x_{i,j}$	4	2	1	5	$\frac{15}{26}$	$\frac{15}{13}$	$\frac{15}{26}$	$\frac{30}{13}$	4

4. NOT FULL RANK SPARSE SYSTEMS

In this part we consider the sparse linear underdetermined system of a special type. Systems of this type appear in non-homogeneous network flow programming problems in the form of systems of restrictions and can be characterized as systems with a large sparse sub-matrix representing the embedded network structure [12, 15–16, 18, 21–22]. We develop a direct method for finding solutions of the system. The algorithm is based on the graph theoretic specificities for the structure of the support [12, 23, 36, 38, 43] and properties of the basis of a solution space of a homogeneous system. One of the key steps is decomposition of the system.

The work on this part is caused, mainly, by the analysis of problems of non-homogeneous network flow optimization [22, 35, 45, 47, 49, 55, 56] on large data files. Our main goal is to develop an effective (direct) method for solving large sparse systems of linear equations with embedded network structure, which appear naturally, e.g. as systems of restrictions, in a broad class of non-homogeneous network flow programming problems [26–27, 45, 49, 55]. In addition, an important application of researched sparse systems of this type is the class of problems of optimal placing of sensors into multigraphs (Sensor Location Problem for the multigraphs), which will be discussed in the Chapter 5.

The 'network nature' of the regarded system allows keeping data in the matrix-free form in the computer memory. The formulas, derived within the chapter, are written in the component (network) form to provide clear approaches towards developing computational algorithms using efficient data structures for graph representation [45, 53].

The general idea of the method is based on the following key steps:

• Distinguishing between the network part of the system and the additional part. The network part of the system represents a network structure and corresponds to the network part of the system of main restrictions of a non-homogeneous network flow programming problem [45], and is given, traditionally, by balance equations or by restrictions to a more general form for generalized networks [49, 54–55] (see Chapter 6), written for the nodes of a network. The additional part of the system corresponds to the additional part of the system of main restrictions and can have a general form. We start the solution by considering the network part of the system only.

• Introduction of the support of the network for a system. The term 'support of the network' (also referred to as network support, or support) is borrowed from optimization theory [10–12] and is used here for further

compatibility with applications in problems of non-homogeneous network flow programming and the Sensor Location Problem for multigraphs. The actual meaning in this chapter is – a set of indices of variables (or, in the network terms, - a set of arcs) corresponding to columns, which form a basis minor of the matrix of a system. In the Sensor Location Problem the term 'support of the system' also includes the nodes with variable intensities. We study the support of the network part of the system, finding the correspondence between the columns of a basis minor and a family of spanning trees for non-homogeneous network flow programming problems or between the basis minor and a forest of the trees with the special properties for the Sensor Location Problem for multigraphs.

• Construction of a general solution for the network part of the system. We compute a basis of a solution space of the corresponding homogeneous system and interpret the basis vectors as characteristic vectors, entailed by non-supporting arcs. For the Sensor Location Problem for multigraphs, the characteristic vectors can also be entailed by nodes with variable intensities.

• Decomposition of the system. We perform column decomposition of the system by separating the variables according to the sets collection of spanning trees, which consist of the arcs of the support for the network part of the sparse system, cyclic arcs and non-supporting arcs respectively; and, finally, sequentially express the unknowns corresponding to the sets of cyclic arcs and the sets collection of spanning trees in terms of the independent variables corresponding to the set non-supporting arcs.

4.1. General form of sparse systems

Let $G = (V,A)$ be a finite oriented connected network without multiple arcs and loops, where V is a set of nodes and A is a set of arcs defined on $V \times V (|V| < \infty, |A| < \infty)$. Let K $(|K| < \infty)$ be a set of different products (types of flow) transported through the network G. For definiteness, we assume the set $K = \{1, \ldots, |K|\}$. Let us denote the connected network corresponding to a certain type k of flow with $S^k = (I^k, U^k)$, where I^k is the set of nodes and U^k is the set of arcs which is available on the flow of type k, $k \in K$. Also, we define for each node $i \in V$ the set of types of flows (products) $K(i) = \{k \in K : i \in I^k\}$ and for each arc $(i,j) \in A$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. In other words, $K(i)$ is the set of types of flows (products) transported through the node $i \in V$ and $K(i,j)$ is the set of types of flows (products) transported through the multiarc $(i,j) \in A$

respectively. Finally, the initial network $G = (V,A)$ may be considered as a union of $|K|$ networks $S^k = (I^k,U^k)$, $k \in K$ and denote with $S = (I,U)$. We call $S = (I,U)$ a multidigraph or multigraph or multinetwork or just a network, if it is clear that we deal with multidigraph $S = (I,U)$. Each multiarc $(i,j) \in U$ of multidigraph $S = (I,U)$ consists from $|K(i,j)|$ arcs:

$$\{(i,j)^k, k \in K(i,j)\}.$$

Let us introduce a subset U_0 of the set U, and let $K_0(i,j) \subseteq K(i,j)$, $(i,j) \in U_0$ be an arbitrary subset of $K(i,j)$ such that $|K_0(i,j)| > 1$.

Consider the following sparse linear underdetermined system

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad i \in I^k, k \in K, \tag{4.1.1}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, p = \overline{1,q}, \tag{4.1.2}$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k = z_{ij}, \quad (i,j) \in U_0, \tag{4.1.3}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j)^k \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j,i)^k \in U^k\}$; $a_i^k, \lambda_{ij}^{kp}, \alpha^p, z_{ij} \in \mathbb{R}$ – constants; $x = (x_{ij}^k, (i,j)^k \in U^k, k \in K)$- vector of unknowns.

The matrix of system (4.1.1) – (4.1.3) has the following block structure:

$$A = \begin{bmatrix} M \\ Q \\ T \end{bmatrix}. \tag{4.1.4}$$

Here M is a sparse submatrix with a block-diagonal structure of size $\sum_{k \in K} |I^k| \times \sum_{k \in K} |U^k|$ such that each block represents a $|I^k| \times |U^k|$ incidence matrix of the network $S^k = (I^k,U^k), k \in K$, namely,

$$M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_{|K|},$$

where $M_k, k = 1, \ldots, |K|$ are blocks of matrix M; Q is a $q \times \sum_{k \in K} |U^k|$ submatrix (dense, in the general case) with elements $\lambda_{ij}^{kp}, (i,j) \in U, k \in K$, $p = \overline{1,q}$; T is a $|U_0| \times \sum_{k \in K} |U^k|$ submatrix consisting of zeros and ones, where all the nonzero elements of the row, corresponding to the multiarc

(i,j) appear in columns corresponding to arcs $(i,j)^k$,$(i,j) \in U_0$,$k \in K_0(i,j)$. Assume that $rank(A) = \sum_{k\in K}(|I^k| - 1) + q + |U_0|$. Since we consider the underdetermined system then the inequality is true:

$$\sum_{k\in K}(|I^k| - 1) + q + |U_0| < \sum_{k\in K}|U^k|.$$

4.2. Network part of system

We start the solution of system (4.1.1) − (4.1.3) by considering the network part of the system.

Definition 4.2.1. *We call system* (4.1.1) *the network part of the system* (4.1.1) − (4.1.3). *Systems* (4.1.2) *and* (4.1.3) *are called the additional part of the system* (4.1.1) − (4.1.3).

Before we proceed, let us recall the following necessary and sufficient condition of consistency for system (4.1.1) implied by Kronecker-Capelli theorem [9!!!!!]. Since M is a sparse submatrix with a block-diagonal structure of size $\sum_{k\in K}|I^k| \times \sum_{k\in K}|U^k|$ such that each block represents a $|I^k| \times |U^k|$ incidence matrix of the network $S^k = (I^k,U^k)$ for each k

$$\sum_{i\in I^k} a_i^k = 0, k \in K.$$

Theorem 4.2.1. *(Rank theorem) The rank of the matrix of system* (4.1.1) *equals* $\sum_{k\in K}|I^k| - |K|$.

Proof. We denote M_k a diagonal block of matrix M where $k = 1,\ldots,|K|$. Since matrix M of the system (4.1.1) has the form $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{|K|}$ and $rank\,(M_k) = |I^k| - 1$ [5!], then $rank(M) = \sum_{k=1}^{|K|} rank(M_k) == \sum_{k\in K}(|I^k| - 1) = \sum_{k\in K}|I^k| - |K|$. □

Remark 4.2.1. We assume, without loss of generality, that the rank of the matrix of the system (4.1.1) − (4.1.3) is $\sum_{k\in K}|I^k| - |K| + q + |U_0|$, where $q + |U_0|$ is a number of equations in the additional part (4.1.2) − (4.1.3).

Since the matrix of system (4.1.1) has the block-diagonal structure, we split the solution of the system into $|K|$ solutions of (independent) systems, each of which corresponds to a separate block, i.e. to a fixed $k \in K$, and has the following form:

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = a_i^k, i \in I^k. \tag{4.2.1}$$

4.3. Multigraph support criterion

Let's define a support of the multigraph $S = (I,U)$ for system (4.1.1).

Definition 4.3.1. *The support of the multigraph $S = (I,U)$ for system (4.1.1) is a set of arcs $U_T = \{U_T^k \subseteq U^k, k \in K\}$, such that the system*

$$\sum_{j \in I_i^+(\hat{U}^k)} x_{ij}^k - \sum_{j \in I_i^-(\hat{U}^k)} x_{ji}^k = 0, i \in I^k, k \in K \tag{4.3.1}$$

has only a trivial solution for $\hat{U}^k = U_T^k$, but has a non-trivial solution for $\hat{U}^k = U_T^k, k \in K \setminus k_0$; $\hat{U}^{k_0} = U_T^{k_0} \cup (i,j)^{k_0}, (i,j)^{k_0} \notin U_T^{k_0}, k_0 \in K$.

We formulate the Multigraph Support Criterion.

Theorem 4.3.1 (Multigraph Support Criterion). *The set of arcs U_T where $U_T = \{U_T^k, k \in K\}$ is a support of the multigraph $S = (I,U)$ for system (4.1.1) iff for each $k \in K$ the set of arcs U_T^k is a spanning tree for the network $S^k = (I^k,U^k)$.*

Proof. Follows directly from the proof [11, 17] for the case when $|K| = 1$ and the block-diagonal structure [9, 57] of the matrix of the system (4.1.1).

□

4.4. Characteristic vectors

Before introducing the definition of a characteristic vector $\delta^k(\tau,\rho)$, let's analyze the structure of a network obtained by appending an arbitrary arc $(\tau,\rho)^k \in U^k \setminus U_T^k$ to the support U_T of multinetwork S for system (4.1.1) where $k \in K$. For a fixed $k \in K$ we consider a network $\hat{S}^k$. The network $\hat{S}^k$ is: $\hat{S}^k = (I^k, U_T^k \cup (\tau,\rho)^k)$, $(\tau,\rho)^k \in U^k \setminus U_T^k$, where the set U_T^k is a spanning

tree of the network S^k. Appending an arc $(\tau,\rho)^k \in U^k \setminus U_T^k$ to the tree U_T^k entails a unique cycle. We denote this cycle with $L_{\tau\rho}^k$.

The set $Z_k = \{L_{\tau\rho}^k,(\tau,\rho)^k \in U^k \setminus U_T^k\}$ is the fundamental set of cycles with respect to the spanning tree U_T^k of the network S^k [17, 59].

Let's consider a cycle $L_{\tau\rho}^k$, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_T^k$. We define the detour direction within the cycle $L_{\tau\rho}^k$ corresponding to the arc $(\tau,\rho)^k$.

Definition 4.4.1. *We call an arc $(i,j)^k \in L_{\tau\rho}^k$, where $k \in K$ is fixed, a forward arc of the cycle $L_{\tau\rho}^k$, if the direction of the arc $(i,j)^k$ is the same as the direction of the arc $(\tau,\rho)^k$ within the cycle $L_{\tau\rho}^k$. Similarly, we call an arc $(i,j)^k \in L_{\tau\rho}^k$, where $k \in K$ is fixed, a backward arc of the cycle $L_{\tau\rho}^k$, if the direction of the arc $(i,j)^k$ is opposite to the direction of the arc $(\tau,\rho)^k$ within the cycle $L_{\tau\rho}^k$.*

We denote the sign of an arc $(i,j)^k$ within a cycle $L_{\tau\rho}^k$ by $\mathrm{sign}(i,j)^{L_{\tau\rho}^k}$:

$$\mathrm{sign}(i,j)^{L_{\tau\rho}^k} = \begin{cases} 1, \ (i,j)^k \in L_{\tau\rho}^{k+}, \\ -1,(i,j)^k \in L_{\tau\rho}^{k-}, \\ 0, (i,j)^k \notin L_{\tau\rho}^k, \end{cases} \tag{4.4.1}$$

where $L_{\tau\rho}^{k+}$ and $L_{\tau\rho}^{k-}$ are the sets of forward and backward arcs of the cycle $L_{\tau\rho}^k$ with a direction corresponding to the arc $(\tau,\rho)^k$.

Let us give a constructive definition of a characteristic vector, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_T^k$.

Definition 4.4.2. *Characteristic vector, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_T^k$ with respect to the spanning tree U_T^k , is the vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho),(i,j)^k \in U^k)$, where is $k \in K$ fixed, constructed according to the following rules:*

- *Add the arc $(\tau,\rho)^k \in U^k \setminus U_T^k$, to the set $U_T^k,k \in K$, which is a spanning tree for the network $S^k = (I^k,U^k)$; and thus create a unique cycle $L_{\tau\rho}^k$.*
- *Let the arc $(\tau,\rho)^k$ set the detour direction within the cycle $L_{\tau\rho}^k$ and $\delta_{\tau\rho}^k(\tau,\rho) = 1$.*
- *For cycle's forward arcs, let $\delta_{ij}^k(\tau,\rho) = 1$.*
- *For cycle's backward arcs, let $\delta_{ij}^k(\tau,\rho) = -1$.*
- *Let $\delta_{ij}^k(\tau,\rho) = 0,(i,j)^k \in U^k \setminus L_{\tau\rho}^k$.*

For briefness, we will call a characteristic vector $\delta^k(\tau,\rho)$, entailed by an arc $(\tau,\rho)^k$, with respect to the spanning tree U_T^k, a characteristic vector $\delta^k(\tau,\rho)$, entailed by an arc $(\tau,\rho)^k$, or, simply, a characteristic vector $\delta^k(\tau,\rho)$.

The next two lemmas state the essential properties of characteristic vectors.

Lemma 4.4.1 (Property 1). *A characteristic vector $\delta^k(\tau,\rho)$, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_T^k$, where $k \in K$ is fixed, is a solution of the homogeneous linear system:*

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} x_{ji}^k = 0, \quad i \in I^k. \tag{4.4.2}$$

P r o o f. Let a support $U_T = \{U_T^k, k \in K\}$ for system (4.1.1) of multinetwork S be defined. For the fixed $k \in K$ we consider the set U_T^k, which is, according to Theorem 4.3.1, a spanning tree of the network S^k and let $L_{\tau\rho}^k$ be the unique cycle of the network $\hat{S}^k = (I^k, U_T^k \cup (\tau,\rho)^k)$, which appears after appending the arc $(\tau,\rho)^k \in U^k \setminus U_T^k$ to the set U_T^k.

Consider the vector $x^k = (x_{ij}^k, (i,j)^k \in U^k)$, of unknowns in system (4.4.2).

Let's $x_{ij}^k = 0, (i,j)^k \in U^k \setminus L_{\tau\rho}^k$. Thus, the system (4.4.2) can be reduced to

$$\sum_{j\in I_i^+(L_{\tau\rho}^k)} x_{ij}^k - \sum_{j\in I_i^-(L_{\tau\rho}^k)} x_{ji}^k = 0, \quad i \in I(L_{\tau\rho}^k), \tag{4.4.3}$$

where $I(L_{\tau\rho}^k)$ denotes all nodes in cycle $L_{\tau\rho}^k$.

Letting $x_{\tau\rho}^k = 1$, from the reduced system (4.4.3), we can easily define the values of the remaining unknowns $x_{ij}^k, (i,j)^k \in L_{\tau\rho}^k \setminus (\tau,\rho)^k$:

$$x_{ij}^k = \mathrm{sign}(i,j)^{L_{\tau\rho}^k}, (i,j)^k \in L_{\tau\rho}^k \setminus (\tau,\rho)^k.$$

Algorithmically, after letting $x_{\tau\rho}^k = 1$, we pass from node ρ to node τ along the cycle $L_{\tau\rho}^k$, consecutively setting the unknowns x_{ij}^k for the arcs $(i,j)^k \in L_{\tau\rho}^k \setminus (\tau,\rho)^k$, to the values of signs of the corresponding arcs within the cycle $L_{\tau\rho}^k$.

Note, the constructed solution vector x^k satisfies all the rules of Definition 4.4.2 of the characteristic vector, entailed by the arc $(\tau,\rho)^k$, and hence

$\delta^k(\tau,\rho) = x^k$ is a solution of the homogeneous linear system (4.4.2). The lemma is proved. □

Lemma 4.4.2 (Property 2). *The set* $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_T^k\}$ *of characteristic vectors, where* k *is fixed, forms the basis of a solution space for the homogeneous system (4.4.2),* $k \in K$.

Proof. According to Lemma 4.4.1, each characteristic vector satisfies the homogeneous system (4.4.2).

By Theorem 4.3.1, for a fixed $k \in K$, the set U_T^k is a spanning tree of the network $S^k = (I^k,U^k)$, hence, $|U_T^k| = = |I^k| - 1$. Thus, the number of characteristic vectors in the set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_T^k\}$ for fixed $k \in K$ equals $|U^k \setminus U_T^k| = |U^k| - |I^k| + 1$.

Now it suffices to show that all the vectors $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_T^k\}$ are linearly independent. Each characteristic vector $\delta^k(\tau,\rho)$, entailed by some arc $(\tau,\rho)^k \in U^k \setminus U_T^k$, always has one and only one component, corresponding to the set $U^k \setminus U_T^k$, that is equal to 1. It corresponds to the arc $(\tau,\rho)^k \in U^k \setminus U_T^k$, that has entailed this vector. All the other components, which correspond to arcs $U^k \setminus L_{\tau\rho}^k$, are equal to 0. This fact implies that any two characteristic vectors, entailed by different arcs, are linearly independent. □

The unknowns of the system (4.1.1) denote x, $x = \{x^k, k \in K\}$ where $x^k = (x_{ij}^k,(i,j)^k \in U^k)$ is the unknowns of system (4.1.1) for a fixed $k \in K$.

Theorem 4.4.1 (The general solution). *The general solution of system (4.1.1) for a fixed* $k \in K$ *can be represented using the following form:*

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} x_{\tau\rho}^k \operatorname{sign}(i,j)^{L_{\tau\rho}^k} +$$

$$+ \left(\tilde{x}_{ij}^k - \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} \tilde{x}_{\tau\rho}^k \operatorname{sign}(i,j)^{L_{\tau\rho}^k} \right), \tag{4.4.4}$$

$$(i,j)^k \in U_T^k, \quad x_{\tau\rho}^k \in \mathbf{R}, \quad (\tau,\rho)^k \in U^k \setminus U_T^k,$$

where $\tilde{x}^k = (\tilde{x}_{ij}^k,(i,j)^k \in U^k)$ *is any partial solution of the (non-homogeneous) system (4.1.1);* $x_{\tau\rho}^k$, $(\tau,\rho)^k \in U^k \setminus U_T^k$ *are independent variables corresponding to arcs* $(\tau,\rho)^k \in U^k \setminus U_T^k$.

Proof. Let $x^k = (x^k_{ij},(i,j)^k \in U^k)$ be a general solution and denote with $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ – a partial solution of the system (4.1.1) for fixed $k \in K$. By Lemma 4.4.1, the set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U^k_T\}$ of characteristic vectors forms the basis of a solution space for the homogeneous system (4.4.2), we can write the expression for x^k in the following vector form:

$$x^k = \sum_{(\tau,\rho)^k \in U^k \setminus U^k_T} \alpha^k_{\tau\rho}\delta^k(\tau,\rho) + \tilde{x}^k, \tag{4.4.5}$$

as a sum of a general solution of the homogeneous system (4.4.2) and a partial solution of the non-homogeneous system (4.1.1) for fixed $k \in K$; $\alpha^k_{\tau\rho} \in \mathbf{R}$ for $(\tau,\rho)^k \in U^k \setminus U^k_T$ are coefficients of the linear combination of characteristic vectors (4.4.5).

Rewriting (4.4.5) in the component form we obtain:

$$x^k_{ij} = \sum_{(\tau,\rho)^k \in U^k \setminus U^k_T} \alpha^k_{\tau\rho}\delta^k_{ij}(\tau,\rho) + \tilde{x}^k_{ij}, \quad (i,j)^k \in U^k_T; \tag{4.4.6}$$

$$x^k_{\tau\rho} = \alpha^k_{\tau\rho} + \tilde{x}^k_{\tau\rho}, \quad (\tau,\rho)^k \in U^k \setminus U^k_T. \tag{4.4.7}$$

From equations (4.4.7) we find $\alpha^k_{\tau\rho} = x^k_{\tau\rho} - \tilde{x}^k_{\tau\rho}, \quad (\tau,\rho)^k \in U^k \setminus U^k_T$ and substitute into (4.4.6). Finally, rewriting components of characteristic vectors according to (4.4.1), we obtain the expression (4.4.4) for the general solution of the system (4.1.1). □

Remark 4.4.1. In practice, for construction of a partial solution $\tilde{x}$, $\tilde{x} = \{\tilde{x}^k, k \in K\}$, $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ of the system (4.1.1), we a priori assume the independent variables equal to zero: $\tilde{x}^k_{\tau\rho} = 0$, $(\tau,\rho)^k \in U^k \setminus U^k_T$ and solve the system

$$\sum_{j \in I^+_i(U^k_T)} \tilde{x}^k_{ij} - \sum_{j \in I^-_i(U^k_T)} \tilde{x}^k_{ji} = a^k_i, \quad i \in I^k.$$

Thus, formula (4.4.4) gets to the form:

$$x^k_{ij} = \sum_{(\tau,\rho)^k \in U^k \setminus U^k_T} x^k_{\tau\rho}\mathrm{sign}(i,j)^{L^k_{\tau\rho}} + \tilde{x}^k_{ij}, (i,j)^k \in U^k_T, \tag{4.4.8}$$

$$x^k_{\tau\rho} \in \mathbf{R}, \quad (\tau,\rho)^k \in U^k \setminus U^k_T.$$

Further, we will use the formula (4.4.8).

4.5. Decomposition of system

Let $U_T = \{U_T^k, k \in K\}$ be a support of the multigraph S for the system (4.1.1). We define a set $U_C = \{U_C^k \subseteq U^k \backslash U_T^k,\ k \in K\}$, $|U_C|$ of cyclic arcs by selecting $q + |U_0|$ arbitrary arcs from the sets $U^k \backslash U_T^k, k \in K$. We denote $U_N = \{U_N^k, k \in K\}, U_N^k = U^k \backslash (U_T^k \cup U_C^k)$ a set of arcs that are not included in support U_T and are not included in the set of cyclic arcs U_C.

Let's substitute the general solution (4.4.8) of the system (4.2.1) for each $k \in K$ into (4.1.2):

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \sum_{k\in K} \sum_{(i,j)^k\in U^k} \lambda_{ij}^{kp} x_{ij}^k =$$

$$= \sum_{k\in K} \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} x_{ij}^k + \sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} \lambda_{\tau\rho}^{kp} x_{\tau\rho}^k =$$

$$= \sum_{k\in K} \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} \left[\sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} x_{\tau\rho}^k \operatorname{sign}(i,j)^{L_{\tau\rho}^k} + \tilde{x}_{ij}^k \right] +$$

$$+ \sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} \lambda_{\tau\rho}^{kp} x_{\tau\rho}^k = \alpha_p, \quad p = \overline{1,q}. \tag{4.5.1}$$

We change the summing order in (4.5.1):

$$\sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} x_{\tau\rho}^k \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} \operatorname{sign}(i,j)^{L_{\tau\rho}^k} + \sum_{k\in K} \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} \tilde{x}_{ij}^k +$$

$$+ \sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} \lambda_{\tau\rho}^{kp} x_{\tau\rho}^k = \alpha_p, \quad p = \overline{1,q}\,. \tag{4.5.2}$$

In equations (4.5.2) we group the variables, corresponding to the sets $U^k \backslash U_T^k$, $k \in K$:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_T^k} x_{\tau\rho}^k \left[\lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} \operatorname{sign}(i,j)^{L_{\tau\rho}^k} \right] =$$

$$= \alpha_p - \sum_{k\in K} \sum_{(i,j)^k\in U_T^k} \lambda_{ij}^{kp} \tilde{x}_{ij}^k\,, p = \overline{1,q}\,. \tag{4.5.3}$$

Definition 4.5.1. *The number*

$$R_p(L_{\tau\rho}^k) = \sum_{(i,j)^k\in L_{\tau\rho}^k} \lambda_{ij}^{kp} \operatorname{sign}(i,j)^{L_{\tau\rho}^k} \tag{4.5.4}$$

is called the determinant of the cycle $L^k_{\tau\rho}$*, entailed by an arc* $(\tau,\rho)^k$ *where* $(\tau,\rho)^k \in U^k \backslash U^k_T$ *, with respect to the equation with the number* p *of the system* (4.1.2).

Let's denote

$$A^p = \alpha_p - \sum_{k\in K} \sum_{(i,j)^k \in U^k_T} \lambda^{kp}_{ij}\, \tilde{x}^k_{ij}\,, p = \overline{1,q}\,. \tag{4.5.5}$$

The equations (4.5.3), according to formulas (4.5.4), (4.5.5), get to the form:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k \backslash U^k_T} R_p(L^k_{\tau\rho}) x^k_{\tau\rho} = A^p, \quad p = \overline{1,q}\,. \tag{4.5.6}$$

In (4.5.6) we group the variables, corresponding to the sets $U^k_C, k \in K$: $U^k_C, k \in K$ for each $p = \overline{1,q}$:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k_C} R_p(L^k_{\tau\rho}) x^k_{\tau\rho} = A^p - \sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k_N} R_p(L^k_{\tau\rho}) x^k_{\tau\rho}. \tag{4.5.7}$$

Now, we apply the similar considerations to the system (4.1.3). Note, that (4.1.3) can be regarded as a particular case of the system (4.1.2) with λ^{kp}_{ij} equal to 0 or 1.

For each $k \in K$ we substitute the general solution (4.4.8) of the system (4.2.1) into equations (4.1.3):

$$\sum_{k\in K_0(i,j)} x^k_{ij} = \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k_T}} x^k_{ij} + \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k \backslash U^k_T}} x^k_{ij} =$$

$$= \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k_T}} \left[\sum_{(\tau,\rho)^k \in U^k \backslash U^k_T} x^k_{\tau\rho} \mathrm{sign}(i,j)^{L^k_{\tau\rho}} + \tilde{x}^k_{ij} \right] +$$

$$+ \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k \backslash U^k_T}} x^k_{ij} = z_{ij}, \quad (i,j) \in U_0, \tag{4.5.8}$$

or

$$\sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k \backslash U^k_T}} x^k_{\tau\rho} \left[\sum_{(i,j)^k \in U^k_T} \mathrm{sign}(i,j)^{L^k_{\tau\rho}} \right] +$$

$$+ \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k\in U^k\setminus U_T^k}} x_{ij}^k = z_{ij} - \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k\in U_T^k}} \tilde{x}_{ij}^k,\ (i,j)\in U_0.$$

Note, that for each arc $(\tau,\rho)^k \in U^k\setminus U_T^k$, $k \in K_0(i,j)$, which entailed a cycle $L_{\tau\rho}^k$, the following equality holds:

$$\sum_{(i,j)^k\in L_{\tau\rho}^k} \operatorname{sign}(i,j)^{L_{\tau\rho}^k} = \delta_{ij}(L_{\tau\rho}^k), \tag{4.5.9}$$

where

$$\delta_{ij}(L_{\tau\rho}^k) = \begin{cases} \operatorname{sign}(i,j)^{L_{\tau\rho}^k}, k\in K_0(i,j), \\ 0, k\notin K_0(i,j), \end{cases} \tag{4.5.10}$$

$$(i,j)\in U_0, (\tau,\rho)^k\in U^k\setminus U_T^k, k\in K.$$

Thus, the equations (4.1.3) take the following form:

$$\sum_{k\in K_0(i,j)} \sum_{(\tau,\rho)^k\in U^k\setminus U_T^k} \delta_{ij}(L_{\tau\rho}^k) x_{\tau\rho}^k = A_{ij}, \quad (i,j)\in U_0, \tag{4.5.11}$$

where

$$A_{ij} = z_{ij} - \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k\in U_T^k}} \tilde{x}_{ij}^k, \quad (i,j)\in U_0. \tag{4.5.12}$$

In (4.5.11) group the variables corresponding to sets $U_C^k, k\in K$:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k\in U_C^k} \delta_{ij}(L_{\tau\rho}^k) x_{\tau\rho}^k =$$

$$= A_{ij} - \sum_{k\in K_0(i,j)} \sum_{(\tau,\rho)^k\in U_N^k} \delta_{ij}(L_{\tau\rho}^k) x_{\tau\rho}^k, \quad (i,j)\in U_0. \tag{4.5.13}$$

We write the equations (4.5.7) and (4.5.13) in matrix form. Let's set an arbitrary numbering of the arcs of the set U_0 and arcs of set U_C, $\xi = \xi(i,j)$ is a number of an arc $(i,j)\in U_0$, $\xi\in\{1,2,\ldots,|U_0|\}$ and let $t = t(\tau,\rho)^k$ is a number of cyclic arc $(\tau,\rho)^k \in U_C^k, k\in K, t\in\{1,2,\ldots,|U_C|\}$. In other words, we number the equations of the system (4.1.3) or (4.5.13), where each equation with a number $\xi(i,j)$ corresponds to an arc $(i,j)\in U_0$. and variables corresponding to a set of cyclic arcs of U_C. Note that introduced the numbering of the cyclic arcs is equivalent to the numbering of elements

set $\{L^k_{\tau\rho},(\tau,\rho)^k \in U^k_C,\ k \in K\}$ cycles entailed by the arcs $(\tau,\rho)^k \in U^k_C$, with respect to spanning trees U^k_T of network $S^k, k \in K$.

Finally, by (4.5.7) and (4.5.13), we can rewrite the system (4.1.2) – (4.1.3) in the following matrix form:

$$Dx_C = \beta, \tag{4.5.14}$$

where $D = \binom{D_1}{D_2}$, $D_1 = (R_p(L^k_{\tau\rho}), p = \overline{1,q}, t(\tau,\rho)^k = \overline{1,|U_C|})$, is the matrix of the size $q \times |U_C|$, $D_2 = (\delta_{ij}(L^k_{\tau\rho}), \xi(i,j) = \overline{1,|U_0|}, t(\tau,\rho)^k = \overline{1,|U_C|})$ is the matrix of the size $|U_0| \times |U_C|$, $x_C = (x^k_{\tau\rho},(\tau,\rho)^k \in U^k_C, k \in K)$ is a vector of unknowns with components are ordered in according to the numbering $t = t(\tau,\rho)^k, t \in \{1,2,\ldots,|U_C|\}$.

The right-hand side of (4.5.14) has the form:

$$\beta = \begin{pmatrix} \beta_p, & p = \overline{1,q}, \\ \beta_{q+\xi(i,j)}, & (i,j) \in U_0 \end{pmatrix}, \tag{4.5.15}$$

where $\beta_p = A^p - \sum\limits_{k\in K}\sum\limits_{(\tau,\rho)^k\in U^k_N} R_p(L^k_{\tau\rho})x^k_{\tau\rho}, p = \overline{1,q}$,

$$\beta_{q+\xi(i,j)} = A_{ij} - \sum_{k\in K_0(i,j)}\sum_{(\tau,\rho)^k\in U^k_N} \delta_{ij}(L^k_{\tau\rho})x^k_{\tau\rho}, \quad (i,j) \in U_0.$$

From (4.5.14), in case of non-singularity of the matrix D we find the unknown variables x_C, corresponding to the set U_C of cyclic arcs:

$$x_C = D^{-1}\beta\,. \tag{4.5.16}$$

Remark 4.5.1. Generally, because of an arbitrary selection of arcs for the set $U_C = \{U^k_C, k \in K\}$, non-singularity of the matrix D is not guaranteed. In the case when $detD = 0$ one should reselect arcs into the set U_C and recompute D, β for the system (4.5.14).

Let $D^{-1} = (\nu_{l,s}; l,s = \overline{1,|U_C|})$. We rewrite (4.5.16) in the component form:

$$x^k_{\tau\rho} = \sum_{p=1}^{q} \nu_{t,p}\beta_p + \sum_{(i,j)\in U_0} \nu_{t,q+\xi(i,j)}\beta_{q+\xi(i,j)}, t = t(\tau,\rho)^k,\ (\tau,\rho)^k \in U^k_C, k \in K.$$

Thus, we have determined all the unknown $x^k = (x^k_{ij},\ (i,j)^k \in U^k)$, $k \in K$ of the system (4.1.1) – (4.1.3):

$$x^k_{\tau\rho} = \sum_{p=1}^{q} \nu_{t,p}\beta_p + \sum_{(i,j)\in U_0} \nu_{t,q+\xi(i,j)}\beta_{q+\xi(i,j)}, \tag{4.5.17}$$

$$t = t(\tau,\rho)^k,\ (\tau,\rho)^k \in U_C^k, k \in K,$$

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U_N^k} x_{\tau\rho}^k \mathrm{sign}(i,j)^{L_{\tau\rho}^k} + \psi_{ij}^k + \tilde{x}_{ij}^k, (i,j)^k \in U_T^k, \tag{4.5.18}$$

$$x_{\tau\rho}^k \in \mathbf{R}\,, (\tau,\rho)^k \in U_N^k, k \in K,$$

where

$$\psi_{ij}^k = \sum_{(\tau,\rho)^k \in U_C^k} x_{\tau\rho}^k \mathrm{sign}(i,j)^{L_{\tau\rho}^k}.$$

Note, the components of the vector $\tilde{x}^k = (\tilde{x}_{ij}^k,\ (i,j)^k \in U^k)$ of a partial solution of the system (4.2.1) for each $k \in K$ are constructed according to the rules in the Remark 4.4.1, i. e. we shall use the formulas (4.4.8).

4.6. Examples of decomposition of linear systems

We perform examples of decomposition of the sparse underdetermined systems by separating the variables according to the sets U_T, U_C and U_N, which consist of the arcs of the support for the network part of the system, cyclic arcs and non-support/non-cyclic arcs respectively; and, finally, sequentially express the unknowns corresponding to the sets U_C and U_T in terms of the independent variables corresponding to the set U_N.

Example 1. Let us consider the example (4.6.1) – (4.6.3) of the system of type (4.1.1) – (4.1.3).

$$\begin{gathered}
x_{12}^1 + x_{13}^1 = 4,\\
x_{23}^1 - x_{12}^1 = 6,\\
-x_{13}^1 - x_{23}^1 = -10,\\
x_{23}^2 + x_{24}^2 = 5,\\
x_{34}^2 - x_{23}^2 - x_{53}^2 = -5,\\
x_{45}^2 - x_{24}^2 - x_{34}^2 = 1,\\
x_{53}^2 - x_{45}^2 = -1,\\
x_{23}^3 + x_{24}^3 = 5,\\
x_{34}^3 - x_{23}^3 - x_{53}^3 = -7,\\
x_{45}^3 - x_{24}^3 - x_{34}^3 = 1,\\
x_{53}^3 - x_{45}^3 = 1,
\end{gathered} \tag{4.6.1}$$

$$\begin{gathered}2x_{12}^1+3x_{13}^1+x_{23}^1+4x_{23}^2+2x_{23}^3+3x_{24}^2-4x_{24}^3+2x_{34}^2+x_{34}^3-\\ -x_{45}^2+7x_{45}^3+x_{53}^2+2x_{53}^3=69,\\ x_{12}^1+2x_{13}^1+2x_{23}^1+5x_{23}^2+3x_{23}^3-x_{24}^2-x_{24}^3+x_{34}^2+x_{34}^3-\\ -2x_{45}^2+3x_{45}^3+2x_{53}^2-x_{53}^3=58,\end{gathered} \tag{4.6.2}$$

$$x_{24}^2+x_{24}^3=1. \tag{4.6.3}$$

We build an example decomposition of the sparse underdetermined system of type (4.1.1) – (4.1.3) for the initial multinetwork $S=(I,U)$, depicted at Figure 4.1, where $I=\{1,2,3,4,5\}$ is the set of nodes and $U=\{(1,2),(1,3),\ (2,3),(2,4),(3,4)\ (4,5),(5,3)\}$ is the set of multiarcs of the multinetwork $S=(I,U)$. In the multinetwork S there are three flow types $k, k\in K$, $K=\{1,2,3\}$. The every set U^k of each network $S^k=\{I^k,U^k\}$ contains arcs: $U^1=\{(1,2)^1,\ (1,3)^1,(2,3)^1\}$, $U^2=\{(2,3)^2,(2,4)^2,(3,4)^2,(4,5)^2,(5,3)^2\}$, $U^3=\{(2,3)^3,(2,4)^3,(3,4)^3,(4,5)^3,(5,3)^3\}$. The initial multinetwork is presented in as a union of networks $S^k=(I^k,U^k), k\in K$ and shown in Figure 4.1.

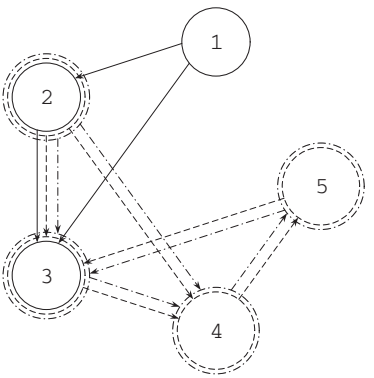

Fig. 4.1. A sample family of network $S^k=(I^k,U^k), k\in K=\{1,2,3\}$

We choose any support U_T for the system (4.6.1) of the multigraph S. By Theorem 4.3.1, we build spanning tree U_T^k of each network $S^k=(I^k,U^k)$, $k\in K=\{1,2,3\}$. Let the support U_T for the system (4.6.1) consists from spanning trees $U_T^k, k\in K=\{1,2,3\}$ where $U_T^1=\{(1,2)^1,(1,3)^1\}$, $U_T^2=$ $=\{(2,3)^2,(2,4)^2,(4,5)^2\}$, $U_T^3=\{(2,4)^3,(3,4)^3,(4,5)^3\}$.

Now, we compute the set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_T^k\}$ of characteristic vectors with respect to the constructed spanning tree U_T^k for each $k \in K$ where $K = \{1,2,3\}$ (see Tables 4.6.1–4.6.3).

Table 4.6.1

The characteristic vector $\delta^1(2,3)$

$(i,j)^1$	$(1,2)^1$	$(1,3)^1$	$(2,3)^1$
$\delta_{ij}^k(\tau,\rho) = \delta_{ij}^1(2,3)$	1	-1	1

Table 4.6.2

The set of characteristic vectors with respect to the spanning tree U_T^2

$(i,j)^2$	$(2,3)^2$	$(2,4)^2$	$(4,5)^2$	$(3,4)^2$	$(5,3)^2$
$\delta_{ij}^k(\tau,\rho) = \delta_{ij}^2(3,4)$	1	-1	0	1	0
$\delta_{ij}^k(\tau,\rho) = \delta_{ij}^2(5,3)$	-1	1	1	0	1

Table 4.6.3

The set of characteristic vectors with respect to the spanning tree U_T^3

$(i,j)^3$	$(2,4)^3$	$(3,4)^3$	$(4,5)^3$	$(2,3)^3$	$(5,3)^3$
$\delta_{ij}^k(\tau,\rho) = \delta_{ij}^3(2,3)$	-1	1	0	1	0
$\delta_{ij}^k(\tau,\rho) = \delta_{ij}^3(5,3)$	0	1	1	0	1

Let's compute the partial solution $\tilde{x}$ of the system (4.6.1) according to the Remark 4.4.1 where $\tilde{x} = \{\tilde{x}^k,\ k \in K\}$, $\tilde{x}^k = (\tilde{x}_{ij}^k,(i,j)^k \in U^k)$, $K = \{1,2,3\}$.

We have

$$\tilde{x}^1 = (\tilde{x}_{12}^1,\ \tilde{x}_{13}^1,\ \tilde{x}_{23}^1) = (-6,\ 10,\ 0),$$

$$\tilde{x}^2 = (\tilde{x}_{23}^2,\ \tilde{x}_{24}^2,\ \tilde{x}_{45}^2,\ \tilde{x}_{34}^2,\ \tilde{x}_{53}^2) = (5,\ 0,\ 1,\ 0,\ 0),$$

$$\tilde{x}^3 = (\tilde{x}^3_{24},\ \tilde{x}^3_{34},\ \tilde{x}^3_{45},\ \tilde{x}^3_{23},\ \tilde{x}^3_{53}) = (5,\ -7,\ -1,\ 0,\ 0).$$

We form the set $U_C = \bigcup\limits_{k=1}^{3} U_C^k = \{(2,3)^1,(3,4)^2,(2,3)^3\}$ of cyclic arcs. The remaining arcs will be included into the set $U_N = \bigcup\limits_{k=1}^{3} U_N^k = \{(5,3)^2,(5,3)^3\}$. Structures representing the union of sets $U_T^k \bigcup U_C^k, k \in K$ are shown in Figure 4.2. With bold arrows we denote the arcs of spanning trees, regular arrows represent cyclic arcs.

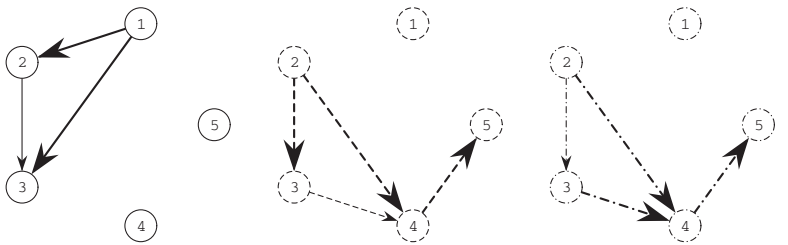

Fig. 4.2. A sample of the sets $U_T^k \bigcup U_C^k$ for networks $S^k, k \in K = \{1,2,3\}$

Using formula (4.5.4) we compute the determinants of the cycles $L^k_{\tau\rho}$, entailed by arcs $(\tau,\rho)^k \in U^k \backslash U_T^k$, for each $k \in K = \{1,2,3\}$, with respect to the equations (4.6.2) with the number $p = 1,2$ (see Table 4.6.4).

Table 4.6.4

Determinants of the cycles $L^k_{\tau\rho}$, , entailed by the arcs $(\tau,\rho)^k \in U^k \backslash U_T^k, k \in K = \{1,2,3\}$

$(\tau,\rho)^k$	$(2,3)^1$	$(3,4)^2$	$(5,3)^2$	$(2,3)^3$	$(5,3)^3$
$R_1(L^k_{\tau\rho})$	0	3	-1	7	10
$R_2(L^k_{\tau\rho})$	1	7	-6	5	3

Now, let's compute the values $\delta_{ij}(L^k_{\tau\rho})$, $(i,j) \in U_0$, $(\tau,\rho)^k \in U^k \setminus U_T^k$, $k \in K = \{1,2,3\}$ according to (4.5.10), for the system (4.6.1) – (4.6.3), $U_0 = = \{(2,4)\}$, $K_0(2,4) = \{2,3\}$ (see Table 4.6.5).

Table 4.6.5

The values $\delta_{ij}(L^k_{\tau\rho}), (i,j) \in U_0, (\tau,\rho)^k \in U^k \setminus U^k_T, k \in K$

$(\tau,\rho)^k$	$(2,3)^1$	$(3,4)^2$	$(5,3)^2$	$(2,3)^3$	$(5,3)^3$
$\delta_{24}(L^k_{\tau\rho})$	0	-1	1	-1	0

We construct the matrix D of the system (4.5.14). For this, we perform the numbering of the arcs the set $U_C = \{(2,3)^1,(3,4)^2,(2,3)^3\}$:

$$t(2,3)^1 = 1,\ t(3,4)^2 = 2,\ t(2,3)^3 = 3.$$

The numbering of the set $U_0 = \{(2,4)\}$ is trivial: $\xi(2,4) = 1$.

We construct the matrix $D_1 = (R_p(L^k_{\tau\rho}), p = \overline{1,2}, t(\tau,\rho)^k = \overline{1,3})$ of the determinants of the cycles $L^k_{\tau\rho}$, entailed by the arcs $(\tau,\rho)^k \in U_C$ by selecting the corresponding columns from the Table 4.6.4: $D_1 = \begin{pmatrix} 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}$.

By analogy, choosing corresponding columns from the Table 4.6.5, we form the matrix $D_2 = (\delta_{24}(L^k_{\tau\rho}), \xi(2,4) = 1, t(\tau,\rho)^k = \overline{1,3})$:

$$D_2 = (0 \quad -1 \quad -1).$$

Thus, joining D_1 and D_2 together, we obtain the matrix D.

$$D = \begin{pmatrix} 0 & 3 & 7 \\ 1 & 7 & 5 \\ 0 & -1 & -1 \end{pmatrix}, \det D \neq 0.$$

Let us compute the vector β in the right hand side of system of type (4.5.14):

$$\beta_1 = A^1 - R_1(L^2_{53})x^2_{53} - R_1(L^3_{53})x^3_{53},$$

$$\beta_2 = A^2 - R_2(L^2_{53})x^2_{53} - R_2(L^3_{53})x^3_{53},$$

$$\beta_3 = A_{24} - \delta_{24}(L^2_{53})x^2_{53} - \delta_{24}(L^3_{53})x^3_{53}.$$

The values $R_p(L^2_{53})$, $R_p(L^3_{53})$, $p = 1,2$, of the determinants of the cycles $L^k_{\tau\rho}$, entailed by the arcs $(\tau,\rho)^k \in U_N$, as well as the values $\delta_{24}(L^2_{53})$,

$\delta_{24}(L^3_{53})$, are already computed and stored within the Tables 4.6.4 and 4.6.5. The numbers A^1,A^2,A_{24} are evaluated using the formulas (4.5.5) and (4.5.12):

$$A^1 = \alpha_1 - \lambda^{11}_{12}\tilde{x}^1_{12} - \lambda^{11}_{13}\tilde{x}^1_{13} - \lambda^{21}_{23}\tilde{x}^2_{23} - \lambda^{21}_{24}\tilde{x}^2_{24} - \lambda^{21}_{45}\tilde{x}^2_{45} - \lambda^{31}_{24}\tilde{x}^3_{24} -$$
$$-\lambda^{31}_{34}\tilde{x}^3_{34} - \lambda^{31}_{45}\tilde{x}^3_{45} = 66,$$

$$A^2 = \alpha_2 - \lambda^{12}_{12}\tilde{x}^1_{12} - \lambda^{12}_{13}\tilde{x}^1_{13} - \lambda^{22}_{23}\tilde{x}^2_{23} - \lambda^{22}_{24}\tilde{x}^2_{24} - \lambda^{22}_{45}\tilde{x}^2_{45} - \lambda^{32}_{24}\tilde{x}^3_{24} -$$
$$-\lambda^{32}_{34}\tilde{x}^3_{34} - \lambda^{32}_{45}\tilde{x}^3_{45} = 36,$$

$$A_{24} = z_{24} - \tilde{x}^2_{24} - \tilde{x}^3_{24} = -4.$$

Thus, we have defined the vector

$$\beta = \begin{pmatrix} 66 + x^2_{53} - 10x^3_{53} \\ 36 + 6x^2_{53} - 3x^3_{53} \\ -4 - x^2_{53} \end{pmatrix}.$$

Since the matrix D turned out to be nonsingular, we can use formula (4.5.16) for finding the components of the vector $x_C = (x^k_{\tau\rho}, (\tau,\rho)^k \in U^k_C, k \in K)$ of solution of the system of type (4.5.14):

$$\begin{pmatrix} x^1_{23} \\ x^2_{34} \\ x^3_{23} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{17}{2} \\ -\frac{1}{4} & 0 & -\frac{7}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 66 + x^2_{53} - 10x^3_{53} \\ 36 + 6x^2_{53} - 3x^3_{53} \\ -4 - x^2_{53} \end{pmatrix}.$$

Finally, using formulas (4.5.17) – (4.5.18), we can define the solution of the system (4.6.1) – (4.6.3) with independent variables x^2_{53}, x^3_{53}:

$$x^1_{23} = 35 - 2x^2_{53} - 8x^3_{53}, \quad x^2_{34} = -\frac{19}{2} + \frac{3}{2}x^2_{53} + \frac{5}{2}x^3_{53},$$

$$x_{23}^3 = \frac{27}{2} - \frac{1}{2}x_{53}^2 - \frac{5}{2}x_{53}^3, \quad x_{12}^1 = 29 - 2x_{53}^2 - 8x_{53}^3,$$
$$x_{13}^1 = -25 + 2x_{53}^2 + 8x_{53}^3,$$

$$x_{23}^2 = -\frac{9}{2} + \frac{1}{2}x_{53}^2 + \frac{5}{2}x_{53}^3,$$
$$x_{24}^2 = \frac{19}{2} - \frac{1}{2}x_{53}^2 - \frac{5}{2}x_{53}^3,$$
$$x_{45}^2 = x_{53}^2 + 1,$$

$$x_{24}^3 = -\frac{17}{2} + \frac{1}{2}x_{53}^2 + \frac{5}{2}x_{53}^3,$$
$$x_{34}^3 = \frac{13}{2} - \frac{1}{2}x_{53}^2 - \frac{3}{2}x_{53}^3,$$
$$x_{45}^3 = x_{53}^3 - 1,$$
$$x_{53}^2, x_{53}^3 \in \mathbf{R}.$$

Example 2. Let us consider the example (4.6.4) − (4.6.6) of construction of sparse underdetermined system of type (4.1.1) − (4.1.3) for the multinetwork $S = (I,U)$, where the set of nodes I is: $I = \{1,2,3,4,5,6\}$ and the set U of the multiarcs is: $U = \{(1,3), (1,4),(1,5),(1,6),(2,1), (2,6),(3,2)(3,4),(3,6),(4,6),(5,2),(5,4),(6,5)\}$. Let $K = \{1,2,3,4,5\}$ be the set of flow kinds. Let

$$U^1 = \{(1,3)^1,(2,1)^1,(3,4)^1,(3,6)^1,(4,6)^1,(5,4)^1,(6,5)^1\},$$

$$U^2 = \{(1,3)^2,(1,4)^2,(1,5)^2,(1,6)^2,(3,2)^2,(3,4)^2,(4,6)^2\},$$

$$U^3 = \{(1,3)^3,(1,5)^3,(2,1)^3, (4,6)^3,(5,2)^3,(5,4)^3,(6,5)^3\},$$

$$U^4 = \{(1,4)^4,(1,5)^4,(1,6)^4,(2,1)^4,(3,2)^4,(3,4)^4,(3,6)^4, (4,6)^4,(5,2)^4,(5,4)^4\},$$

$$U^5 = \{(1,5)^5,(2,1)^5,(2,6)^5, (3,2)^5,(3,4)^5,(4,6)^5,(6,5)^5\},$$

be sets of arcs, carrying the flows of types $k, k \in K, K = \{1,2,3,4,5\}$.

$$
\begin{gathered}
x^1_{13} - x^1_{21} = -6 \\
x^1_{21} = 10 \\
x^1_{34} + x^1_{36} - x^1_{13} = 6 \\
x^1_{46} - x^1_{34} - x^1_{54} = -6 \\
x^1_{54} - x^1_{65} = 2 \\
x^1_{65} - x^1_{36} - x^1_{46} = -6
\end{gathered}
$$

$$
\begin{gathered}
x^2_{13} + x^2_{14} + x^2_{15} + x^2_{16} = 14 \\
-x^2_{32} = -8 \\
x^2_{32} + x^2_{34} - x^2_{13} = 7 \\
x^2_{46} - x^2_{14} - x^2_{34} = -3 \\
-x^2_{15} = -4 \\
-x^2_{16} - x^2_{46} = -6
\end{gathered}
$$

$$
\begin{gathered}
x^3_{13} + x^3_{15} - x^3_{21} = 8 \\
x^3_{21} - x^3_{52} = -6 \\
-x^3_{13} = -7 \\
x^3_{46} - x^3_{54} = 7 \\
x^3_{52} + x^3_{54} - x^3_{15} - x^3_{65} = 3 \\
x^3_{65} - x^3_{46} = -5
\end{gathered}
\tag{4.6.4}
$$

$$
\begin{gathered}
x^4_{14} + x^4_{15} + x^4_{16} - x^4_{21} = 16 \\
x^4_{21} - x^4_{32} - x^4_{52} = -8 \\
x^4_{32} + x^4_{34} + x^4_{36} = 7 \\
x^4_{46} - x^4_{14} - x^4_{34} - x^4_{54} = 2 \\
x^4_{52} + x^4_{54} - x^4_{15} = -2 \\
-x^4_{16} - x^4_{36} - x^4_{46} = -15
\end{gathered}
$$

$$
\begin{gathered}
x^5_{15} - x^5_{21} = -5 \\
x^5_{21} + x^5_{26} - x^5_{32} = 4 \\
x^5_{32} + x^5_{34} = 9 \\
x^5_{46} - x^5_{34} = -1 \\
-x^5_{15} - x^5_{65} = -4 \\
x^5_{65} - x^5_{26} - x^5_{46} = -3
\end{gathered}
$$

$$
\begin{gathered}
8x^1_{13} + 9x^2_{13} + x^3_{13} + 9x^2_{14} + +5x^4_{14} + 7x^2_{15} + 5x^3_{15} + 7x^4_{15} + \\
+3x^5_{15} + 2x^4_{16} + 9x^1_{21} + 9x^3_{21} + 2x^4_{21} + 2x^5_{21} + 6x^5_{26} + 2x^5_{32} + \\
+8x^1_{34} + 6x^2_{34} + 2x^4_{34} + 5x^5_{34} + \\
+6x^1_{36} + 3x^1_{46} + 7x^2_{46} + 8x^3_{46} + 8x^4_{46} + 2x^5_{46} + \\
+8x^3_{52} + 9x^4_{52} + 4x^3_{54} + 6x^4_{54} + 9x^1_{65} + 6x^3_{65} + 7x^5_{65} = 948 \\
9x^1_{13} + 5x^2_{13} + 10x^3_{13} + 5x^2_{14} + 9x^4_{14} + 5x^2_{15} + 3x^3_{15} + 7x^4_{15} + \\
+4x^2_{16} + 10x^4_{16} + 2x^1_{21} + 5x^3_{21} + 6x^4_{21} + 2x^5_{21} + 4x^5_{26} + 9x^2_{32} + \\
+7x^5_{32} + 7x^1_{34} + 2x^2_{34} + 7x^4_{34} + x^5_{34} + 10x^1_{36} + 6x^4_{36} + 4x^1_{46} + \\
+7x^2_{46} + 6x^3_{46} + 6x^4_{46} + 2x^3_{52} + 9x^4_{52} + 3x^1_{54} + 6x^3_{54} + 10x^4_{54} + \\
+7x^3_{65} + 6x^5_{65} = 964
\end{gathered}
\tag{4.6.5}
$$

$$
\begin{gathered}
x^2_{13} + x^3_{13} = 13 \\
x^2_{15} + x^3_{15} + x^5_{15} = 8 \\
x^2_{16} + x^4_{16} = 7
\end{gathered}
\tag{4.6.6}
$$

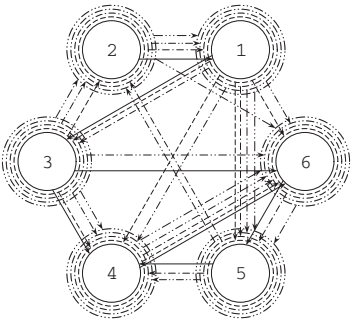

Fig. 4.3. The union of networks $S^k = (I^k, U^k), k \in K = \{1,2,3,4,5\}$

Let's consider a multigraph $S = (I,U)$ (see Figure 4.3) which is represented in Figures 4.4 – 4.6 as the set of the graphs.

We choose a support of the multinetwork $S = (I,U)$ for the system (4.6.4). By Theorem 4.3.1, the chosen support consist of spanning trees $U^k_T, k \in K = \{1,2,3,4.5\}$:

$$U^1_T = \{(1,3)^1, (2,1)^1, (3,4)^1, (4,6)^1, (6,5)^1\},$$

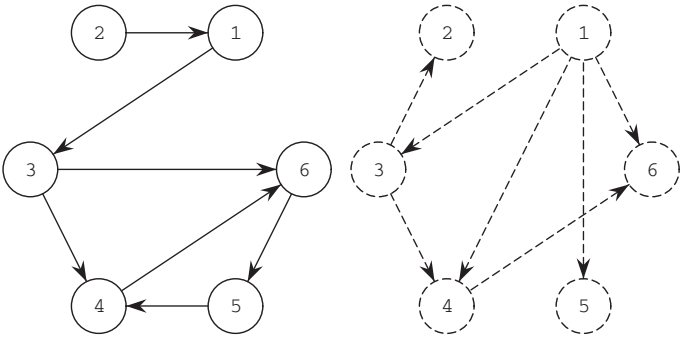

Fig. 4.4. The graphs G^1, G^2

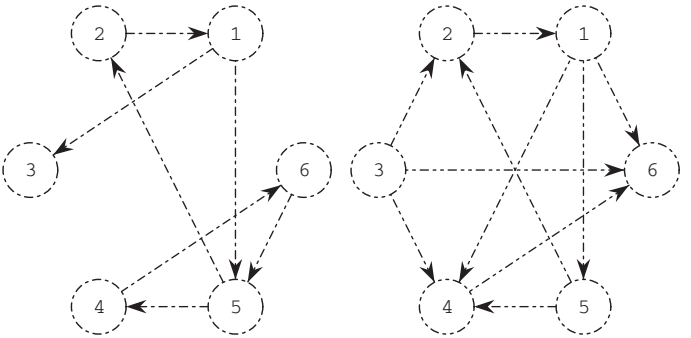

Fig. 4.5. The graphs G^3, G^4

$$U_T^2 = \{(1,3)^2,(1,5)^2,(3,2)^2,(3,4)^2,(4,6)^2\},$$

$$U_T^3 = \{(1,3)^3,(1,5)^3,(4,6)^3,(5,2)^3,(5,4)^3\},$$

$$U_T^4 = \{(1,4)^4,(3,2)^4,(3,6)^4,(4,6)^4,(5,2)^4\},$$

$$U_T^5 = \{(1,5)^5,(2,6)^5,(3,2)^5,(3,4)^5,(6,5)^5\}.$$

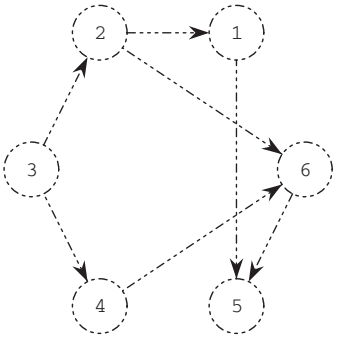

Fig. 4.6. The graph G^5

We build the set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_T^k\}$ of characteristic vectors with respect to the chosen spanning tree U_T^k for every $k \in K = \{1,2,3,4,5\}$. The set of characteristic vectors is presented in the Tables 4.6.6 – 4.6.10.

Table 4.6.6

Characteristic vectors with respect to the spanning tree U_T^1

$(i,j)^1$	$(1,3)^1$	$(2,1)^1$	$(3,4)^1$	$(3,6)^1$	$(4,6)^1$	$(5,4)^1$	$(6,5)^1$
$\delta_{ij}^1(3,6)$	0	0	1	1	-1	0	0
$\delta_{ij}^1(5,4)$	0	0	0	0	1	1	1

Let's build the partial solution of the system (4.6.4) for every $k \in K$ where $K = \{1,2,3,4,5\}$ according to the rules of Remark 4.4.1.

$$\tilde{x}^1 = (\tilde{x}_{13}^1, \tilde{x}_{21}^1, \tilde{x}_{34}^1, \tilde{x}_{36}^1, \tilde{x}_{46}^1, \tilde{x}_{54}^1, \tilde{x}_{65}^1) = (4, 10, 10, 0, 4, 0, -2);$$
$$\tilde{x}^2 = (\tilde{x}_{13}^2, \tilde{x}_{14}^2, \tilde{x}_{15}^2, \tilde{x}_{16}^2, \tilde{x}_{32}^2, \tilde{x}_{34}^2, \tilde{x}_{46}^2) = (10, 0, 4, 0, 8, 9, 6);$$
$$\tilde{x}^3 = (\tilde{x}_{13}^3, \tilde{x}_{15}^3, \tilde{x}_{21}^3, \tilde{x}_{46}^3, \tilde{x}_{52}^3, \tilde{x}_{54}^3, \tilde{x}_{65}^3) = (7, 1, 0, 5, 6, -2, 0);$$
$$\tilde{x}^4 = (\tilde{x}_{14}^4, \tilde{x}_{15}^4, \tilde{x}_{16}^4, \tilde{x}_{21}^4, \tilde{x}_{32}^4, \tilde{x}_{34}^4, \tilde{x}_{36}^4, \tilde{x}_{46}^4, \tilde{x}_{52}^4, \tilde{x}_{54}^4) =$$
$$= (16, 0, 0, 0, 10, 0, -3, 18, -2, 0);$$
$$\tilde{x}^5 = (\tilde{x}_{15}^5, \tilde{x}_{21}^5, \tilde{x}_{26}^5, \tilde{x}_{32}^5, \tilde{x}_{34}^5, \tilde{x}_{46}^5, \tilde{x}_{65}^5) = (5, 0, 12, 8, 1, 0, 9).$$

Table 4.6.7

Characteristic vectors with respect to the spanning tree U_T^2

$(i,j)^2$	$(1,3)^2$	$(1,4)^2$	$(1,5)^2$	$(1,6)^2$	$(3,2)^2$	$(3,4)^2$	$(4,6)^2$
$\delta_{ij}^2(1,4)$	-1	1	0	0	0	-1	0
$\delta_{ij}^2(1,6)$	-1	0	0	1	0	-1	-1

Table 4.6.8

Characteristic vectors with respect to the spanning tree U_T^3

$(i,j)^3$	$(1,3)^3$	$(1,5)^3$	$(2,1)^3$	$(4,6)^3$	$(5,2)^3$	$(5,4)^3$	$(6,5)^3$
$\delta_{ij}^3(2,1)$	0	1	1	0	1	0	0
$\delta_{ij}^3(6,5)$	0	0	0	1	0	1	1

We form the set of cyclic arcs U_C :

$$U_C = \bigcup_{k=1}^{5} U_C^k = \{(3,6)^1,(5,4)^1,(1,4)^2,(1,6)^2,(2,1)^3\}.$$

The structures, representing the union of the sets $U_T^k \cup U_C^k$ are shown at the Figures 4.7, 4.8 where $k \in K = \{1,2,3,4,5\}$.

Let's build the set U_N :

$$U_N = \bigcup_{k=1}^{5} U_N^k = \{(6,5)^3,(1,5)^4, (1,6)^4,(2,1)^4,(3,4)^4,(5,4)^4,(2,1)^5,(4,6)^5\}.$$

According to (4.5.4) we compute determinants of the cycles $L_{\tau\rho}^k$, entailed by the arcs $(\tau,\rho)^k \in U^k \backslash U_T^k$ for every $k \in K = \{1,2,3,4,5\}$, with respect to the equations (4.6.5) with the numbers $p = 1,2$ (see Tables 4.6.11 and 4.6.12).

According to the formula (4.5.10) we compute the values $\delta_{ij}(L_{\tau\rho}^k), (i,j) \in U_0, (\tau,\rho)^k \in U^k \setminus U_T^k, k \in K = \{1,2,3,4,5\}$, $U_0 = \{(1,3),(1,5),(1,6)\}$, $K_0(1,3) = \{2,3\}$, $K_0(1,5) = \{2,3,5\}$, $K_0(1,6) = \{2,4\}$ (see Tables 4.6.13, 4.6.14).

Table 4.6.9

Characteristic vectors with respect to the spanning tree U_T^4

$(i,j)^4$	$(1,4)^4$	$(1,5)^4$	$(1,6)^4$	$(2,1)^4$	$(3,2)^4$
$\delta_{ij}^4(1,5)$	-1	1	0	0	-1
$\delta_{ij}^4(1,6)$	-1	0	1	0	0
$\delta_{ij}^4(2,1)$	1	0	0	1	1
$\delta_{ij}^4(3,4)$	0	0	0	0	0
$\delta_{ij}^4(5,4)$	0	0	0	0	1
$(i,j)^4$	$(3,4)^4$	$(3,6)^4$	$(4,6)^4$	$(5,2)^4$	$(5,4)^4$
$\delta_{ij}^4(1,5)$	0	1	-1	1	0
$\delta_{ij}^4(1,6)$	0	0	-1	0	0
$\delta_{ij}^4(2,1)$	0	-1	1	0	0
$\delta_{ij}^4(3,4)$	1	-1	1	0	0
$\delta_{ij}^4(5,4)$	0	-1	1	-1	1

In order to build the matrix D of the system (4.5.14) we enumerate arcs of the set $U_C = \{(3,6)^1,(5,4)^1,(1,4)^2,(1,6)^2,(2,1)^3\}$:

$$t(3,6)^1 = 1, t(5,4)^1 = 2, t(1,4)^2 = 3, t(1,6)^2 = 4, t(2,1)^3 = 5.$$

Also we enumerate multiarcs of the set $U_0 = \{(1,3),(1,5),(1,6)\}$:

$$\xi(1,3) = 1, \xi(1,5) = 2, \xi(1,6) = 3.$$

Let's form the matrix $D_1 = (R_p(L_{\tau\rho}^k), p = \overline{1,2}, t(\tau,\rho)^k = \overline{1,5})$ of determinants of the cycles $L_{\tau\rho}^k$ entailed by the arcs $(\tau,\rho)^k \in U_C$:

$$D_1 = \begin{pmatrix} R_1(L_{36}^1) & R_1(L_{54}^1) & R_1(L_{14}^2) & R_1(L_{16}^2) & R_1(L_{21}^3) \\ R_2(L_{36}^1) & R_2(L_{54}^1) & R_2(L_{14}^2) & R_1(L_{16}^2) & R_1(L_{21}^3) \end{pmatrix} =$$

$$= \begin{pmatrix} -5 & 12 & -6 & -22 & 22 \\ -1 & 7 & -2 & -10 & 10 \end{pmatrix}$$

Table 4.6.10

Characteristic vectors with respect to the spanning tree U_T^5

$(i,j)^5$	$(1,5)^5$	$(2,1)^5$	$(2,6)^5$	$(3,2)^5$	$(3,4)^5$	$(4,6)^5$	$(6,5)^5$
$\delta_{ij}^5(2,1)$	1	1	-1	0	0	0	-1
$\delta_{ij}^2(4,6)$	0	0	-1	-1	1	1	0

Table 4.6.11

Determinants of the cycles $L_{\tau\rho}^k$, $(\tau,\rho)^k \in U_C^k, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(3,6)^1$	$(5,4)^1$	$(1,4)^2$	$(1,6)^2$	$(2,1)^3$
$R_1(L_{\tau\rho}^k)$	-5	12	-6	-22	22
$R_2(L_{\tau\rho}^k)$	-1	7	-2	-10	10

Now we form the matrix D_2 :

$$D_2 = \begin{pmatrix} \delta_{13}(L_{36}^1) & \delta_{13}(L_{54}^1) & \delta_{13}(L_{14}^2) & \delta_{13}(L_{16}^2) & \delta_{13}(L_{21}^3) \\ \delta_{15}(L_{36}^1) & \delta_{15}(L_{54}^1) & \delta_{15}(L_{14}^2) & \delta_{15}(L_{16}^2) & \delta_{15}(L_{21}^3) \\ \delta_{16}(L_{36}^1) & \delta_{16}(L_{54}^1) & \delta_{16}(L_{14}^2) & \delta_{16}(L_{16}^2) & \delta_{16}(L_{21}^3) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus by joining D_1 and D_2 we obtain the matrix of the system (4.5.14):

$$D = \begin{pmatrix} -5 & 12 & -6 & -22 & 22 \\ -1 & 7 & -2 & -10 & 10 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \det D \neq 0.$$

Let's compute the numbers $A^1, A^2, A_{13}, A_{15}, A_{16}$ by using the formulas (4.5.5) and (4.5.12):

$$A^1 = 99; A^2 = 142; A_{13} = -4; A_{15} = 8; A_{16} = 7.$$

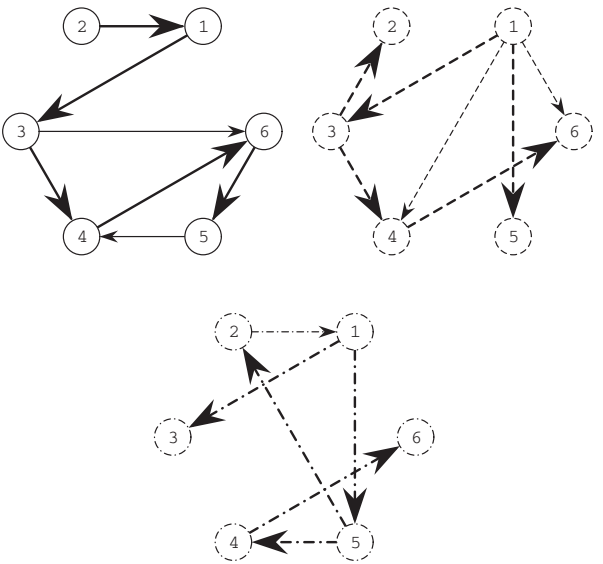

Fig. 4.7. The sets $U_T^k \bigcup U_C^k$ of the networks S^k, $k = \{1,2,3\}$

Now we compute the vector β of the right-side part of the system (4.5.14):

$$\beta = \begin{pmatrix} 99 - 18x_{65}^3 - 3x_{15}^4 + 11x_{16}^4 - 15x_{21}^4 - 10x_{34}^4 - 5x_{54}^4 + 8x_{21}^5 + x_{46}^5 \\ 142 - 19x_{65}^3 - 7x_{15}^4 + 5x_{16}^4 - 15x_{21}^4 - 7x_{34}^4 - x_{54}^4 + 8x_{21}^5 + 10x_{46}^5 \\ -4 \\ 8 - x_{21}^5 \\ 7 - x_{16}^4 \end{pmatrix}.$$

Since the matrix D is nonsingular we apply the formula (4.5.16) for obtaining the components of the vector $x_C = (x_{\tau\rho,(\tau,\rho)}^k)^k \in U_C^k, k \in K)$ of the system (4.5.14):

$$x_{36}^1 = \frac{1}{23}(1099 - 102x_{65}^3 - 63x_{15}^4 - x_{16}^4 - 75x_{21}^4 - 14x_{34}^4 + 23x_{54}^4 + 6x_{21}^5 + 113x_{46}^5),$$

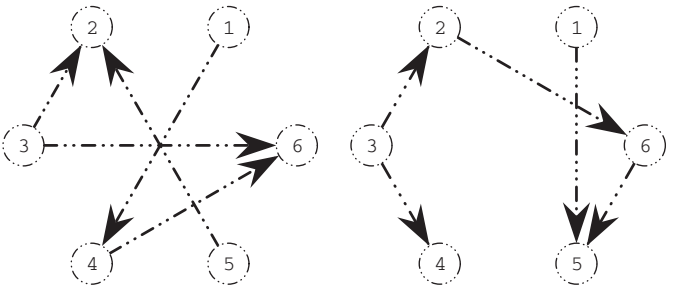

Fig. 4.8. The sets $U_T^k \bigcup U_C^k$ of the networks S^k, $k = \{4,5\}$

Table 4.6.12

Determinants of the cycles $L^k_{\tau\rho}$, $(\tau,\rho)^k \in U_C^k, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(6,5)^3$	$(1,5)^4$	$(1,6)^4$	$(2,1)^4$	$(3,4)^4$	$(5,4)^4$	$(2,1)^5$	$(4,6)^5$
$R_1(L^k_{\tau\rho})$	185	3	-11	15	10	5	-8	-1
$R_1(L^k_{\tau\rho})$	19	7	-5	15	7	1	-8	-10

$$x^1_{54} = \frac{1}{23}(571 - 77x^3_{65} - 32x^4_{15} - 10x^4_{16} - 60x^4_{21} - 25x^4_{34} + 60x^5_{21} + 49x^5_{46}),$$

$$x^2_{14} = -3 + x^4_{16},$$
$$x^2_{16} = 7 - x^4_{16},$$
$$x^3_{21} = 8 - x^5_{21}.$$

Components of the vector $x_T = (x^k_{\tau\rho}, (\tau,\rho)^k \in U_T^k, k \in K)$, corresponding to arcs of the set $U_T = \bigcup\limits_{k=1}^{5} U_T^k$, are equal to

$$x^1_{21} = 10,$$

$$x^1_{13} = 4,$$

$$x^1_{34} = \frac{1}{23}(-869+102x^3_{65}+63x^4_{15}+x^4_{16}+75x^4_{21}+14x^4_{34}-23x^4_{54}-6x^5_{21}-113x^5_{46}),$$

Table 4.6.13

Values of $\delta_{ij}(L^k_{\tau\rho}), (i,j) \in U_0, (\tau,\rho)^k \in U^k_C, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(3,6)^1$	$(5,4)^1$	$(1,4)^2$	$(1,6)^2$	$(2,1)^3$
$\delta_{13}(L^k_{\tau\rho})$	0	0	-1	-1	0
$\delta_{15}(L^k_{\tau\rho})$	0	0	0	0	1
$\delta_{16}(L^k_{\tau\rho})$	0	0	0	1	0

Table 4.6.14

Values of $\delta_{ij}(L^k_{\tau\rho}), (i,j) \in U_0, (\tau,\rho)^k \in U^k_C, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(6,5)^3$	$(1,5)^4$	$(1,6)^4$	$(2,1)^4$	$(3,4)^4$	$(5,4)^4$	$(2,1)^5$	$(4,6)^5$
$\delta_{13}(L^k_{\tau\rho})$	0	0	0	0	0	0	0	0
$\delta_{15}(L^k_{\tau\rho})$	0	0	0	0	0	0	1	0
$\delta_{16}(L^k_{\tau\rho})$	0	0	1	0	0	0	0	0

$$x^1_{46} = \frac{1}{23}(-436+25x^3_{65}+31x^4_{15}-9x^4_{16}+15x^4_{21}-11x^4_{34}-23x^4_{54}+54x^5_{21}-64x^5_{46}),$$
$$x^1_{65} = \frac{1}{23}(525 - 77x^3_{65} - 32x^4_{15} - 10x^4_{16} - 60x^4_{21} - 25x^4_{34} + 60x^5_{21} + 49x^5_{46});$$

$$x^2_{13} = 6, x^2_{15} = 4, x^2_{32} = 8, x^2_{34} = 5, x^2_{46} = -1 + x^2_{16};$$

$$x^3_{13} = 7, x^3_{15} = 9 - x^5_{21}, x^3_{46} = 5 + x^3_{65},$$
$$x^3_{52} = 14 - x^5_{21}, x^3_{54} = -2 + x^3_{65};$$

$$x^4_{14} = 16 - x^4_{15} - x^4_{16} + x^4_{21},$$
$$x^4_{32} = 10 - x^4_{15} + x^4_{21} + x^4_{54},$$
$$x^4_{36} = -3 + x^4_{15} - x^4_{21} - x^4_{34} - x^4_{54},$$
$$x^4_{46} = 18 - x^4_{15} - x^4_{16} + x^4_{21} + x^4_{34} + x^4_{54},$$
$$x^4_{52} = -2 + x^4_{15} - x^4_{54};$$

$$x_{15}^5 = -5 + x_{21}^5,$$
$$x_{26}^5 = 12 - x_{21}^5 - x_{46}^5,$$
$$x_{32}^5 = 8 - x_{46}^5,$$
$$x_{34}^5 = 1 + x_{46}^5,$$
$$x_{65}^5 = 9 - x_{21}^5,$$
$$x_{ij}^k \in \mathbf{R}, (i,j)^k \in U_N = \bigcup_{k=1}^{5} U_N^k =$$
$$= \{(6,5)^3,(1,5)^4,(1,6)^4,(2,1)^4,(3,4)^4,(5,4)^4,(2,1)^5,(4,6)^5\}.$$

4.7. Implementation in Wolfram Mathematica

For the multinetwork $S = (I,U)$, $I = \{1,2,3,4,5\}$, $U = \{(1,2),(2,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,4)\}$ we consider sparse underdetermined system of linear algebraic equations of type (4.1.1) − (4.1.3) which has the form (4.7.1) − (4.7.3). The multinetwork $S = (I,U)$ is presented in the form of three connected networks $S^k = (I^k,U^k)$, $k \in K$ where $K = \{1,2,3\}$ (see Figure 4.9).

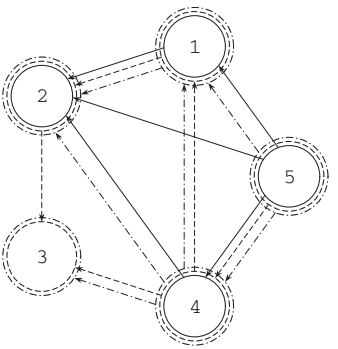

Fig. 4.9. The union of networks $S^k = (I^k,U^k)$, $k \in K = \{1,2,3\}$

The sets of nodes I^1,I^2,I^3 of the networks S^1, S^2, S^3 respectively, are defined as follows:

$$I^1 = \{1, 2, 4, 5\},\ I^2 = \{1, 2, 3, 4, 5\},\ I^3 = \{1, 2, 3, 4, 5\},$$

The sets of arcs U^1,U^2,U^3 of the networks S^1, S^2, S^3 respectively are defined as follows:

$$U^1 = \{(1,2)^1,(4,2)^1,(5,1)^1,(5,2)^1,(5,4)^1\},$$

$$U^2 = \{(1,2)^2, (2,3)^2, (4,1)^2, (4,3)^2, (5,4)^2\},$$
$$U^3 = \{(1,2)^3, (4,1)^3, (4,2)^3, (5,1)^3, (5,4)^3\}.$$

The sparse system of type (4.1.1) has the form:

$$\begin{gathered}
x^1_{1,2} - x^1_{5,1} = 3 \\
-x^1_{1,2} - x^1_{4,2} - x^1_{5,2} = -15 \\
x^1_{4,2} - x^1_{5,4} = -8 \\
x^1_{5,1} + x^1_{5,2} + x^1_{5,4} = 20
\end{gathered}$$

$$\begin{gathered}
x^2_{1,2} - x^2_{4,1} = 0 \\
x^2_{2,3} - x^2_{1,2} = -6 \\
-x^2_{2,3} - x^2_{4,3} = -5 \\
x^2_{4,1} + x^2_{4,3} - x^2_{5,4} = 7 \\
x^2_{5,4} = 4
\end{gathered} \tag{4.7.1}$$

$$\begin{gathered}
x^3_{1,2} - x^3_{4,1} - x^3_{5,1} = -10 \\
-x^3_{1,2} - x^3_{4,2} = -10 \\
-x^3_{4,3} = -7 \\
x^3_{4,1} + x^3_{4,2} + x^3_{4,3} - x^3_{5,4} = 9 \\
x^3_{5,1} + x^3_{5,4} = 18
\end{gathered}$$

The supporting set of arcs $U^k_T \cup U^k_C$, $k \in K = \{1,2,3\}$ of multinetwork S for the system (4.7.1) – (4.7.3) is shown in Figure 4.10, where $U_T = \bigcup\limits_{k=1}^{3} U^k_T$, $U^1_T = \{(1,2)^1, (4,2)^1, (5,4)^1\}$, $U^2_T = \{(1,2)^2, (2,3)^2, (4,3)^2, (5,4)^2\}$, $U^3_T = \{(1,2)^3, (4,2)^3, (4,3)^3, (5,4)^3\}$ are the sets of arcs of spanning trees U^1_T,

U_T^2, U_T^3 respectively, $U_C = \bigcup_{k=1}^{3} U_C^k$, $U_C^1 = \{(5,1)^1, (5,2)^1\}$, $U_C^2 = \{(4,1)^2\}$, $U_C^3 = \{(4,1)^3\}$ are the sets of the cyclic arcs U_C^1, U_C^2, U_C^3 respectively.

$$\begin{gathered}
7x_{1,2}^1 + 4x_{1,2}^2 + 2x_{1,2}^3 + 8x_{2,3}^2 + 9x_{4,1}^2 + 9x_{4,1}^3 + \\
+9x_{4,2}^1 + 4x_{4,2}^3 + 2x_{4,3}^2 + 5x_{4,3}^3 + 9x_{5,1}^1 + 9x_{5,1}^3 + \\
+5x_{5,2}^1 + 9x_{5,4}^1 + 3x_{5,4}^2 + 7x_{5,4}^3 = 556 \\
\\
6x_{1,2}^1 + 7x_{1,2}^2 + 6x_{1,2}^3 + 3x_{2,3}^2 + 7x_{4,1}^2 + \\
+5x_{4,2}^1 + 2x_{4,2}^3 + 2x_{4,3}^3 + 10x_{5,1}^1 + 7x_{5,1}^3 + \\
+8x_{5,2}^1 + 9x_{5,4}^1 + 5x_{5,4}^2 + 7x_{5,4}^3 = 501 \\
\\
2x_{1,2}^1 + x_{1,2}^2 + 5x_{1,2}^3 + 5x_{2,3}^2 + 2x_{4,1}^2 + \\
+6x_{4,1}^3 + 3x_{4,2}^1 + 3x_{4,3}^2 + 2x_{4,3}^3 + 8x_{5,1}^1 + \\
+x_{5,1}^3 + 8x_{5,2}^1 + 2x_{5,4}^1 + 4x_{5,4}^2 + 8x_{5,4}^3 = 307
\end{gathered} \tag{4.7.2}$$

$$x_{5,4}^1 + x_{5,4}^3 = 13 \tag{4.7.3}$$

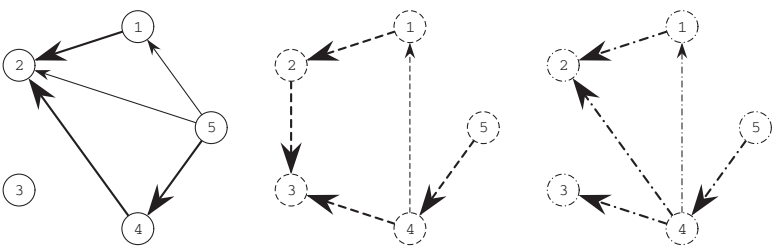

Fig. 4.10. The supporting set of arcs $U_T^k \cup U_C^k$, $k \in K = \{1,2,3\}$

We construct the general solution of the system (4.7.1) − (4.7.3) with respect to a support of the multinetwork $S = (I,U)$ (see Figure 4.10) for

the system (4.7.1) – (4.7.3) using the algorithms of decomposition of the support of multigraph (multinetwork) $S = (I,U)$. For this perform the following steps:

- *Step 1.* Select the data structures to represent the support of the network part of sparse system (4.7.1). For the representation of spanning trees $U_T^k, k \in K = \{1,2,3\}$ we use corresponding rooted trees [2, 53]. Construct a system of characteristic vectors with respect to the support $U_T^k, k \in K = \{1,2,3\}$ of multinetwork $S = (I,U)$ for system (4.7.1) (see Figure 4.10). The number of operations to compute each characteristic vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k)$ is proportional to $|U^k|, (\tau,\rho)^k \in U^k \backslash U_T^k, k \in K = \{1,2,3\}$ in the worst case.

- *Step 2.* Compute the partial solution of the system (4.7.1) with block-diagonal matrix A. We compute the partial solution according to the rules of Remark 4.4.1. The number of operations for computation of the partial solution of the system (4.7.1) for each block of the size $|I^k| \times |U^k|$, which is represented by the incidence matrix of the network $S^k = (I^k,U^k)$ is equal to $O(|U^k|), k \in K = \{1,2,3\}$ in the worst case.

- *Step 3.* Form the matrix of determinants and the vector β of the right-hand side of the system (4.5.14). Compute the unknowns of the system (4.7.1) – (4.7.3), which correspond to the cyclic arcs: $U_C = \bigcup_{k=1}^{3} U_C^k$, $U_C^1 = \{(5,1)^1, (5,2)^1\}$, $U_C^2 = \{(4,1)^2\}$, $U_C^3 = \{(4,1)^3\}$, using the principle of decomposition.

- *Step 4.* Compute the remaining unknowns of the linear system (4.7.1) – (4.7.3), which correspond to the arcs $U_T = \bigcup_{k=1}^{3} U_T^k$ of spanning trees $U_T^1 = \{(1,2)^1, (4,2)^1, (5,4)^1\}$, $U_T^2 = \{(1,2)^2, (2,3)^2, (4,3)^2, (5,4)^2\}$, $U_T^3 = \{(1,2)^3, (4,2)^3, (4,3)^3, (5,4)^3\}$, using the graph theoretical properties of the support of the multinetwork $S = (I,U)$ for the system (4.7.1).

- *Step 5.* Test the obtained rational solution of the underdetermined sparse system of linear algebraic equations (4.7.1) – (4.7.3) using the built-in *Simplify* function of CAS *Wolfram Mathematica*.

In the code in Listing 1 we presented an implementation in CAS *Wolfram Mathematica* of decomposition algorithms for solution of the sparse underdetermined system (4.7.1) – (4.7.3). The support is presented in Figure 4.10. The computations is performed in a rational arithmetic.

Listing 1

```
System = {
x1_{1,2} − x1_{5,1} == 3,
−x1_{1,2} − x1_{5,2} − x1_{4,2} == −15,

x1_{4,2} − x1_{5,4} == −8,
x1_{5,1} + x1_{5,4} + x1_{5,2} == 20,

x2_{1,2} − x2_{4,1} == 0,
−x2_{1,2} + x2_{2,3} == −6,
−x2_{2,3} − x2_{4,3} == −5,
x2_{4,1} + x2_{4,3} − x2_{5,4} == 7,
x2_{5,4} == 4,

x3_{1,2} − x3_{4,1} − x3_{5,1} == −10,
−x3_{1,2} − x3_{4,2} == −10,
−x3_{4,3} == −7,
x3_{4,1} + x3_{4,3} + x3_{4,2} − x3_{5,4} == 9,
x3_{5,1} + x3_{5,4} == 18,

7x1_{1,2} + 4x2_{1,2} + 2x3_{1,2} + 8x2_{2,3} + 9x2_{4,1} + 9x3_{4,1} + 9x1_{4,2} + 4x3_{4,2}+
+2x2_{4,3} + 5x3_{4,3} + 9x1_{5,1} + 9x3_{5,1} + 5x1_{5,2} + 9x1_{5,4} + 3x2_{5,4} + 7x3_{5,4} == 556,


6x1_{1,2} + 7x2_{1,2} + 6x3_{1,2} + 3x2_{2,3} + 7x2_{4,1} + 5x1_{4,2} + 2x3_{4,2} + 2x3_{4,3}+
+10x1_{5,1} + 7x3_{5,1} + 8x1_{5,2} + 9x1_{5,4} + 5x2_{5,4} + 7x3_{5,4} == 501,

2x1_{1,2} + x2_{1,2} + 5x3_{1,2} + 5x2_{2,3} + 2x2_{4,1} + 6x3_{4,1} + 3x1_{4,2} + 3x2_{4,3} + 2x3_{4,3}+

+8x1_{5,1} + x3_{5,1} + 8x1_{5,2} + 2x1_{5,4} + 4x2_{5,4} + 8x3_{5,4} == 307,

x1_{5,4} + x3_{5,4} == 13
};
```

```
system1 = {x1_{1,2} - x1_{5,1} == 0,
-x1_{1,2} - x1_{5,2} - x1_{4,2} == 0,
0 == 0,
x1_{4,2} - x1_{5,4} == 0,
x1_{5,1} + x1_{5,4} + x1_{5,2} == 0
};
(*thread*)
t = {5,4,2,1};
(*pred*)
p = {0,1,0,2,4};
(*dir*)
d = {0,1,0, - 1, - 1};
system1a = system1;
system1a[[5]] = system1a[[5]]/. {x1_{5,1} → 1} ;
system1a[[1]] = system1a[[1]]/. {x1_{5,1} → 1} ;
system1a[[5]] = system1a[[5]]/. {x1_{5,2} → 0} ;
system1a[[2]] = system1a[[2]]/. {x1_{5,2} → 0} ;
δ1_{5,1} = {x1_{5,1} → 1,x1_{5,2} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1a[[t[[i]]]],x1_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system1a[[t[[i]]]],x1_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system1a[[p[[t[[i]]]]]] = system1a[[p[[t[[i]]]]]]/.δ];
δ1_{5,1} = Join [δ1_{5,1},δ] ;
}];


system1b = system1;
system1b[[5]] = system1b[[5]]/. {x1_{5,1} → 0} ;
system1b[[1]] = system1b[[1]]/. {x1_{5,1} → 0} ;
system1b[[5]] = system1b[[5]]/. {x1_{5,2} → 1} ;
system1b[[2]] = system1b[[2]]/. {x1_{5,2} → 1} ;
δ1_{5,2} = {x1_{5,1} → 0,x1_{5,2} → 1} ;

For[i = 1,i ⩽ 3,++i,
```

```
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1b[[t[[i]]]],x1_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system1b[[t[[i]]]],x1_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system1b[[p[[t[[i]]]]]] = system1b[[p[[t[[i]]]]]]/.δ];
δ1_{5,2} = Join [δ1_{5,2},δ] ;
}];
system2 = {x2_{1,2} − x2_{4,1} == 0,
−x2_{1,2} + x2_{2,3} == 0,
−x2_{2,3} − x2_{4,3} == 0,
x2_{4,1} + x2_{4,3} − x2_{5,4} == 0,
x2_{5,4} == 0
};
(*thread*)
t = {5,4,3,2,1};
(*pred*)
p = {0,1,2,3,4};
(*dir*)
d = {0,1,1, − 1, − 1};
system2a = system2;
system2a[[4]] = system2a[[4]]/. {x2_{4,1} → 1} ;
system2a[[1]] = system2a[[1]]/. {x2_{4,1} → 1} ;
δ2_{4,1} = {x2_{4,1} → 1} ;

For[i = 1,i ⩽ 4,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system2a[[t[[i]]]],x2_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system2a[[t[[i]]]],x2_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system2a[[p[[t[[i]]]]]] = system2a[[p[[t[[i]]]]]]/.δ];
δ2_{4,1} = Join [δ2_{4,1},δ] ;
}];

system3 = {x3_{1,2} − x3_{4,1} − x3_{5,1} == 0,
−x3_{1,2} − x3_{4,2} == 0,
−x3_{4,3} == 0,
```

```
x3_{4,1} + x3_{4,3} + x3_{4,2} - x3_{5,4} == 0,
x3_{5,1} + x3_{5,4} == 0
};
(*thread*)
t = {5,3,4,2,1};
(*pred*)
p = {0,1,4,2,4};
(*dir*)
d = {0,1,1, - 1, - 1};
system3a = system3;
system3a[[4]] = system3a[[4]]/. {x3_{4,1} → 1} ;
system3a[[1]] = system3a[[1]]/. {x3_{4,1} → 1} ;
system3a[[5]] = system3a[[5]]/. {x3_{5,1} → 0} ;
system3a[[1]] = system3a[[1]]/. {x3_{5,1} → 0} ;
δ3_{4,1} = {x3_{4,1} → 1,x3_{5,1} → 0} ;

For[i = 1,i ⩽ 4,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3a[[t[[i]]]],x3_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system3a[[t[[i]]]],x3_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system3a[[p[[t[[i]]]]]] = system3a[[p[[t[[i]]]]]]/.δ];
δ3_{4,1} = Join [δ3_{4,1},δ] ;
}];


system3b = system3;
system3b[[4]] = system3b[[4]]/. {x3_{4,1} → 0} ;
system3b[[1]] = system3b[[1]]/. {x3_{4,1} → 0} ;
system3b[[5]] = system3b[[5]]/. {x3_{5,1} → 1} ;
system3b[[1]] = system3b[[1]]/. {x3_{5,1} → 1} ;
δ3_{5,1} = {x3_{4,1} → 0,x3_{5,1} → 1} ;

For[i = 1,i ⩽ 4,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3b[[t[[i]]]],x3_{p[[t[[i]]]],t[[i]]}] [[1]],
```

```
δ = Solve [system3b[[t[[i]]]],x3_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system3b[[p[[t[[i]]]]]] = system3b[[p[[t[[i]]]]]]/.δ];
δ3_{5,1} = Join [δ3_{5,1},δ] ;
}];

(*particular solution*)
system1 = {x1_{1,2} − x1_{5,1} == 3,
−x1_{1,2} − x1_{5,2} − x1_{4,2} == −15,
0 == 0,
x1_{4,2} − x1_{5,4} == −8,
x1_{5,1} + x1_{5,4} + x1_{5,2} == 20
};
(*thread*)
t = {5,4,2,1};
(*pred*)
p = {0,1,0,2,4};
(*dir*)
d = {0,1,0, − 1, − 1};

system1a = system1;
system1a[[5]] = system1a[[5]]/. {x1_{5,1} → 0} ;
system1a[[1]] = system1a[[1]]/. {x1_{5,1} → 0} ;
system1a[[5]] = system1a[[5]]/. {x1_{5,2} → 0} ;
system1a[[2]] = system1a[[2]]/. {x1_{5,2} → 0} ;
δ1 = {x1_{5,1} → 0,x1_{5,2} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1a[[t[[i]]]],x1_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system1a[[t[[i]]]],x1_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system1a[[p[[t[[i]]]]]] = system1a[[p[[t[[i]]]]]]/.δ];
δ1 = Join [δ1,δ] ;
}];
```

```
system2 = {x2_{1,2} - x2_{4,1} == 0,
-x2_{1,2} + x2_{2,3} == -6,
-x2_{2,3} - x2_{4,3} == -5,
x2_{4,1} + x2_{4,3} - x2_{5,4} == 7,
x2_{5,4} == 4
};
(*thread*)
t = {5,4,3,2,1};
(*pred*)
p = {0,1,2,3,4};
(*dir*)
d = {0,1,1, - 1, - 1};
system2a = system2;
system2a[[4]] = system2a[[4]]/. {x2_{4,1} → 0} ;
system2a[[1]] = system2a[[1]]/. {x2_{4,1} → 0} ;
δ2 = {x2_{4,1} → 0} ;

For[i = 1,i ⩽ 4,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system2a[[t[[i]]]],x2_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system2a[[t[[i]]]],x2_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system2a[[p[[t[[i]]]]]] = system2a[[p[[t[[i]]]]]]/.δ];
δ2 = Join [δ2,δ] ;
}];


system3 = {x3_{1,2} - x3_{4,1} - x3_{5,1} == -10,
-x3_{1,2} - x3_{4,2} == -10,
-x3_{4,3} == -7,
x3_{4,1} + x3_{4,3} + x3_{4,2} - x3_{5,4} == 9,
x3_{5,1} + x3_{5,4} == 18
};
(*thread*)
t = {5,3,4,2,1};
(*pred*)
p = {0,1,4,2,4};
```

```
(*dir*)
d = {0,1,1, − 1, − 1};
system3a = system3;
system3a[[4]] = system3a[[4]]/. {x3_{4,1} → 0} ;
system3a[[1]] = system3a[[1]]/. {x3_{4,1} → 0} ;
system3a[[5]] = system3a[[5]]/. {x3_{5,1} → 0} ;
system3a[[1]] = system3a[[1]]/. {x3_{5,1} → 0} ;
δ3 = {x3_{4,1} → 0,x3_{5,1} → 0} ;

For[i = 1,i ⩽ 4,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3a[[t[[i]]]],x3_{p[[t[[i]]]],t[[i]]}] [[1]],
δ = Solve [system3a[[t[[i]]]],x3_{t[[i]],p[[t[[i]]]]}] [[1]]] ;
If[p[[ p[[t[[i]]]]]] ≠ 0,
system3a[[p[[t[[i]]]]]] = system3a[[p[[t[[i]]]]]]/.δ];
δ3 = Join [δ3,δ] ;
}];

R11_{5,1} = 7x1_{1,2} + 9x1_{4,2} + 9x1_{5,1} + 9x1_{5,4}/.δ1_{5,1};
R21_{5,1} = 6x1_{1,2} + 5x1_{4,2} + 10x1_{5,1} + 9x1_{5,4}/.δ1_{5,1};
R31_{5,1} = 2x1_{1,2} + 3x1_{4,2} + 8x1_{5,1} + 2x1_{5,4}/.δ1_{5,1};
R41_{5,1} = 0x1_{1,2} + 0x1_{4,2} + 0x1_{5,1} + x1_{5,4}/.δ1_{5,1};

R11_{5,2} = 7x1_{1,2} + 9x1_{4,2} + 5x1_{5,2} + 9x1_{5,4}/.δ1_{5,2};
R21_{5,2} = 6x1_{1,2} + 5x1_{4,2} + 8x1_{5,2} + 9x1_{5,4}/.δ1_{5,2};
R31_{5,2} = 2x1_{1,2} + 3x1_{4,2} + 8x1_{5,2} + 2x1_{5,4}/.δ1_{5,2};
R41_{5,2} = 0x1_{1,2} + 0x1_{4,2} + 0x1_{5,2} + x1_{5,4}/.δ1_{5,2};

R12_{4,1} = 4x2_{1,2} + 8x2_{2,3} + 9x2_{4,1} + 2x2_{4,3} + 3x2_{5,4}/.δ2_{4,1};
R22_{4,1} = 7x2_{1,2} + 3x2_{2,3} + 7x2_{4,1} + 0x2_{4,3} + 5x2_{5,4}/.δ2_{4,1};
R32_{4,1} = x2_{1,2} + 5x2_{2,3} + 2x2_{4,1} + 3x2_{4,3} + 4x2_{5,4}/.δ2_{4,1};
R42_{4,1} = 0x2_{1,2} + 0x2_{2,3} + 0x2_{4,1} + 0x2_{4,3} + 0x2_{5,4}/.δ2_{4,1};

R13_{4,1} = 2x3_{1,2} + 9x3_{4,1} + 4x3_{4,2} + 5x3_{4,3} + 7x3_{5,4}/.δ3_{4,1};
R23_{4,1} = 6x3_{1,2} + 0x3_{4,1} + 2x3_{4,2} + 2x3_{4,3} + 7x3_{5,4}/.δ3_{4,1};
R33_{4,1} = 5x3_{1,2} + 6x3_{4,1} + 0x3_{4,2} + 2x3_{4,3} + 8x3_{5,4}/.δ3_{4,1};
R43_{4,1} = 0x3_{1,2} + 0x3_{4,1} + 0x3_{4,2} + 0x3_{4,3} + x3_{5,4}/.δ3_{4,1};
```

$$R13_{5,1} = 2x3_{1,2} + 4x3_{4,2} + 5x3_{4,3} + 9x3_{5,1} + 7x3_{5,4}/.\delta 3_{5,1};$$
$$R23_{5,1} = 6x3_{1,2} + 2x3_{4,2} + 2x3_{4,3} + 7x3_{5,1} + 7x3_{5,4}/.\delta 3_{5,1};$$
$$R33_{5,1} = 5x3_{1,2} + 0x3_{4,2} + 2x3_{4,3} + x3_{5,1} + 8x3_{5,4}/.\delta 3_{5,1};$$
$$R43_{5,1} = 0x3_{1,2} + 0x3_{4,2} + 0x3_{4,3} + 0x3_{5,1} + x3_{5,4}/.\delta 3_{5,1};$$

$$DD = \begin{pmatrix} R11_{5,1} & R11_{5,2} & R12_{4,1} & R13_{4,1} \\ R21_{5,1} & R21_{5,2} & R22_{4,1} & R23_{4,1} \\ R31_{5,1} & R31_{5,2} & R32_{4,1} & R33_{4,1} \\ R41_{5,1} & R41_{5,2} & R42_{4,1} & R43_{4,1} \end{pmatrix};$$

$$A_1 = 556 - (7x1_{1,2} + 4x2_{1,2} + 2x3_{1,2} + 8x2_{2,3} + 9x2_{4,1} + 9x3_{4,1} +$$
$$+9x1_{4,2} + 4x3_{4,2} + 5x3_{4,3} + 9x1_{5,1} + 9x3_{5,1} + 5x1_{5,2} +$$
$$+9x1_{5,4} + 3x2_{5,4} + 7x3_{5,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$
$$A_2 = 501 - (6x1_{1,2} + 7x2_{1,2} + 6x3_{1,2} + 3x2_{2,3} + 7x2_{4,1} + 5x1_{4,2} +$$
$$+2x3_{4,2} + 2x3_{4,3} + 10x1_{5,1} + 7x3_{5,1} + 8x1_{5,2} + 9x1_{5,4} +$$
$$+5x2_{5,4} + 7x3_{5,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$
$$A_3 = 307 - (2x1_{1,2} + x2_{1,2} + 5x3_{1,2} + 5x2_{2,3} + 2x2_{4,1} + 6x3_{4,1} +$$
$$+3x1_{4,2} + 3x2_{4,3} + 2x3_{4,3} + 8x1_{5,1} + x3_{5,1} + 8x1_{5,2} +$$
$$+2x1_{5,4} + 4x2_{5,4} + 8x3_{5,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$
$$A_4 = 13 - (x1_{5,4} + x3_{5,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$

$$\beta_1 = A_1 - \left(R13_{5,1}y3_{5,1}\right);$$
$$\beta_2 = A_2 - \left(R23_{5,1}y3_{5,1}\right);$$
$$\beta_3 = A_3 - \left(R33_{5,1}y3_{5,1}\right);$$
$$\beta_4 = A_4 - \left(R43_{5,1}y3_{5,1}\right);$$

$$Y = \text{Simplify}\left[\text{Inverse}[DD].\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}\right];$$

$$\text{rule1} = \{x1_{5,1} \to Y[[1]][[1]], x1_{5,2} \to Y[[2]][[1]], x2_{4,1} \to Y[[3]][[1]],$$
$$x3_{4,1} \to Y[[4]][[1]], x3_{5,1} \to y3_{5,1}\};$$

$$\text{rule2} = \{x1_{1,2} \to x1_{5,1}\left(x1_{1,2}/.\delta 1_{5,1}\right) + x1_{5,2}\left(x1_{1,2}/.\delta 1_{5,2}\right) +$$
$$\left(x1_{1,2}/.\overline{\delta 1}\right),$$
$$x1_{4,2} \to x1_{5,1}\left(x1_{4,2}/.\delta 1_{5,1}\right) + x1_{5,2}\left(x1_{4,2}/.\delta 1_{5,2}\right) +$$
$$\left(x1_{4,2}/.\overline{\delta 1}\right),$$

```
x1_{5,4} → x1_{5,1} (x1_{5,4}/.δ1_{5,1}) + x1_{5,2} (x1_{5,4}/.δ1_{5,2}) +
(x1_{5,4}/.δ̄1) ,
x2_{1,2} → x2_{4,1} (x2_{1,2}/.δ2_{4,1}) + (x2_{1,2}/.δ̄2) ,
x2_{2,3} → x2_{4,1} (x2_{2,3}/.δ2_{4,1}) + (x2_{2,3}/.δ̄2) ,
x2_{4,3} → x2_{4,1} (x2_{4,3}/.δ2_{4,1}) + (x2_{4,3}/.δ̄2) ,
x2_{5,4} → x2_{4,1} (x2_{5,4}/.δ2_{4,1}) + (x2_{5,4}/.δ̄2) ,

x3_{1,2} → x3_{4,1} (x3_{1,2}/.δ3_{4,1}) + x3_{5,1} (x3_{1,2}/.δ3_{5,1}) +
(x3_{1,2}/.δ̄3) ,
x3_{4,2} → x3_{4,1} (x3_{4,2}/.δ3_{4,1}) + x3_{5,1} (x3_{4,2}/.δ3_{5,1}) +
(x3_{4,2}/.δ̄3) ,
x3_{4,3} → x3_{4,1} (x3_{4,3}/.δ3_{4,1}) + x3_{5,1} (x3_{4,3}/.δ3_{5,1}) +
(x3_{4,3}/.δ̄3) ,
x3_{5,4} → x3_{4,1} (x3_{5,4}/.δ3_{4,1}) + x3_{5,1} (x3_{5,4}/.δ3_{5,1}) +
(x3_{5,4}/.δ̄3)
};

solution = Simplify[Join[rule1,rule2/.rule1]];

Print[Simplify[System/.solution]];
Print[solution];
```

The general solution of the underdetermined system of linear algebraic equations (4.7.1) – (4.7.3) with respect to a supporting set of arcs $U_T^k \cup U_C^k$, $k \in K = \{1,2,3\}$ (see Figure 4.10) is:

$$x_{5,1}^1 \to -\frac{2}{359}\left(-6406 + 323y_{5,1}^3\right),$$

$$x_{5,2}^1 \to \frac{1}{359}\left(-3837 + 287y_{5,1}^3\right),$$

$$x_{4,1}^2 \to \frac{1}{359}\left(73 + 30y_{5,1}^3\right),$$

$$x_{4,1}^3 \to \frac{1}{359}\left(-1612 + 267y_{5,1}^3\right),$$

$$x_{5,1}^3 \to y_{5,1}^3,$$

$$x_{1,2}^1 \to -\frac{323}{359}\left(-43 + 2y_{5,1}^3\right),$$

$$x_{4,2}^1 \to -13 + y_{5,1}^3,$$

$$x_{5,4}^1 \to -5 + y_{5,1}^3,$$

$$x_{1,2}^2 \to \frac{1}{359}\left(73 + 30y_{5,1}^3\right),$$

$$x_{2,3}^2 \to \frac{1}{359}\left(-2081 + 30y_{5,1}^3\right),$$

$$x_{4,3}^2 \to -\frac{6}{359}\left(-646 + 5y_{5,1}^3\right),$$

$$x_{5,4}^2 \to 4,$$

$$x_{1,2}^3 \to \frac{2}{359}\left(-2601 + 313y_{5,1}^3\right),$$

$$x_{4,2}^3 \to -\frac{2}{359}\left(-4396 + 313y_{5,1}^3\right),$$

$$x_{4,3}^3 \to 7,$$

$$x_{5,4}^3 \to 18 - y_{5,1}^3.$$

5. SLP FOR MULTIGRAPHS

In Chapter 4, we considered not full rank sparse linear systems with embedded network structure. Their structure was inherited from the inhomogeneous network flow programming problems without nodes of variable intensities. In this chapter we consider applications of discussed algorithms in Chapter 4 and algorithms [32, 44] for constructing of solutions sparse underdetermined systems to the Sensor Location Problem (SLP) for multigraphs with nodes of variable intensities.

5.1. Introduction

Let $G = (I,U)$ is a finite oriented connected multigraph without loops with set of nodes I and set of multiarcs U, $|U| \gg| I |$. Let K ($|K| < \infty$) be a set of different types of flow transported through the multinetwork G. We assume that $K = \{1, \cdots ,|K|\}$. By analogy with the multigraph discussed in the section 4.1, we introduce the connected network G^k where $G^k = (I^k,U^k)$ corresponds to a certain type k, I^k is the set of nodes and U^k is the set of arcs which is available on the flow of type k, $k \in K$. Also, we define for each node $i \in I$ the set of types of flows (products) $K(i) = \{k \in K : i \in I^k\}$ and for each multiarc $(i,j) \in U$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. In other words, $K(i)$ is the set of types of flows (products) transported through the node $i \in I$ and $K(i,j)$ is the set of types of flows (products) transported through the multiarc $(i,j) \in U$ respectively.

Consider the following sparse underdetermined system of linear algebraic equations:

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} x_{ji}^k = \begin{cases} a_i^k, & i \in I^k \backslash I_k^*, \\ x_i^k \cdot sign[i], & i \in I_k^*, k \in K; \end{cases} \tag{5.1.1}$$

$$\sum_{k\in K} \sum_{(i,j)^k\in U^k} \lambda_{ij}^{k,p} x_{ij}^k + \sum_{k\in K} \sum_{i\in I_k^*} \lambda_i^{k,p} x_i^k = \beta_p, \quad \text{for} \quad p = \overline{1,q}, \tag{5.1.2}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j)^k \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j,i)^k \in U^k\}$, x_{ij}^k – a flow along the arc $(i,j)^k$. Nodes $i \in I_k^*$, $I_k^* \subseteq I$, $k \in K$ are named the nodes with variable intensities x_i^k, $sign[i^k] = 1$, if $i^k \in I_{k+}^*$, $sign[i^k] = -1$, if $i^k \in I_{k-}^*$,I_{k+}^*,$I_{k-}^* \subseteq I_k^*$, $I_{k+}^* \cap I_{k-}^* = \oslash$, a_i^k, $\lambda_{ij}^{k,p}$, $\lambda_i^{k,p}$, β_p – rational numbers.

The matrix of system (5.1.1) – (5.1.2) has the following block structure:

$$A = \begin{bmatrix} M & B \\ Q & T \end{bmatrix}$$

Here M is a sparse matrix with a block-diagonal structure of size $\sum\limits_{k\in K} |I^k| \times \sum\limits_{k\in K} |U^k|$ such that each block represents a $|I^k| \times |U^k|$ incidence matrix of the network $G^k = (I^k,U^k)$ for each $k \in K$, namely, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{|K|}$, where $M_k, k = 1,\ldots,|K|$ are blocks of matrix M; Q is a $q \times \sum\limits_{k\in K} |U^k|$ matrix (dense, in the general case) with elements $\lambda_{ij}^{k,p},(i,j) \in U$, $k \in K(i,j)$, $p = \overline{1,q}$. Matrix B is a sparse matrix with a block-diagonal structure of size $\sum\limits_{k\in K} |I^k| \times \sum\limits_{k\in K} |I_k^*|$ such that each block B_k represents a $|I^k| \times |I_k^*|$ matrix of the network $G^k = (I^k,U^k), k \in K$, namely, $B = B_1 \oplus B_2 \oplus \cdots \oplus B_{|K|}$, where $B_k,\ k = 1,\ldots,|K|$ are blocks of matrix B. Each column of matrix B_k has the unique non-zero element. This non-zero element is placed at the intersection of the row and the column both of which correspond to the node $i \in I_k^*$. This non-zero element is equal to $-sign[i]$, $i \in I_k^*$, $k \in K$. T is the matrix of size $q \times \sum\limits_{k\in K} |I_k^*|$ and consists of elements $\lambda_i^{k,p}$ for $i \in I_k^*,k \in K,\ p = \overline{1,q}$.

According to Theorem 2.2.1 the rank of the matrix $\begin{bmatrix} M_k & B_k \end{bmatrix}$ of system (5.1.1) for fixed $k \in K$ for a connected graph $G^k = (I^k,U^k)$, $I_k^* \neq \emptyset$, is equal to $|I^k|$. If for fixed $k \in K$ the connected graph $G^k = (I^k,U^k)$ does not contain of nodes from set I_k^*, ie $I_k^* = \emptyset$, then the rank of the matrix $\begin{bmatrix} M_k & B_k \end{bmatrix}$ of system (5.1.1) is equal to $|I^k| - 1$ [11, 45]. We assume that each connected graph $G^k = (I^k,U^k)$ contains at least one node with a variable intensity: $I_k^* \neq \emptyset$.

We assume $rank(A) = \sum\limits_{k\in K} |I^k| + q$. Since we consider the underdetermined systems of linear algebraic equations then the inequality is true:

$$\sum_{k\in K} |I^k| + q < \sum_{k\in K} |U^k| + \sum_{k\in K} |I_k^*|.$$

The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc are constructed for fixed $k \in K$ according to the rules in Section 2.7.

Theorems 5.1.1 and 5.1.2 indicate the main properties of characteristic vectors.

Theorem 5.1.1. *The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc for fixed $k \in K$ satisfy the system (5.1.3).*

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = \begin{cases} 0, & i \in I^k \backslash I_k^*, \\ x_i^k \cdot sign[i], & i \in I_k^*, k \in K. \end{cases} \tag{5.1.3}$$

Theorem 5.1.2. *Any solution of system (5.1.3) for fixed $k \in K$ is a linear combination of characteristic vectors.*

Definition 5.1.1. We call an aggregate of sets

$$R = \{U_R^k, I_R^{*k}, k \in K\}, U_R^k \subseteq U^k, I_R^{*k} \subseteq I_k^*$$

a support of multigraph G for system (5.1.1) if for $\widetilde{R} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}$, $\widetilde{U}^k = U_R^k$, $\tilde{I}_k^* = I_R^{*k}$ the system

$$\sum_{j \in I_i^+(\widetilde{U}^k)} x_{ij}^k - \sum_{j \in I_i^-(\widetilde{U}^k)} x_{ji}^k = \begin{cases} 0, & i \in I^k \backslash \tilde{I}_k^*, \\ x_i^k \cdot sign[i], & i \in \tilde{I}_k^*, k \in K \end{cases} \tag{5.1.4}$$

has only a trivial solution, but has a nontrivial solution for any of the following set aggregations:

- $\widetilde{R} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}, \widetilde{U}^{k_0} = U_R^{k_0} \cup (i_0, j_0)^{k_0}$, $(i_0, j_0)^{k_0} \in U^{k_0} \backslash U_R^{k_0}$; $\widetilde{U}^k = U_R^k$ for $k \in K \setminus k_0$ and $\tilde{I}_k^* = I_R^{*k}$ for $k \in K$;

- $\widetilde{R} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}$, $\widetilde{U}^k = U_R^k$ for $k \in K$; $\tilde{I}_{k_0}^* = I_R^{*k_0} \cup \{i_0\}$, $i_0 \in I_{k_0}^* \backslash I_R^{*k_0}$; $\tilde{I}_k^* = I_R^{*k}$ for $k \in K \setminus k_0$.

We construct $\tilde{t}_k$ connected component $T^{k,t_k} = \left(I_T^{k,t_k}, U_T^{k,t_k}\right)$, $t_k = \overline{1, \tilde{t}_k}$ of the graph $G^k = (I^k, U^k)$ for every $k \in K$, $I_T^{k,t_k} = I(U_T^{k,t_k})$, $U_T^{k,t_k} \subseteq U_R^k$. For each $k \in K$ we form the following set:

$$U_R^k = \bigcup_{t_k=1}^{\tilde{t}_k} U_T^{k,t_k}.$$

Theorem 5.1.3. ***(Network Support Criterion)*** *An aggregate of sets $R = \{U_R^k, I_R^{*k}, k \in K\}$, where $U_R^k \subseteq U^k$, and $I_R^{*k} \subseteq I_k^*$ is a support of the multigraph G for system (5.1.1) iff for each $k \in K$ the following conditions are be carried out:*

- *Each connected component $T^{k,t_k} = \left(I(U_T^{k,t_k}), U_T^{k,t_k}\right)$ of the graph G^k, $t_k = \overline{1,\tilde{t}_k}$ is a tree, $U_R^k = \cup_{t_k=1}^{\tilde{t}_k} U_T^{k,t_k}$.*
- *The set of nodes $\cup_{t_k=1}^{\tilde{t}_k} I_T^{k,t_k}$ covers all set of the nodes of the graph $G^k = (I^k, U^k)$:*
$$\bigcup_{t_k=1}^{\tilde{t}_k} I_T^{k,t_k} = I^k.$$
- *Each tree $T^{k,t_k} = \left(I(U_T^{k,t_k}), U_T^{k,t_k}\right)$ contains unique node $\{u_{t_k}\}$ from the set I_R^{*k} : $|I_R^{*k} \cap I(U_T^{k,t_k})| = 1$, $I_R^{*k} \cap I(U_T^{k,t_k}) = \{u_{t_k}\}$, where $t_k = \overline{1,\tilde{t}_k}$,*
$$I_R^{*k} = \bigcup_{t_k=1}^{\tilde{t}_k} \{u_{t_k}\}.$$

After the support $R = \{U_R^k, I_R^{*k}, k \in K\}$ of system (5.1.1) is chosen, we determine what structures can be obtained after adding one non-supporting element to the support R.

Definition 5.1.2. The **characteristic vector entailed by an arc** $(\tau,\rho)^k \in U^k \backslash U_R^k$ is the vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k; \delta_i^k(\tau,\rho), i \in I_k^*)$ constructed according to the following rules for fixed k:

- If the set $U_R^k \cup \{(\tau,\rho)^k\}$ has a cycle $L_k = \{I_L^k, U_L^k\}$, then the entailed characteristic vector is the characteristic vector of that cycle, and the arc $(\tau,\rho)^k$ is chosen to define the detour direction of the cycle.
- If the set $U_R^k \cup \{(\tau,\rho)^k\}$ has a chain $C_k = \{I_C^k, U_C^k\}$ that connects nodes $u,v \in I_R^{*k}$, then the entailed characteristic vector is the characteristic vector of that chain, and the arc that defines the detour direction is chosen to be $(\tau,\rho)^k$.

Definition 5.1.3. The **characteristic vector entailed by a node** $\gamma \in I_k^* \backslash I_R^{*k}$ is the characteristic vector $\delta^k(\gamma) = (\delta_{ij}^k(\gamma), (i,j)^k \in U^k; \delta_i^k(\gamma), i \in I_k^*)$ of the chain for fixed k that connects nodes γ and $v \in I_R^{*k}$ with node γ being chosen as the beginning of the chain.

Theorem 5.1.4. *The general solution of the system (5.1.1) for fixed $k \in K$ may be uniquely represented in the following way:*

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_R^k} x_{\tau,\rho}^k \delta_{ij}^k(\tau,\rho) + \sum_{\gamma \in I_k^* \setminus I_R^{*k}} x_\gamma^k \delta_{ij}^k(\gamma) + \widetilde{x}_{ij}^k, \qquad (5.1.5)$$

for $(i,j)^k \in U_R^k$;

$$x_i^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_R^k} x_{\tau,\rho}^k \delta_i^k(\tau,\rho) + \sum_{\gamma \in I_k^* \setminus I_R^{*k}} x_\gamma^k \delta_i^k(\gamma) + \widetilde{x}_i^k, \qquad (5.1.6)$$

for $i \in I_R^{*k}$, *where* $\widetilde{x}^k = (\widetilde{x}_{ij}^k, (i,j)^k \in U^k, \widetilde{x}_i^k, i \in I_k^*)$ *is a partial solution of the nonhomogeneous system (5.1.1).*

The general solution of the nonhomogeneous system (5.1.1) is a sum of the general solution of the homogeneous system generated from the system (5.1.1) and a partial solution of the nonhomogeneous system (5.1.1).

Remark. The formulas (5.1.5) and (5.1.6) are correct, if the partial solution $\widetilde{x}^k = (\widetilde{x}_{ij}^k, (i,j)^k \in U^k, \widetilde{x}_i^k, i \in I_k^*)$ for fixed $k \in K$ is constructed according to the rules:

$$\widetilde{x}_{\tau\rho}^k = 0, (\tau,\rho) \in U^k \setminus U_R^k,\ \widetilde{x}_\gamma^k = 0, \gamma \in I_k^* \setminus I_R^{*k}$$

and solves system (5.1.1).

Further, we shall use formulas (5.1.5) and (5.1.6) where the partial solution $\widetilde{x}^k$, $k \in K$ is constructed according to the above rules.

5.2. Decomposition of the multigraph

Let $R = \{U_R^k, I_R^{*k}, k \in K\}, U_R^k \subseteq U^k$, $I_R^{*k} \subseteq I_k^*$ be a support of the multigraph $G = \{I, U\}$ of system (5.1.1). In arbitrary order, we choose sets $W = \{U_W^k, I_W^{*k}, k \in K\}, |W| = q, U_W^k \subseteq U^k \setminus U_R^k$: $I_W^{*k} \subseteq I_k^* \setminus I_R^{*k}$. We denote by U_W the subset of arcs from W: $U_W = \{U_W^k,\ k \in K\}$. We denote by I_W^* the subset of nodes with variable intensities from W: $I_W^* = \{I_W^{*k}, k \in K\}$. After substituting the general solution of system (5.1.1), which has the form (5.1.5) – (5.1.6), into (5.1.2), the system (5.1.2) takes the form:

$$\sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_R^k} \Lambda_{\tau\rho}^{k,p} x_{\tau\rho}^k + \sum_{k \in K} \sum_{\gamma \in I_k^* \setminus I_R^{*k}} \Lambda_\gamma^{k,p} x_\gamma^k = A_p \quad p = \overline{1,q}, \qquad (5.2.1)$$

where

$$\Lambda^{k,p}_{\tau\rho} = \lambda^{k,p}_{\tau\rho} + \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \delta^k_{ij}(\tau,\rho) + \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \delta^k_i(\tau,\rho),$$

$$\Lambda^{k,p}_{\gamma} = \lambda^{k,p}_{\gamma} + \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \delta^k_{ij}(\gamma) + \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \delta^k_i(\gamma),$$

$$A_p = \beta_p - \sum_{k \in K} \sum_{(i,j)^k \in U^k_R} \lambda^{k,p}_{ij} \widetilde{x}^k_{ij} - \sum_{k \in K} \sum_{i \in I^{*k}_R} \lambda^{k,p}_i \widetilde{x}^k_i.$$

In system (5.2.1), we separate variables that correspond to the set W and then we obtain (5.2.2).

$$\sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k_W} \Lambda^{k,p}_{\tau\rho} x^k_{\tau\rho} + \sum_{k \in K} \sum_{\gamma \in I^{*k}_W} \Lambda^{k,p}_{\gamma} x^k_{\gamma} = \tag{5.2.2}$$

$$= A_p - \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus (U^k_W \cup U^k_R)} \Lambda^{k,p}_{\tau\rho} x^k_{\tau\rho} - \sum_{k \in K} \sum_{\gamma \in I^*_k \setminus (I^{*k}_W \cup I^{*k}_R)} \Lambda^{k,p}_{\gamma} x^k_{\gamma}$$

for $p = \overline{1,q}$.

We can rewrite the system (5.2.2) in the following matrix form:

$$Dx_W = \beta, \tag{5.2.3}$$

where $D = [D_1 \;\; D_2]$, $D_1 = (\Lambda^{k,p}_{\tau\rho}, p = \overline{1,q};\ (\tau,\rho)^k \in U_W)$ is the matrix of the size $q \times |U_W|$; $D_2 = (\Lambda^{k,p}_{\gamma},\ p = \overline{1,q};\ \gamma \in I^*_W)$ is the matrix of the size $q \times |I^*_W|$, $x_W = (x^k_{\tau\rho},(\tau,\rho)^k \in U_W;\ x^k_{\gamma},\ \gamma \in I^*_W)$ is a vector of unknowns of the system (5.2.3).

The right-hand side of (5.2.3) has the form:

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_q \end{pmatrix}, \tag{5.2.4}$$

where

$$\beta_p = A_p - \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus (U^k_W \cup U^k_R)} \Lambda^{k,p}_{\tau\rho} x^k_{\tau\rho} - \sum_{k \in K} \sum_{\gamma \in I^*_k \setminus (I^{*k}_W \cup I^{*k}_R)} \Lambda^{k,p}_{\gamma} x^k_{\gamma}, p = \overline{1,q}$$

Note that each column of the matrix D corresponds to an element (arc or node with variable intensity) of W. The components of the column vector x_W of unknowns of the system (5.2.3) follow in the same order as the columns of the matrix D.

5.3. Support of the multigraph

In section 5.1 (Theorem 5.1.3) we defined the support of multigraph for the sparse system (5.1.1). Now we define the support of multigraph G for the system (5.1.1) – (5.1.2). The matrix of the system (5.1.1) – (5.1.2) along with the sparse matrix of the system (5.1.1) contains the matrix of the system (5.1.2) of general form.

Definition 5.3.1. We call a support of multigraph G for system (5.1.1) – (5.1.2) such an aggregate of sets $Z = \{U_Z^k, I_Z^{*k}, k \in K\}, U_Z^k \subseteq U^k$ and $I_Z^{*k} \subseteq I_k^*$, that for given $\widetilde{Z} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}, \widetilde{U}^k = U_Z^k, \tilde{I}_k^* = I_Z^{*k}$ the system

$$\sum_{j \in I_i^+(\widetilde{U}^k)} x_{ij}^k - \sum_{j \in I_i^-(\widetilde{U}^k)} x_{ji}^k = \begin{cases} 0, & i \in I^k \backslash \tilde{I}_k^*, \\ x_i^k \cdot sign[i], & i \in \tilde{I}_k^*, k \in K; \end{cases}$$

$$\sum_{k \in K} \sum_{(i,j)^k \in \widetilde{U}^k} \lambda_{ij}^{k,p} x_{ij}^k + \sum_{k \in K} \sum_{i \in \tilde{I}_k^*} \lambda_i^{k,p} x_i^k = 0, \text{for} \quad p = \overline{1,q} \tag{5.3.1}$$

has only a trivial solution. But it has a nontrivial solution for any of the following aggregations sets:

- $\widetilde{Z} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}$, $\widetilde{U}^{k_0} = U_Z^{k_0} \cup (i_0, j_0)^{k_0}$, $(i_0, j_0)^{k_0} \in U^{k_0} \backslash U_Z^{k_0}$; $\widetilde{U}^k = U_Z^k$ for $k \in K \setminus k_0$ and $\tilde{I}_k^* = I_Z^{*k}$, for $k \in K$;

- $\widetilde{Z} = \{\widetilde{U}^k, \tilde{I}_k^*, k \in K\}$, $\widetilde{U}^k = U_Z^k$, for $k \in K$; $\tilde{I}_{k_0}^* = I_Z^{*k_0} \cup \{i_0\}$, $i_0 \in I_{k_0}^* \backslash I_Z^{*k_0}$ and $\tilde{I}_k^* = I_Z^{*k}$, $k \in K \setminus k_0$.

Theorem 5.3.1. *The aggregation of sets $Z = \{U_Z^k, I_Z^{*k}, k \in K\}$, where $U_Z^k \subseteq U^k$ and $I_Z^{*k} \subseteq I_k^*$ is a support of multigraph $G = \{I, U\}$ for system (5.1.1) – (5.1.2) if and only if*

- *the aggregation of sets $Z = \{U_Z^k, I_Z^{*k}, k \in K\}$ may be divided into two aggregations: $R = \{U_R^k, I_R^{*k}, k \in K\}$ and $W = \{U_W^k, I_W^{*k}, k \in K\}$, such that $R \cup W = Z$, $R \cap W = \emptyset$ and the aggregation of sets R is a support of the multigraph $G = \{I, U\}$ for the sparse system (5.1.1);*
- *matrix D of the system (5.2.3), which consists of determinants $\Lambda_{\tau\rho}^p$, Λ_γ^p of the structures entailed by the arcs and nodes of the aggregation W, is nonsingular matrix.*

We now investigate theoretical-graphical properties of the structure of the support of multigraph $G = \{I,U\}$ for system (5.1.1) – (5.1.2). According to Theorem 5.3.1, the aggregation of the sets $Z = \{U_Z^k, I_Z^{*k}, k \in K\}$ includes the support $R = \{U_R^k, I_R^{*k}, k \in K\}$ of multigraph G for the sparse system (5.1.1). Supporting elements that correspond to the aggregate R make up a forest of trees that covers all the nodes of the set I^k for every $k \in K$ if $I_k^* \neq \emptyset$. Every tree of the forest has exactly one node from the set I_R^{*k}. We make a cycle or a chain after adding each additional element from W or N, where $W = \{U_W^k, I_W^{*k}, k \in K\}$, $N = \{U_N^k, I_N^{*k}, k \in K\}$, $U_N^k = U^k \setminus (U_R^k \cup U_W^k)$, $I_N^{*k} = I_k^* \setminus (I_R^{*k} \cup I_W^{*k})$ to the elements of the set $R = \{U_R^k, I_R^{*k}, k \in K\}$.

5.4. Modeling of multigraphs

Let $G = (I,U)$ is a finite oriented connected multigraph without loops with set of nodes I and set of multiarcs U, $|U| \gg |\ I\ |$. By analogy with the multigraph discussed in the section 5.1 we present G through the set of $|K|$ connected networks $G^k = (I^k, U^k)$ corresponding to a certain type k, where I^k is the set of nodes and U^k is the set of arcs which is available on the flow of type k, $k \in K$, $K = \{1, \cdots, |K|\}$. We define for each node $i \in I$ the set of types of flows (products) $K(i) = \{k \in K : i \in I^k\}$ and for each arc $(i,j) \in U$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. In other words, $K(i)$ is the set of types of flows (products) transported through the node $i \in I$ and $K(i,j)$ is the set of types of flows (products) transported through the multiarc $(i,j) \in U$ respectively.

We represent the traffic flow by a network flow function $x : U \to \mathbb{R}$ that satisfies the following system:

$$\sum_{j \in I_i^+(U^k)} x_{i,j}^k - \sum_{j \in I_i^-(U^k)} x_{j,i}^k = \begin{cases} x_i^k, & i \in I_k^*, \\ 0, & i \in I^k \setminus I_k^*, \quad k \in K, \end{cases} \tag{5.4.1}$$

where $I_k^* \subseteq I^k$ is the set of nodes with variable intensities, x_i^k is the variable intensity of node $i \in I_k^*$, $k \in K$.

If for fixed k the variable intensity x_i^k of node $i \in I_k^*$ is positive, the node i is a source; if it is negative, this node i is a sink.

For system (5.4.1) for each $k \in K$ is true the following condition:

$$\sum_{i \in I^k} x_i^k = 0.$$

According to [32, 53] if $I_k^* \neq \emptyset$, then the rank of the matrix of system (5.4.1) for a connected graph $G^k = (I^k, U^k)$ for fixed $k \in K$ is equal to $|I^k|$.

In order to obtain information about the unknown x_{ij}^k for the arcs $(i,j)^k \in U^k$ and unknown variables intensities x_i^k of nodes $i \in I_k^*$, $k \in K$ sensors are locate at the nodes. The nodes in the multigraph with sensors we call monitored ones and denote the set of monitored nodes M. The set of monitored nodes in the network $G^k = (I^k, U^k)$ we denote M_k, where

$$M_k = I^k \bigcap M,\ k \in K.$$

If a node $i \in M$ is monitored, then we know the values of flows on all outgoing and all incoming arcs for each node $i \in M$, $k \in K(i)$:

$$\begin{aligned} x_{ij}^k &= f_{ij}^k,\ j \in I_i^+(U^k), \\ x_{ji}^k &= f_{ji}^k,\ j \in I_i^-(U^k), \\ k \in K(i) &\quad \text{for each} \quad i \in M. \end{aligned} \tag{5.4.2}$$

If the set M includes the nodes from the set I_k^*, $k \in K(i)$ then we know the values of flows for all incoming and outgoing arcs for the nodes of the set M and besides, we know also the values:

$$x_i^k = f_i^k, \quad i \in M \cap I_k^*, \quad k \in K(i). \tag{5.4.3}$$

Consider any node $i \in I$. For every outgoing arc $(i,j)^k \in U^k$ for this node i we determinate a real number $p_{ij}^k \in (0,1]$, which denotes the corresponding part of the total outgoing flow:

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k$$

from i which leaves along this arc $(i,j)^k$, for each $k \in K(i)$.

That is,

$$x_{ij}^k = p_{ij}^k \sum_{j \in I_i^+(U^k)} x_{ij}^k,$$
$$0 < p_{ij}^k \leqslant 1, \quad \sum_{j \in I_i^+(U^k)} p_{ij}^k = 1.$$

If $|I_i^+(U^k)| \geqslant 2$ for the node $i \in I$ then we can write the flow along all outgoing arcs from node i in terms of a single outgoing arc, for example, $(i,v_i)^k, v_i \in I_i^+(U^k)$:

$$x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} x_{i,v_i}^k, \quad j \in I_i^+(U^k) \setminus v_i. \tag{5.4.4}$$

We continue this process for each node $i \in I$ if it is the case that: $|I_i^+(U^k)| \geqslant 2$ for every $k \in K(i)$.

So we shall formulate the Sensor Location Problem for the multigraph: **what is the minimum number of monitored nodes M such that system (5.4.1) has an unique solution and in which nodes of multigraph G should we place the sensors?**

Combinatory properties of algorithm of the solution of the Sensor Location Problem for the graph are considered in [46, 50, 52, 53].

Let's substitute (5.4.2) and (5.4.3) to the equations of system (5.4.1).

If $|I_i^+(U^k)| \geqslant 2$ for the node $i \in I^k$ then we can write the flow along all outgoing arcs from node i in terms of a single known outgoing arc flow f_{i,v_i}^k for the arc $(i,v_i)^k, v_i \in I_i^+(U^k)$, where x_{i,v_i}^k is known and equal to $x_{i,v_i}^k = f_{i,v_i}^k$:

$$\begin{gathered} x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} f_{i,v_i}^k, \quad j \in I_i^+(U^k) \setminus v_i, \\ |I_i^+(U^k)| \geqslant 2, \quad i \in I, \quad k \in K(i). \end{gathered} \tag{5.4.5}$$

Also, we substitute known arcs flow (5.4.5) to the equations of system (5.4.1). Let's delete from graphs $G^k = (I^k, U^k), k \in K$ the set of the arcs and nodes on which the arc flows and values x_i^k are known.

Then we have a new multigraph $\overline{G} = (\overline{I}, \overline{U})$, which consists from the set of graphs $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$, $k \in \overline{K}$, where each $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$ is, in general, a disconnected graph, corresponding to a certain type of flow $k \in \overline{K}$. We denote for each multiarc $(i,j) \in \overline{U}$ of multigraph $\overline{G}$ the set:

$$\overline{K}(i,j) = \{k \in \overline{K} : (i,j)^k \in \overline{U}^k\}$$

of types of flow transported through an multiarc (i,j). We denote, also, for each node $i \in I$ the set of types of flows:

$$\overline{K}(i) = \{k \in \overline{K} : i \in \overline{I}^k\}$$

transported through the node $i \in I$. We note, that $|\overline{K}| \leqslant |K|$ in general case, because after we have defined the set M, for some types $k \in K$, we could have obtained the complete information about arc flows and variable intensities for the network flow of type k. So, for this type k there is no subnetwork in the new multigraph $\overline{G}$.

The new multigraph $\overline{G}$ consists of connected components. Some connected components may contain no nodes of the set $\overline{I}_k^*$, where $\overline{I}_k^*$ is the set of nodes with variable intensities of graph $\overline{G}^k$, $k \in \overline{K}$.

Relations (5.4.4) are true for all arcs in new multigraph $\overline{G}$. So, if $|I_i^+(\overline{U}^k)| \geqslant 2$ for the node $i \in \overline{I}^k$ then we can write the flow along all outgoing arcs from node i in terms of a single unknown outgoing arc flow x_{i,v_i}^k, for example, for the arc $(i,v_i)^k$, where x_{i,v_i}^k is an unknown flow:

$$x_{i,j}^k = \frac{p_{i,j}^k}{p_{i,v_i}^k} x_{i,v_i}^k, \quad j \in I_i^+(\overline{U}^k) \setminus v_i, \quad i \in \overline{I}^k, k \in \overline{K}. \tag{5.4.6}$$

The system (5.4.1) and (5.4.6) for multigraph $\overline{G} = (\overline{I}, \overline{U})$ will be transformed into the following one:

$$\sum_{j \in I_i^+(\overline{U}^k)} x_{i,j}^k - \sum_{j \in I_i^-(\overline{U}^k)} x_{j,i}^k = \begin{cases} x_i^k + b_i, & i \in \overline{I}_k^*, \\ a_i^k, & i \in \overline{I}^k \setminus \overline{I}_k^*, k \in \overline{K}. \end{cases} \tag{5.4.7}$$

$$\sum_{(i,j) \in \overline{U}} \sum_{k \in \overline{K}(i,j)} \overline{\lambda}_{ij}^{k,p} x_{ij}^k = 0, p = \overline{1,q}, \tag{5.4.8}$$

where a_i, b_i, $\overline{\lambda}_{ij}^{k,p}$ – are constants.

Let's state the steps of the algorithm for modeling of sets $I^k \setminus M_k^*$ for the given set M_k for new multigraph $\overline{G}$.

Step 1. Construct the cut $CC(M_k)$ of the graph G^k for the set M_k of monitored nodes for each $k \in K$.

Step 2. Find for each $k \in K$ the nodes of the set $I(CC(M_k))$.

Step 3. Construct sets $M_k^+ = I(CC(M_k)) \setminus M_k, k \in K$.

Step 4. Form sets

$$I^k \setminus M_k^*, \quad \text{where} \quad M_k^* = M_k \bigcup M_k^+, \quad k \in K.$$

The part of the unknowns of the system (5.4.7) – (5.4.8) are the flows for outgoing arcs from the nodes of the set $I^k \setminus M_k^*, k \in \overline{K}$. Also the unknowns in the system (5.4.7) – (5.4.8) are the variable intensities x_i^k, where $i \in \overline{I}_k^*, k \in \overline{K}$ for the new multigraph $\overline{G} = (\overline{I}, \overline{U})$.

The new multigraph $\overline{G}$ consists from a connected components. If the fixed connected component of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ contains nodes of the set $\overline{I}_k^*$, then, according to Theorem 2.2.1 and Theorem 5.1.3, we compute the rank of the matrix of the sparse system (5.4.7) since system (5.4.7) – (5.4.8) is a partial case of the system (5.1.1) – (5.1.2) for that connected component.

If the fixed connected component of the new multigraph $\overline{G} = (\overline{I}, \overline{U})$ don't include the nodes of the set $\overline{I}_k^*$, we use the theory of decomposition [39], [41–42, 45, 48, 53] for that connected component.

The system (5.4.7) – (5.4.8) has an unique solution for the given set M if and only if the rank of the matrix of system (5.4.7) – (5.4.8) is equal to the number of unknowns of the system (5.4.7) – (5.4.8). For computing the rank of the matrix of system (5.4.7) – (5.4.8) we use theoretical-graphical properties of the structure of the support according to Theorem 5.3.1.

6. FULL RANK SPARSE SYSTEMS

In this part we consider the linear underdetermined system for generalized multinetwork. Systems of this type appear in non-homogeneous network flow programming problems [11–13, 24] in the form of systems of restrictions and can be characterized as systems with a large sparse submatrix representing the embedded network structure. We develop a direct method for finding solutions of the system. The algorithm is based on the theoretic-graph specificities for the structure of the support for the generalized network and properties of the basis of a solution space of a homogeneous system of special type. One of the key steps is decomposition of the generalized multinetwork and, as a result, decomposition of the sparse underdetermined system.

The work on the Chapter 6 was caused, mainly, by the analysis of problems of non-homogeneous network flow optimization for the generalized network [12, 33, 35, 38, 45, 49, 53, 54–55] on large data files.

By analogy to Chapter 4, where we considered not full rank systems, the general idea of the method is based on the following key steps:

• Distinguishing between the network part of the system and the additional part. The network part of the system represents a network structure and corresponds to the network part of the system of main restrictions of a non-homogeneous network flow programming generalized problem [12], [13, 45, 51, 54, 55]. The additional part of the system corresponds to the additional part of the system of main restrictions and can have a general form. We start the solution by considering the network part of the system only.

• Introduction of the support of the generalized network for a system. The term 'support of the generalized network' (also referred to as network support, or support) is borrowed from optimization theory [12, 13] and is used here for further compatibility with applications in problems of non-homogeneous generalized network flow programming [45, 51].

• Construction of a general solution for the network part of the system. We compute a basis of a solution space of the corresponding homogeneous system and interpret the basis vectors as characteristic vectors, entailed by non-support arcs. A effective algorithms for finding a partial solution of the (non-homogeneous) system is considered.

• Decomposition of the system. We perform the decomposition of the system by separating the variables according to the sets collection of connected components of the special structure, which consist of the arcs of the

support for the network part of the sparse system, bicyclic arcs and non-support/non-bicyclic arcs respectively; and, finally, sequentially express the unknowns corresponding to the sets of bicyclic arcs and the sets of arcs in connected components of the special structure, which consist of the arcs of the support for the network part of the sparse system in terms of the independent variables corresponding to the set non-support/non-bicyclic arcs.

6.1. Sparse systems for generalized multinetwork

Let $G = (V,A)$ be a finite oriented connected network without multiple arcs and loops, where V is a set of nodes and A is a set of arcs defined on $V \times V(|V| < \infty, |A| < \infty)$. Let K ($|K| < \infty$) be a set of different products (types of flow) transported through the network G. For definiteness, we assume the set $K = \{1,\ldots,|K|\}$. Let us denote the connected network corresponding to a certain type k of flow with $S^k = (I^k,U^k)$, where I^k is the set of nodes and U^k is the set of arcs which is available on the flow of type k, $k \in K$. Also, we define for each node $i \in V$ the set of types of flows (products) $K(i) = \{k \in K : i \in I^k\}$ and for each arc $(i,j) \in A$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. In other words, $K(i)$ is the set of types of flows (products) transported through the node $i \in V$ and $K(i,j)$ is the set of types of flows (products) transported through the multiarc $(i,j) \in A$ respectively. Finally, the initial network $G = (V,A)$ may be considered as a union of $|K|$ networks $S^k = (I^k,U^k)$, $k \in K$ and denote with $S = (I,U)$.

For each arc $(i,j)^k \in U^k$ arc flow for the arc $(i,j)^k \in U^k$ of the value x_{ij}^k comes from the node i in the network $S^k = (I^k,U^k)$ and enters to the node j in the transformed form: $\mu_{ij}^k x_{ij}^k$, where μ_{ij}^k is a coefficient of arc flow transformation. Each network $S^k = (I^k, U^k)$ is called a generalized network, $k \in K$, and multinetwork $S = (I, U)$ is called a generalized multinetwork or a generalized multigraph, (hereinafter referred to simply network if the context clearly that it is a generalized multinetwork).

Each multiarc $(i,j) \in U$ of multidigraph $S = (I,U)$ consists from $|K(i,j)|$ arcs:

$$\{(i,j)^k, k \in K(i,j)\}.$$

Let us introduce a subset U_0 of the set U, and let $K_0(i,j) \subseteq K(i,j)$, $(i,j) \in U_0$ be an arbitrary subset of $K(i,j)$ such that $|K_0(i,j)| > 1$.

For the multinetwork $S = (I, U)$ consider the following linear underdetermined system

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, \quad i \in I^k, k \in K, \tag{6.1.1}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \; p = \overline{1,q}, \tag{6.1.2}$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k = z_{ij}, \quad (i,j) \in U_0, \tag{6.1.3}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j)^k \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j,i)^k \in U^k\}$; $a_i^k, \lambda_{ij}^{kp}, \alpha_p, z_{ij}, \mu_{ij}^k \in R$ – parameters of the system, $\mu_{ij}^k > 0$, $x = (x_{ij}^k, (i,j)^k \in U^k, k \in K)$ – vector of unknowns.

Conditions for existence of solutions of system (6.1.1) follow from Kronecker – Capelli theorem [9, 14].

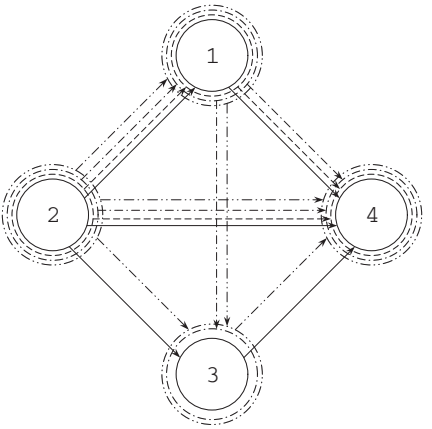

Fig. 6.1. The union of networks $S^k = (I^k, U^k), k \in K = \{1,2,3,4\}$

For the multinetwork $S = (I, U)$, which is presented in Figure 6.1 as the union of $|K| = 4$ of connected networks $S^k = (I^k, U^k), k \in K = \{1,2,3,4\}$, $U_0 = \varnothing$:

$$U^1 = \{(1,4)^1, (2,1)^1, (2,3)^1, (2,4)^1, (3,4)^1\},$$
$$U^2 = \{(1,4)^2, (2,1)^2, (2,4)^2\},$$

$$U^3 = \{(1,3)^3,(1,4)^3,(2,1)^3,(2,4)^3\},$$
$$U^4 = \{(1,3)^4,(2,1)^4,(2,3)^4,(2,4)^4,(3,4)^4\}.$$

The system of type (6.1.1) − (6.1.3) for the networks $S^k = (I^k,U^k)$ where $k \in K = \{1,2,3,4\}$ (see Figure 6.1) has the form (6.1.4) − (6.1.5):

$$\begin{gathered}
x^1_{1,4} - x^1_{2,1} = 0 \\
x^1_{2,1} + x^1_{2,3} + x^1_{2,4} = 15 \\
x^1_{3,4} - 0.9x^1_{23} = 3.2 \\
-0.6x^1_{1,4} - 0.3x^1_{2,4} - 0.3x^1_{3,4} = -6.3 \\
\\
x^2_{1,4} - 0.6x^2_{2,1} = -0.2 \\
x^2_{2,1} + x^2_{2,4} = 4 \\
-0.7x^2_{1,4} - 0.5x^2_{2,4} = -1.7 \\
\\
x^3_{1,3} + x^3_{1,4} - 0.1x^3_{2,1} = 9.4 \\
x^3_{2,1} + x^3_{2,4} = 14 \\
-0.5x^3_{1,3} = -3.5 \\
-0.1x^3_{1,4} - x^3_{2,4} = -8.3 \\
\\
x^4_{1,3} - 0.3x^4_{2,1} = 5.7 \\
x^4_{2,1} + x^4_{2,3} + x^4_{2,4} = 2 \\
x^4_{3,4} - 0.7x^4_{1,3} - 0.9x^4_{2,3} = -3.2 \\
-0.8x^4_{2,4} - 0.9x^4_{3,4} = -1.7
\end{gathered} \tag{6.1.4}$$

$$\begin{gathered}
3x^3_{1,3} + 4x^4_{1,3} + x^1_{1,4} + 8x^1_{2,1} + 5x^2_{2,1} + 3x^3_{2,1} + 4x^4_{2,1} + x^1_{2,3} + \\
+6x^4_{2,3} + 9x^1_{2,4} + x^2_{2,4} + 2x^3_{2,4} + 9x^4_{2,4} + 2x^1_{3,4} + x^4_{3,4} = 234
\end{gathered} \tag{6.1.5}$$

Let us assume that the inequality $\sum\limits_{k\in K} |I^k| + q + |U_0| < \sum\limits_{k\in K} |U^k|$ is true, since we investigate underdetermined systems. The system (6.1.1) are called the network part and the system (6.1.2) − (6.1.3) are called the additional part of the system (6.1.1) − (6.1.3).

The matrix of system (6.1.1) − (6.1.3) has the following block structure:

$$A = \begin{bmatrix} M \\ Q \\ T \end{bmatrix}. \tag{6.1.6}$$

Here M is a sparse matrix of size $\sum\limits_{k\in K}|I^k| \times \sum\limits_{k\in K}|U^k|$ with a block-diagonal structure , such that each block with the number k corresponds to the matrix M_k with the size $|I^k| \times |U^k|$ and the network $S^k = (I^k,U^k), k \in K$, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{|K|}$. Each column of the matrix M_k corresponds to an arc $(i,j)^k$ of the network $S^k = (I^k,U^k)$, and the nonzero elements of the specified column are the two elements: element of the row with the number i, is equal to 1, and element of the row with the number j, equal to $-\mu_{ij}^k$. Q is a $q \times \sum\limits_{k\in K}|U^k|$ matrix (dense, in the general case) with elements $\lambda_{ij}^{kp}, (i,j) \in U, k \in K(i,j)\,, p = \overline{1,q}\,; T$ is a $|U_0| \times \sum\limits_{k\in K}|U^k|$ matrix, consisting of zeros and ones, where all the nonzero elements appear in columns corresponding to arcs $(i,j)^k, (i,j) \in U_0, k \in K_0(i,j)$.

6.2. The basis of the solutions space

Theorem 6.2.1. *The rank of the matrix of system (6.1.1) for the multinetwork $S = (I,U)$ equals $\sum\limits_{k\in K}|I^k|$.*

Proof. Since matrix M has the form $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{|K|}$, where M_k is a diagonal block of matrix $M, k = 1, \ldots, |K|$ and rank of matrix M_k is equal to $|I^k|$ [11, 12], then:

$$rank\,(M) = \sum_{k=1}^{|K|} rank\,(M_k) = \sum_{k\in K}|I^k|.$$

□

Remark 6.2.1. We assume, without loss of generality, that the rank of the system (6.1.1) – (6.1.3) is $\sum\limits_{k\in K}|I^k| + q + |U_0|$.

Since the matrix of system (6.1.1) has the block-diagonal structure, we split the solution of the system (6.1.1) into $|K|$ solutions of (independent) systems, each of which corresponds to a separate block, i.e. to a fixed $k \in K$, and has the following form:

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, \quad i \in I^k \tag{6.2.1}$$

6.2.1. Support for generalized multinetwork

Let's define a support of generalized multinetwork $S = (I,U)$ for system (6.1.1).

Definition 6.2.1. *The support of generalized multinetwork S, $S = (I,U)$ for system (6.1.1) is a set of arcs $U_L = \{U_L^k \subseteq U^k, k \in K\}$, such that the system*

$$\sum_{j \in I_i^+(\hat{U}^k)} x_{ij}^k - \sum_{j \in I_i^-(\hat{U}^k)} \mu_{ji}^k x_{ij}^k = 0, \quad i \in I^k, k \in K \tag{6.2.2}$$

has only a trivial solution for $\hat{U}^k = U_L^k$, but has a non-trivial solution for $\hat{U}^k = \{U_L^k, k \in K \setminus k_0;\ U_L^{k_0} \cup (i,j)^{k_0}\}$, $(i,j)^{k_0} \notin U_L^{k_0}, k_0 \in K$.

For fixed $k \in K$ we consider any cycle of network $S^k = (I^k,U^k)$. An arbitrary way we choose in the cycle the detour direction. The arcs of cycle, which coincide with the direction the detour of the are called a forward arcs of the cycle. Remaining arcs of the cycle are called backward arcs of the cycle.

Definition 6.2.2. *The cycle is called non-degenerate if the product of coefficients $\mu_{i,j}^k$ for a forward arcs of the cycle is not equal to product of the coefficients $\mu_{i,j}^k$ for a backward arcs of the cycle.*

Let us formulate of theoretic-graphical properties of support for the generalized multinetwork $S = (I,U)$ for the system (6.1.1).

Theorem 6.2.2. *The set of arcs $U_L = \{U_L^k, k \in K\}$ is the support for generalized multinetwork $S = (I,U)$ for the system (6.1.1) iff for each $k \in K$ a network $S_L^k = (I^k,U_L^k)$ is the union of $\tilde{t}_k$ connected component $S_L^{k,t_k} = (I(U_L^{k,t_k}),U_L^{k,t_k})$ each of which contains a unique non-degenerate cycle, $S_L^k = \bigcup_{t_k} S_L^{k,t_k}$, $U_L^k = \bigcup_{t_k} U_L^{k,t_k}$ and $I^k = \bigcup_{t_k} I(U_L^{k,t_k})$, $t_k = \overline{1,\tilde{t}_k}$.*

Proof. Follows from theoretic-graphical properties of support of the system (6.1.1) of a generalized network for a homogeneous flow ($|K| = 1$) and block-diagonal structure of matrix M [11, 18, 45]. □

We present an example of support $U_L = \{U_L^k, k \in K = \{1,2,3,4\}\}$ of the generalized multinetwork $S = (I, U)$ (see Figure 6.1) of the system (6.1.4).

In Figure 6.2 we show the set of arcs $U_L = \{U_L^k, k \in K = \{1,2,3,4\}\}$ where $U_L^1 = \{(1,4)^1, (2,1)^1, (2,3)^1, (2,4)^1\}$, $U_L^2 = \{(1,4)^2, (2,1)^2, (2,4)^2\}$,

$$U_L^3 = \{(1,3)^3, (1,4)^3, (2,1)^3, (2,4)^3\},\ U_L^4 = \{(1,3)^4, (2,1)^4, (2,4)^4, (3,4)^4\},$$

which is a support of the generalized multinetwork, which is shown in Figure 6.1 for the sparse system (6.1.4).

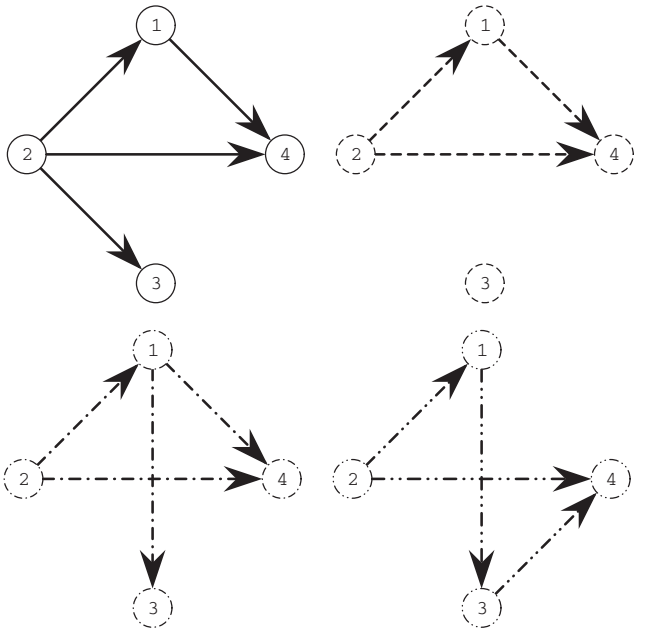

Fig. 6.2. The set of arcs $U_L = \{U_L^k, k \in K = \{1,2,3,4\}\}$

6.2.2. Characteristic vectors

Consider the homogeneous system of linear algebraic equations, generated from the system (6.2.1):

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = 0, \quad i \in I^k. \tag{6.2.3}$$

We introduce the characteristic vector $\delta^k(\tau,\rho)$, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U_L^k$ with respect to the support U_L^k of network $S^k = \left(I^k, U^k\right)$, where $k \in K$ is fixed, components of vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho),(i,j)^k \in U^k)$ are of the solution of the system (6.2.4) – (6.2.5).

$$\sum_{j\in I_i^+(B_{\tau\rho}^k)} \delta_{ij}^k(\tau,\rho) - \sum_{j\in I_i^-(B_{\tau\rho}^k)} \mu_{ji}^k \delta_{ji}^k(\tau,\rho) = 0, \tag{6.2.4}$$
$$i \in I^k, B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k,$$

$$\delta_{\tau\rho}^k(\tau,\rho) = 1, \delta_{ij}^k(\tau,\rho) = 0,\ (i,j)^k \in U^k \setminus (U_L^k \cup (\tau,\rho)^k). \tag{6.2.5}$$

Definition 6.2.3. *The set of arcs $B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k$ is called a bicycle, entailed by an arc $(\tau,\rho)^k$ with respect to the support U_L.*

Let us show that the system (6.2.4) – (6.2.5) has unique solution. The system (6.2.4) – (6.2.5) is equivalent to the system (6.2.6) – (6.2.7):

$$\sum_{j\in I_i^+(B_{\tau\rho}^k\setminus(\tau,\rho)^k)} \delta_{ij}^k(\tau,\rho) - \sum_{j\in I_i^-(B_{\tau\rho}^k\setminus(\tau,\rho)^k)} \mu_{ji}^k \delta_{ji}^k(\tau,\rho) =$$
$$= \begin{cases} 0, & i \in I^k \setminus \{\tau,\rho\}; \\ -1, & i = \tau; \\ \mu_{\tau,\rho}^k, & i = \rho; \end{cases} \qquad i \in I^k, B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k, \tag{6.2.6}$$

$$\delta_{\tau\rho}^k(\tau,\rho) = 1, \delta_{ij}^k(\tau,\rho) = 0,\ (i,j)^k \in U^k \setminus (U_L^k \cup (\tau,\rho)^k). \tag{6.2.7}$$

The definition of the support implies that the matrix of the system (6.2.6) – (6.2.7) is nonsingular and full rank, therefore the system (6.2.6) – (6.2.7) has a unique solution.

Theorem 6.2.3. *The set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_L^k\}$ of characteristic vectors, where k is fixed, constitutes the basis of a solution space for the homogeneous system (6.2.3), generated by the system (6.2.1).*

Proof. Since the rank of the system (6.2.3) is $|I^k|$ and number of variables is $|U^k|$, then the dimension of the solution space for this system equals to $|U^k| - |I^k|$. Hence, any $|U^k| - |I^k|$ linearly independent vectors

of solutions of this system form the basis of its the solution space. From the definition of the characteristic vectors, it follows that it is the solution of the system (6.2.3). Now it suffices to show that all the vectors in the set $\delta^k(\tau,\rho)$, $\delta^k(\tau,\rho) = (\delta^k_{ij}(\tau,\rho),(i,j)^k \in U^k)$, $(\tau,\rho)^k \in U^k \setminus U^k_L$ are linearly independent. We form the matrix M, whose rows are the vectors $\delta^k(\tau,\rho)$, $(\tau,\rho)^k \in U^k \setminus U^k_L$. Obviously, the permutation of rows and columns of the matrix M can be reduced to $[A|E]$, where E is the identity matrix of size $(|U^k| - |I^k|)(|U^k| - |I^k|)$, A is some matrix. Consequently, the rank of M is equal $|U^k| - |I^k|$. This means that that all its rows, i.e. the vectors $\delta^k(\tau,\rho)$, $(\tau,\rho)^k \in U^k \setminus U^k_L$ are linearly independent. The theorem is proved. □

We present an example of the basis of a solution space for the homogeneous system, generated by the system (6.1.4). The basis of a solution space for the homogeneous system, generated by the system (6.1.4), consists of vectors $\delta^1(3,4)$, $\delta^4(2,3)$ for $k = 1$ and $k = 4$ respectively:

$$\delta^1(3,4) = (\delta^1_{1,4}, \delta^1_{2,1}, \delta^1_{2,3}, \delta^1_{2,4}, \delta^1_{3,4}) = \left(\frac{1}{9}, \frac{1}{9}, \frac{10}{9}, -\frac{11}{9}, 1\right);$$

$$\delta^4(2,3) = (\delta^4_{1,3}, \delta^4_{2,1}, \delta^4_{2,3}, \delta^4_{2,4}, \delta^4_{3,4}) = \left(\frac{3}{611}, \frac{10}{611}, 1, -\frac{621}{611}, \frac{552}{611}\right).$$

Denote by U^k_Z the set of arcs, which correspond to nonzero components of the characteristic vector $\delta^k(\tau,\rho)$.

Theorem 6.2.4. *Among the components of the characteristic vector $\delta^k(\tau,\rho)$, entailed by an arc $(\tau,\rho)^k \in U^k \setminus U^k_L$, different from zero are the components of the characteristic vector $\delta^k(\tau,\rho)$, which correspond to the following sets of arcs:*

- *arcs of chains, constructed in network $S^k = (I^k, U^k_L)$ from the node τ and from the node ρ of arc $(\tau,\rho)^k$ to the corresponding nearest nodes a non-degenerate of the cycle (of non-degenerate cycles);*
- *arcs of a non-degenerate cycle (non-degenerate cycles) of network $S^k = (I^k, U^k_L)$, corresponding to the nearest nodes of chains of nodes τ and ρ (the arc $(\tau,\rho)^k$ generates a characteristic vector $\delta^k(\tau,\rho)$);*
- *the arc $(\tau,\rho)^k$, which entails the characteristic vector of $\delta^k(\tau,\rho)$.*

Proof. From the system (6.2.4) – (6.2.5) it follows that arcs incident to leaf nodes of the graph S^k, the values of the corresponding components of the characteristic vector $\delta^k(\tau, \rho)$ are equal to 0. Thus, all arcs of hanging incident nodes can be excluded from the graph S^k. After we use the described procedure for finite number times a graph won't contain leaf nodes, i. e. it will only consist of cycles and connecting them circuits. The theorem is proved. □

To construct the characteristic vectors it is necessary to perform the following steps:

- Construct for each bicycle $B^k_{\tau\rho} = U^k_L \cup (\tau,\rho)^k$ chains from beginning of τ and the end of ρ of each arc $(\tau,\rho)^k$ to appropriate nodes a nondegenerate cycle (non-degenerate cycles) and obtain the components of the characteristic vector, which correspond to the arcs of these chains.
- For each arc a non-degenerate cycle (non-degenerate cycles) of the support U^k_L calculate the components of characteristic vector $\delta^k(\tau,\rho)$, $\delta^k(\tau,\rho) = (\delta^k_{ij}(\tau,\rho),(i,j)^k \in U^k)$, entailed by arc $(\tau,\rho)^k \in U^k \setminus U^k_L$ respect to support U^k_L of network $S^k = \{I^k, U^k\}$, where $k \in K$ is fixed, which correspond to the arcs of this a non-degenerate cycle (non-degenerate cycles).

By Theorem 6.2.4 the set U^k_Z includes the following arcs:

- arcs of the chains constructed in network $S^k = (I^k,U^k_L \cup (\tau, \rho)^k)$ from the nodes τ and ρ of the arc $(\tau,\rho)^k$ to the appropriate node a non-degenerate cycle (non-degenerate cycles);
- the arcs of a non-degenerate cycle (non-degenerate cycles) of network S^k, in which the nearest nodes connect these chains with nodes τ and ρ (the arc $(\tau,\rho)^k$ entails a characteristic vector $\delta^k(\tau,\rho)$);
- the arc $(\tau,\rho)^k$, which entails the characteristic vector $\delta^k(\tau,\rho)$.

To component of $\delta^k_{\tau,\rho}(\tau,\rho)$ of the characteristic vector $\delta^k(\tau,\rho)$, entailed by arc $(\tau,\rho)^k$, assign 1: $\delta^k_{\tau,\rho}(\tau,\rho) = 1$ (send along the arc $(\tau,\rho)^k$ of the network $S^k = (I^k,U^k_L \cup (\tau,\rho)^k)$ one unit of flow). For the remaining arcs of the network corresponding to components of the characteristic vector $\delta^k(\tau,\rho)$, $\delta^k(\tau,\rho) = (\delta^k_{ij}(\tau,\rho),(i,j)^k \in U^k)$ we temporarily put them to zero.

For each current node i of the chain, leading to a corresponding nearest node a non-degenerate cycle (non-degenerate cycles) solve the equation for the component $\delta^k_{i,h}(\tau,\rho)$ of the characteristic vector $\delta^k(\tau,\rho)$, where $(i,h)^k$

is the arc that connects node i with the parent of node i to the root structure of a network S^k, $S^k = (I^k, U_L^k)$. Thus, the component $\delta_{i,h}^k(\tau,\rho)$ of the characteristic vector is equal:

$$\delta_{i,h}^k(\tau,\rho) = -\sum_{j \in I_i^+(U_Z^k \setminus (i,h)^k)} \delta_{ij}^k(\tau,\rho) + \sum_{j \in I_i^-(U_Z^k \setminus (i,h)^k)} \mu_{ji}^k \delta_{ji}^k(\tau,\rho), \qquad (6.2.8)$$

where U_Z^k is the set of arcs, that correspond to the nonzero components of the characteristic vector $\delta^k(\tau,\rho)$.

If the arc $(h,i)^k$ is the arc that connects node i with the parent of node i to the root structure of a network $S^k = (I^k, U_L^k)$, then the component of $\delta_{h,i}^k(\tau,\rho)$ of characteristic vector $\delta^k(\tau,\rho)$ is:

$$\delta_{h,i}^k(\tau,\rho) = \sum_{j \in I_i^+(U_Z^k \setminus (h,i)^k)} \frac{1}{\mu_{hi}^k} \delta_{ij}^k(\tau,\rho) -$$

$$- \sum_{j \in I_i^-(U_Z^k \setminus (h,i)^k)} \frac{1}{\mu_{hi}^k} \mu_{ji}^k \delta_{ji}^k(\tau,\rho). \qquad (6.2.9)$$

So, in the beginning let's set all the components of the characteristic vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k)$ to zero. Because addition is associative, while traversing through arcs of chains from node τ and node ρ of arc $(\tau,\rho)^k$ to the corresponding nearest nodes of non-degenerate cycle (or non-degenerate cycles) of network $S^k = (I^k, U_L^k)$, we will not get, but change values of components of characteristic vector to summands, which depend on the modifications of the components of the previous arc. That means we can get the value of the component of the characteristic vector not through browsing the current node once, while using the definition of depth of node in the basis [2, 53], but possibly through browsing the current node i: twice while building the chain from node τ and while building the chain from node ρ, where $(\tau,\rho)^k$ is the arc which generate the characteristic vector $\delta^k(\tau,\rho)$. Every browsing contributes one summand to the resulting sums of the previous formulas (6.2.8), (6.2.9). This way we simplify the algorithm of generating chains and to make it uniform for every node, and to get rid of three variants of the correspondence of the depths of nodes and putting the values of depths into basis.

By analogy we can modify one or two (depending on the configuration of the network $S^k = (I^k, U_L^k \cup (\tau,\rho)^k)$) components of characteristic vector of

the arcs from cycle. To compute the components of characteristic vector, which correspond to arcs of the non-degenerate cycle, let's use the algorithm of the solving of the system of linear algebraic equations (SLAE) of the special type.

Let's denote through I_C^k, U_C^k the set of nodes and the set of arcs, correspondingly, for some non-degenerate cycle, which is included into the support U_L^k for the fixed k. Let's consider the following SLAE:

$$\sum_{j\in I_i^+(U_C^k)} x_{ij}^k - \sum_{j\in I_i^-(U_C^k)} \mu_{ji}^k x_{ji}^k = b_i^k, \quad i \in I_C^k. \tag{6.2.10}$$

The system (6.2.10) is a special case of the following linear system:

$$\begin{gathered} l_{1,1}x_1 + l_{1,2}x_2 = l_{1,3}, \\ l_{2,1}x_2 + l_{2,2}x_3 = l_{2,3}, \\ \cdots \\ l_{n-1,1}x_{n-1} + l_{n-1,2}x_n = l_{n-1,3}, \\ l_{n,1}x_n + l_{n,2}x_1 = l_{n,3}, \end{gathered} \tag{6.2.11}$$

where $l_{i,1}$, $l_{i,2}$, $l_{i,3}$, are given numbers, $l_{i,1} \neq 0$, $l_{i,2} \neq 0$, $i = \overline{1,n}$.

Let's suppose that the conditions of the Kronecker-Capelli theorem are met and the matrix of the system (6.2.11) is non-degenerate. Therefore the system (6.2.11) has unique solution.

Let's point to the effective algorithm for the solution of system (6.2.11) with the linear estimate in the worst case. Let's fix $x_1 = \widetilde{x}_1$. While consequentially solving the first $n-1$ equations of the system (6.2.11), we will get that x_n can be expressed in terms of $\widetilde{x}_1$, i.e.

$$x_n = l_1\widetilde{x}_1 + l_2, \tag{6.2.12}$$

where l_1, l_2 are some coefficients. After we substituted (6.2.12) to the last equation of the system (6.2.11), we will get

$$x_1 = l_3\widetilde{x}_1 + l_4, \tag{6.2.13}$$

where l_3, l_4 are some coefficients. Let's consider the residual $\psi = x_1 - \widetilde{x}_1$. Given the equality (6.2.13), we will get:

$$\psi = a\widetilde{x}_1 + b, \tag{6.2.14}$$

where a, b are some coefficients.

In the case, when $\widetilde{x}_1$ is equal to the component x_1 of the solution of the system (6.2.11), then $\psi = 0$. Moreover, based on the way we got the equality (6.2.14), we can state, that the converse is true as well. Thus, the component x_1 is the solution of the system (6.2.11) iff the condition $ax_1 + b = 0$ holds.

According to condition if the solution x_1 is unique, then $a \neq 0$, therefore

$$\boldsymbol{x}_1 = -\frac{b}{a}. \tag{6.2.15}$$

Let's point to the method of the computing of the coefficients a and b of the equation (6.2.15). Let's set $\widetilde{x}_1 = 0$. By consequentially solving the equations of the system (6.2.11), we will compute the numerical value of the residual ψ_0. Thus, considering (6.2.14), we will get $\psi_0 = b$. For $\widetilde{x}_1 = 1$, by analogy, we will get a numerical value for the residual ψ_1. Considering (6.2.14), we have $\psi_1 = a + b$. So, to compute coefficients a and b of the equation (6.2.15) we have the following equalities:

$$\begin{aligned} \psi_0 &= b; \\ \psi_1 &= a + b \end{aligned} \tag{6.2.16}$$

For the system (6.2.16) we compute coefficients a and b:

$$\begin{aligned} a &= \psi_1 - \psi_0; \\ b &= \psi_0. \end{aligned} \tag{6.2.17}$$

By substituting the values of the coefficients a and b, which are computed through the residual (6.2.17) to formula (6.2.15):

$$x_1 = \frac{\psi_0}{\psi_0 - \psi_1}. \tag{6.2.18}$$

So, we have proposed an algorithm for the solution of the sparse SLAE of the type (6.2.11) in the linear time in the worst case. Let's describe the basic steps of the proposed algorithm:

- Put $\widetilde{x}_1 = 0$.
- By solving consequentially the equations of the system in the order they are in (6.2.11), from the last equation of the system (6.2.11) compute x_1.
- Compute residual ψ_0.

- Put $\tilde{x}_1 = 1$.
- Compute residual ψ_1.
- Compute the real value of x_1 according to the formula (6.2.18).
- By consequentially solving $n - 1$ first equations of the system (6.2.11), compute the real values of the remaining unknowns of the system (6.2.11).

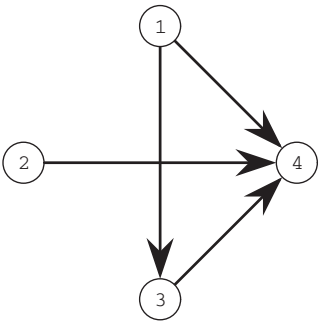

Fig. 6.3. The set of arcs U_L^1

The system (6.2.10) is the special case of the SLAE of type (6.2.11). To compute the components of the characteristic vector, which correspond to the arcs of the cycle, we have to solve sparse systems of the linear algebraic equations of type (6.2.11). Let's consider the following example on the graph shown in Figure 6.3. We compute the components of the characteristic vector, entailed by the arc $(\tau,\rho)^k = (2,1)^k$, which correspond to the arcs of the cycle $(I_C^k,\ U_C^k)$ where

$$I_C^k = \{1, 3, 4\},\ \ U_C^k = \{(1,3)^1,(1,4)^1,(3,4)^1\}.$$

For the brevity, we will not show the arc $(\tau,\rho)^k$, entailing the characteristic vector $\delta^k(\tau,\rho)$, if it is clear which vector we are talking about. Let's denote the components of the characteristic vector $\delta^1(2,1)$ with the following notation: $\delta^1(2,1) = (\delta^1_{1,3},\ \delta^1_{1,4},\ \delta^1_{2,1},\ \delta^1_{2,3},\ \delta^1_{2,4},\ \delta^1_{3,4})$. We will send one unit of the flow in the network $S^1 = (I^1,U_L^1 \cup (2,1)^1)$ along the arc $(2,1)^1$: $\delta^1_{2,1} = 1$. The nonzero components of the characteristic vector

$\delta^k(\tau,\rho) = \delta^1(2,1)$, entailed by the arc $(\tau,\rho)^1 = (2,1)^1$, corresponding to arcs $(1,3)^1,(1,4)^1,(2,1)^1,(2,4)^1,(3,4)^1$ are computed from the system:

$$\begin{gathered}
\delta^1_{1,3} + \delta^1_{1,4} - \frac{1}{10}\delta^1_{2,1} = 0,\\
\delta^1_{2,1} + \delta^1_{2,4} = 0,\\
\delta^1_{3,4} - \frac{6}{10}\delta^1_{1,3} = 0,\\
-\frac{9}{10}\delta^1_{1,4} - \frac{1}{2}\delta^1_{2,4} - \frac{8}{10}\delta^1_{3,4} = 0,\\
\delta^1_{2,1} = 1.
\end{gathered} \tag{6.2.19}$$

Let's provide the data structure which allows to compute the nonzero components of every characteristic vector $\delta^k(\tau,\rho)$ in $O(n)$ complexity in the worst case, where $n = |I^k|$. To construct the solution of the system (6.2.19) with the given estimate for the computation of the non-zero components of the characteristic vector $\delta^1(2,1)$ it's necessary to provide the following information: $t = \{2,3,4,1\}$ – these are the elements of the list $\{t[i], i \in I^k\}$ for the connected component U^1_L. The detailed description of the elements of the list $\{t[i],\ i \in I^k\}$ is given below. The list $\{t[i],\ i \in I^k\}$ defines the order of the solution of the equations of the system (6.2.19); the list $p = \{-3, 4, -4, -1\}$, for every node $i \in I^1$, defines the value $|p[i]|$, which is the parent node of the node i in the rooted structure; the minus sign defines the root of the node of the rooted structure; the list $d = \{-1, -1, -1, 1\}$ defines for every node $i \in I^1$ the direction of the arc in the rooted structure: if the node i is the root, then $d[i] = 0$; if $(|p[i]|,i) \in U^1$, then $d[i] = 1$; if $(i,|p[i]|) \in U^1$, then $d[i] = -1$.

To compute the nonzero components of the characteristic vector $\delta^1(2,1)$, which correspond to the arcs of the cycle, we have to solve the sparse SLAE of the type: (6.2.20):

$$\begin{gathered}
\delta^1_{1,3} + \delta^1_{1,4} = \frac{1}{10},\\
\delta^1_{3,4} - \frac{6}{10}\delta^1_{1,3} = 0,\\
-\frac{9}{10}\delta^1_{1,4} - \frac{8}{10}\delta^1_{3,4} = -\frac{1}{2}.
\end{gathered} \tag{6.2.20}$$

Let's select any component of the characteristic vector $\delta^1(2,1)$, which correspond to the arc in the cycle, for example, $\delta^1_{1,3}$. Let $\delta^1_{1,3} = 0$. We solve the equations of the system (6.2.20) for each of the unknowns which correspond to the arcs in the cycle, in the following order:

$$3,\ 4,\ 1.$$

Let's compute the residual ψ_0, which we got as the result of the first traversal of the cycle:

$$\psi_0 = \frac{41}{90}.$$

Let's do analogically for the initial value $\delta^1_{1,3} = 1$, which is not equal to the component $\delta^1_{1,3}$ during the first traversal of the cycle, i.e. $\delta^1_{1,3} \neq 0$. We solve the equations of the system (6.2.20) for every node of the cycle in the following order:

$$3,\ 4,\ 1.$$

Let's compute the residual ψ_1, we got as the result of the second traversal of the cycle:

$$\psi_1 = \frac{83}{90}.$$

Let's note, that after the first and the second traversal of the cycle we do not need to keep the values of the computed components of the characteristic vector.

According to the formula (6.2.18) we compute the desired value of the component $\delta^1_{1,3}$ of the characteristic vector $\delta^1(2,1)$ (corresponding to the selected arc $(1,3)^1$ of the cycle):

$$\delta^1_{1,3} = -\frac{41}{42}.$$

We traverse the cycle for the third time, keeping the desired values of components of the characteristic vector $\delta^1(2,1)$, which correspond to the arcs of the cycle. These values we get as the result of the solution of the equations of the system (6.2.20) for each node in the cycle in the same order: $[\,3,\ 4,\ 1]$.

After we have solved the equation of the system (6.2.20) for the node $i = 3$

$$\delta^1_{3,4} - \frac{6}{10}\delta^1_{1,3} = 0,$$

we have computed the real value of the component $\delta^1_{3,4}$ of the characteristic vector $\delta^1(2,1)$:

$$\delta^1_{3,4} = -\frac{41}{70}.$$

After we have solved the equation of the system (6.2.20), for the node $i = 4$

$$-\frac{9}{10}\delta^1_{1,4} - \frac{8}{10}\delta^1_{3,4} = -\frac{1}{2},$$

we have computed the real value of the component $\delta^1_{1,4}$ of the characteristic vector $\delta^1(2,1)$:

$$\delta^1_{1,4} = \frac{113}{105}.$$

After we substitute the computed values of the components of the characteristic vector into the equation

$$\delta^1_{1,3} + \delta^1_{1,4} = \frac{1}{10}$$

of the system (6.2.20) for the node $i = 1$, $\delta^1(2,1)$, we will get identity.

So we have constructed the characteristic vector $\delta^1(2,1)$, entailed by the arc $(\tau,\rho)^1 = (2,1)^1$:

$$\delta^1(2,1) = (\delta^1_{2,1}(2,1),\ \delta^1_{2,3}(2,1),\ \delta^1_{2,4}(2,1),\ \ \delta^1_{3,4}(2,1),\ \delta^1_{1,4}(2,1),\ \delta^1_{1,3}(2,1)),$$

where

$$\delta^1_{2,1}(2,1) \to 1, \delta^1_{2,3}(2,1) \to 0, \delta^1_{2,4}(2,1) \to -1,$$

$$\delta^1_{3,4}(2,1) \to -\frac{41}{70}, \delta^1_{1,4}(2,1) \to \frac{113}{105}, \delta^1_{1,3}(2,1) \to -\frac{41}{42}.$$

As we have initially kept the contribution of the chains of the rooted structure to the equations for the arcs of the cycle, so we didn't have to browse all the adjacent arcs while recomputing the components of the characteristic vector. That means the amount of the browsed arcs doesn't exceed the amount of the browsed nodes. So, the complexity of the algorithm of computation of the non-zero components of the characteristic vector $\delta^k(\tau,\rho)$ is $O(|I^k|)$ in the worst case for the fixed $k \in K$.

The support $U_L = \{U_L^k, k \in K\}$ of the network $S = (I,U)$ for the system (6.1.1) is union of the connected components $S_L^{k,t} = (I(U_L^{k,t}), U_L^{k,t})$:

$$S_L^k = \bigcup_t S_L^{k,t}.$$

Each connected component $S_L^{k,t}$, $S_L^{k,t} = (I(U_L^{k,t}), U_L^{k,t})$ has the unique non-degenerate cycle [6, 17, 26, 34], where the set of arcs and the set of nodes are $U_L^k = \bigcup_t U_L^{k,t}$, $I^k = \bigcup_t I(U_L^{k,t})$. As the result of addition of arc $(\tau,\rho)^k \in U^k \setminus U_L^k$, where $k \in K$ is fixed, to the support U_L we get network, which is composed of connected components, the only one among them is a bicycle $B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k$. Each of the remaining connected components has the graph-theoretical properties of the support of generalized multinetwork $S = (I,U)$ for the system (6.1.1). These graph-theoretical properties are given above.

While using the graph-theoretical properties of the support U_L and the corresponding data structures and the algorithms for finding a nonzero components of the characteristic vector of bicycle $\delta^k(\tau,\rho)$ entailed by the arc $(\tau,\rho)^k \in U^k \setminus U_L^k$ by respect to U_L^k, where $k \in K$ is fixed, we compute in $O(n)$ time in the worst case, where n is the amount of the nodes in the bicycle $B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k$.

$$\begin{gathered}
\delta_{1,4}^1 - \delta_{2,1}^1 = 0, \\
\delta_{2,1}^1 + \delta_{2,3}^1 + \delta_{2,4}^1 = 0, \\
\delta_{3,4}^1 - \frac{9}{10}\delta_{2,3}^1 = 0, \\
-\frac{3}{5}\delta_{1,4}^1 - \frac{3}{10}\delta_{2,4}^1 - \frac{3}{10}\delta_{3,4}^1 = 0, \\
\delta_{3,4}^1 = 1.
\end{gathered} \tag{6.2.21}$$

Let's consider the example of the construction of the characteristic vector $\delta^k(\tau,\rho) = \delta^1(3,4)$, entailed by the arc $(3,4)^1$ by respect to U_L, presented in the Figure. 6.2. For brevity, in the components of the characteristic vector $\delta^k(\tau,\rho)$ we do not show the arc, which entails it, if it is clear, which vector we are talking about. The components of the characteristic vector $\delta^1(3,4)$, $\delta^1(3,4) = (\delta_{1,4}^1, \delta_{2,1}^1, \delta_{2,3}^1, \delta_{2,4}^1, \delta_{3,4}^1)$ satisfy the sparse SLAE (6.2.21).

Let's describe the data structures for computing the nonzero components of each characteristic vector $\delta^k(\tau,\rho)$ in $O(n)$ time in the worst case,

where $n = |I^k|$. To construct the solution of the system (6.2.21) with the given estimate we have to provide the following information: elements $t = \{3,1,2,4\}$ of the list $\{t[i], i \in I^k\}$ the connected component U_L^1 to compute the nonzero components of the characteristic vector $\delta^1(3,4)$. Detailed description of the components of the list $\{t[i], i \in I^k\}$ is given below. The list $\{t[i], i \in I^k\}$ defines the order of solution of the equations of the system (6.2.21); the list $p = \{-2, -4, 2, -1\}$ defines for each node $i \in I^k$ the value $|p[i]|$, which is the parent node of the node i in the rooted structure; the minus sign defines the root of the node of the rooted structure; the list $d = \{1, -1, 1, 1\}$ defines for each node $i \in I^k$ the direction of the arc in the rooted structure: if the node i is the root, then $d[i] = 0$; if $(|p[i]|, i) \in U^k$, then $d[i] = 1$; if $(i, |p[i]|) \in U^k$, then $d[i] = -1$.

Let's note that if for each i the element $p[i]$ of the list $\{p[i], i \in I^k\}$ of the rooted structure contains information about the its parent arc (i,j), which connects the node i with the node j, where j is the parent node for i, then there's no necessity to keep the list: $\{d[i], i \in I^k\}$.

The forming of the elements of the list $\{t[i], i \in I^k\}$ for the arbitrary rooted tree of the collection of trees with the root in the nodes U_L of the non-degenerate cycle of the support, is made in the following way:

- for fixed $t = 1$, where t is the index of the connected component U_L^{kt}, we perform a dynastic traversal of the arbitrary rooted tree from the collection of rooted trees with the root in the node of the only cycle of the connected component U_L^{kt};
- in the tree under consideration, we delete the root;
- we invert the list of nodes we got;
- we place the inverted list of nodes (without a root) to the resulting list $\{t[i], i \in I^k\}$.

We perform analogically for every tree in the collection of the rooted trees with roots in the nodes of the unique non-degenerate cycle of the connected component U_L^{kt}. After we traversed all rooted trees of the collection, we place the resulting lists in the arbitrary order to the final list $\{t[i], i \in I^k\}$. Then, we place the nodes of the non-degenerate cycle to the list $\{t[i], i \in I^k\}$, which belong to the connected component U_L^{kt} in the following order: $j, |p[j]|, |p[|p[j]|]|, \cdots, v$, where j is an arbitrary node of the non-degenerate cycle, v is the node of the cycle, for which holds: $|p[v]| = j$.

We perform the described steps of the formation of list $\{t[i], i \in I^k\}$ for the connected components U_L^{kt}, $t = 2, \cdots, s$, where s is the amount of the

connected components of the support U_L of the graph S. The formed list $\{t[i], i \in I^k\}$ defines the order of solution of the equations of the system for computation of the characteristic vector or partial solution of the system for fixed k.

The nonzero components of the characteristic vector $\delta^k(\tau,\rho)$, entailed by any other arc $(\tau,\rho)^k \in U^k \setminus U_L^k$, are:

- a component $\delta^k_{\tau\rho}$, which corresponds to arc $(\tau,\rho)^k$;
- the components, which correspond to the arcs of the chains, constructed from nodes τ and ρ of the arc $(\tau,\rho)^k$ in the network $(I^k, U_L^k \cup (\tau,\rho)^k)$ to the nodes of the cycle;
- the components of the characteristic vector, which correspond to the arcs of non-degenerate cycle in the network (I^k, U_L^k).

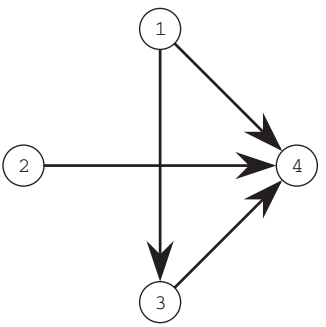

Fig. 6.4. The set of arcs U_L^1

In the Table 6.2.1 we present the rooted structure, required for storing the elements of the set U_L^1 (see Figure. 6.4), which allows to construct the nonzero components of the characteristic vector $\delta^1(2,1)$ in $O(n)$ time in the worst case, where $n = |I^1|$.

Theorem 6.2.5. *The general solution of the system (6.1.1) for fixed $k \in K$ may be presented in the following form:*

$$x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \left(\tilde{x}_{ij}^k - \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \tilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) \right), \quad (6.2.22)$$

$$(i,j)^k \in U_L^k; \quad x_{\tau\rho}^k \in \mathbf{R}, \quad (\tau,\rho)^k \in U^k \setminus U_L^k,$$

Table 6.2.1

The rooted structure for U_L^1

i	1	2	3	4
$p[i]$	-3	4	-4	-1
$d[i]$	-1	-1	-1	1
$t[i]$	2	3	4	1
$depth[i]$	0	1	0	0

where $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ – any partial solution of the system (6.1.1) for the fixed $k \in K$; $x^k_{\tau\rho}$ are independent variables, which correspond to the arcs $(\tau,\rho)^k \in U^k \setminus U^k_L$.

Proof. Let's denote the general solution of the system (6.1.1) through $x^k = (x^k_{ij},(i,j)^k \in U^k)$ and let $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ be some partial solution of the system (6.1.1). As, according to the Theorem 6.2.3 the set $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U^k_L\}$ of the characteristic vectors comprises the basis of the space of solutions of the homogeneous system, generated by the system (6.1.1), then we give the general solution x^k in the following vector form:

$$x^k = \sum_{(\tau,\rho)^k \in U^k \setminus U^k_L} \alpha^k_{\tau\rho}\delta^k(\tau,\rho) + \tilde{x}^k \tag{6.2.23}$$

as the sum of the general solution of the homogenous system, generated by the system (6.1.1) and some partial solution of the non-homogenous system (6.1.1); $\alpha^k_{\tau\rho} \in \mathbf{R}$ – the coefficients of the linear combination of the characteristic vectors in (6.2.23). Let's give (6.2.23) in the component form:

$$x^k_{ij} = \sum_{(\tau,\rho)^k \in U^k \setminus U^k_L} \alpha^k_{\tau\rho}\delta^k_{ij}(\tau,\rho) + \tilde{x}^k_{ij}, \quad (i,j)^k \in U^k_L; \tag{6.2.24}$$

$$x^k_{\tau\rho} = \alpha^k_{\tau\rho} + \tilde{x}^k_{\tau\rho}, \quad (\tau,\rho)^k \in U^k \setminus U^k_L. \tag{6.2.25}$$

From the equations (6.2.25) we find $\alpha^k_{\tau\rho} = x^k_{\tau\rho} - \tilde{x}^k_{\tau\rho}$, $(\tau,\rho)^k \in U^k \setminus U^k_L$ and substitute them into (6.2.24). So, we got the expression (6.2.22) for the general solution of the system (6.1.1). □

Remark 6.2.2. In practice, when constructing a partial solution $\tilde{x}^k$, $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ of the system (6.1.1) we will consider $\tilde{x}^k_{\tau\rho} = 0$, for the arcs $(\tau,\rho)^k \in U^k \backslash U^k_L$, and then the system (6.1.1) for finding a particular solution $\tilde{x}^k$ is presentable in the form:

$$\sum_{j \in I_i^+(U_L^k)} \tilde{x}^k_{ij} - \sum_{j \in I_i^-(U_L^k)} \mu^k_{ji} \tilde{x}^k_{ji} = a^k_i, \quad i \in I^k.$$

In this case, the formulas (6.2.22) have the following form:

$$x^k_{ij} = \sum_{(\tau,\rho)^k \in U^k \backslash U^k_L} x^k_{\tau\rho} \delta^k_{ij}(\tau,\rho) + \tilde{x}^k_{ij}, (i,j)^k \in U^k_L, \tag{6.2.26}$$

$$x^k_{\tau\rho} \in \mathbf{R}, \quad (\tau,\rho)^k \in U^k \backslash U^k_L.$$

Later on, we will use the formulas (6.2.26).

In the Table 6.2.2 we present the rooted structure, required for the storage of the elements of the set U^2_L (Figure 6.5). This structure allows to compute the components of a partial solution of the system of type (6.1.1), which correspond to the arcs of the set U^k_L for each k, in $O(n)$ time in the worst case, where $n = |I^k|$. In the code in Listing 3 we show the use of the rooted structure for constructing components of a partial solution, which correspond to the arcs in the set U^2_L (see Figure 6.5).

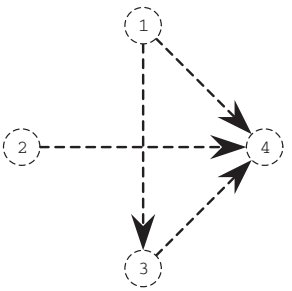

Fig. 6.5. The set of arcs U^2_L

In the code in Listing 2 we show an implementation of the algorithm for the computation of non-zero components of each characteristic vector in *Wolfram Mathematica*. The data structures to effectively compute the nonzero components of the characteristic vector $\delta^1_{2,1}$ in $O(n)$ time $n = |I^k|, k = 1$ are given in Table 6.2.2.

Table 6.2.2

The rooted structure for U_L^2

i	1	2	3	4
$p[i]$	-3	4	-4	-1
$d[i]$	-1	-1	-1	1
$t[i]$	2	3	4	1
$depth[i]$	0	1	0	0

Listing 2

```
system1 = {x1_{1,3} + x1_{1,4} - 1/10 x1_{2,1} == 0,
x1_{2,1} + x1_{2,3} + x1_{2,4} == 0,
x1_{3,4} - 6/10 x1_{1,3} - 9/10 x1_{2,3} == 0,
-9/10 x1_{1,4} - 1/2 x1_{2,4} - 8/10 x1_{3,4} == 0};

t = {2,3,4,1};
p = {-3,4, - 4, - 1};
d = {-1, - 1, - 1,1};

system1a = system1;
system1a[[1]] = system1a[[1]]/. {x1_{2,1} → 1};
system1a[[2]] = system1a[[2]]/. {x1_{2,1} → 1};
system1a[[2]] = system1a[[2]]/. {x1_{2,3} → 0};
system1a[[3]] = system1a[[3]]/. {x1_{2,3} → 0};

δ1_{2,1} = {x1_{2,1} → 1,x1_{2,3} → 0};

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1a[[t[[i]]]],x1_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system1a[[t[[i]]]],x1_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]];
```

```
δl2,1 = Join [δl2,1,δ] ;
If[i ≠ 3,
system1a[[Abs[p[[t[[i]]]]]]] = system1a[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x1Abs[p[[t[[4]]]]],t[[4]]/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1Abs[p[[t[[4]]]]],t[[4]] → 0}) ,x1Abs[p[[t[[4]]]]],t[[4]]] [[1]];
φ1 = (x1Abs[p[[t[[4]]]]],t[[4]]/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1Abs[p[[t[[4]]]]],t[[4]] → 1}) ,x1Abs[p[[t[[4]]]]],t[[4]]] [[1]]) − 1;
δ = {x1Abs[p[[t[[4]]]]],t[[4]] → φ0/(φ0−φ1)};
},
{
φ0 = x1t[[4]],Abs[p[[t[[4]]]]]/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1t[[4]],Abs[p[[t[[4]]]]] → 0}) ,x1t[[4]],Abs[p[[t[[4]]]]]] [[1]];
φ1 = (x1t[[4]],Abs[p[[t[[4]]]]]/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1t[[4]],Abs[p[[t[[4]]]]] → 1}) ,x1t[[4]],Abs[p[[t[[4]]]]]] [[1]]) − 1;
δ = {x1t[[4]],Abs[p[[t[[4]]]]] → φ0/(φ0−φ1)};
}];
δl2,1 = Join [δl2,1/.δ,δ] ;
}];
}];
δl2,1 →
{x1_{2,1} → 1, x1_{2,3} → 0, x1_{2,4} → −1, x1_{3,4} → −41/70, x1_{1,4} → 113/105, x1_{1,3} → −41/42}
```

The algorithm of the computation of the components of a partial solution of the system of type (6.2.1), which correspond to the set of arcs U_L^k for fixed k equal of $O(n)$ time in the worst case, $n = |I^k|$ [53].

6.3. Decomposition algorithms

Let $U_L = \{U_L^k, k \in K\}, U_L^k \subseteq U^k$ be some support of multinetwork S for the system (6.1.1). We define the set $U_B = \{U_B^k \subseteq U^k \backslash U_L^k, k \in K\}$ of the bicycle arcs $|U_B| = q + |U_0|$, selecting $q + |U_0|$ arbitrary arcs from the set $U^k \backslash U_L^k, k \in K$. We denote the set of arcs, which are not in the support of the multinetwork S for the system (6.1.1) and which are not bicyclic,

through $U_N : U_N = \{U_N^k, k \in K\}, U_N^k = U^k \backslash (U_L^k \cup U_B^k), k \in K$. Thus, $U^k = U_L^k \cup U_B^k \cup U_N^k$, where U_L^k, U_B^k, U_N^k are not intersecting sets of arcs.

We substitute the general solution (6.2.26) of the system (6.2.1) to the equations (6.1.2):

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \sum_{k\in K} \sum_{(i,j)^k\in U^k} \lambda_{ij}^{kp} x_{ij}^k =$$

$$= \sum_{k\in K} \left(\sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \left[\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \tilde{x}_{ij}^k \right] + \sum_{(i,j)^k\in U^k\backslash U_L^k} \lambda_{ij}^{kp} x_{ij}^k \right) =$$

$$= \sum_{k\in K} \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \left[\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \tilde{x}_{ij}^k \right] +$$

$$+ \sum_{k\in K} \sum_{(i,j)^k\in U^k\backslash U_L^k} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad p = \overline{1,q}. \tag{6.3.1}$$

We will change the order of summation in (6.3.1):

$$\sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_L^k} x_{\tau\rho}^k \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \delta_{ij}^k(\tau,\rho) + \sum_{k\in K} \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \tilde{x}_{ij}^k +$$

$$+ \sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_L^k} \lambda_{\tau\rho}^{kp} x_{\tau\rho}^k = \alpha_p, \quad p = \overline{1,q}. \tag{6.3.2}$$

In equations (6.3.2) we group the variables, which correspond to the sets of arcs $U^k \backslash U_L^k$, $k \in K$:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k\in U^k\backslash U_L^k} x_{\tau\rho}^k \left[\lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \delta_{ij}^k(\tau,\rho) \right] =$$

$$= \alpha_p - \sum_{k\in K} \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \tilde{x}_{ij}^k. \tag{6.3.3}$$

Definition 6.3.1. *The number*

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp} \delta_{ij}^k(\tau,\rho), \quad (\tau,\rho)^k \in U^k \backslash U_L^k, \tag{6.3.4}$$

we will call a determinant of the bicycle $B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k$, entailed by the arc $(\tau,\rho)^k \in U^k \backslash U_L^k$ by respect to equation (6.1.2) with number p.

We denote:

$$A^p = \alpha_p - \sum_{k\in K} \sum_{(i,j)^k \in U_L^k} \lambda_{ij}^{kp} \tilde{x}_{ij}^k. \tag{6.3.5}$$

The equations (6.3.3) in accordance with the formulas (6.3.4), (6.3.5) look like the following:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \Lambda_{\tau\rho}^{kp} x_{\tau\rho}^k = A^p, \quad p = \overline{1,q}. \tag{6.3.6}$$

In equations (6.3.6) we group the variables, corresponding to the sets $U_B^k, k \in K$:

$$\sum_{k\in K} \sum_{(\tau,\rho)^k \in U_B^k} \Lambda_{\tau\rho}^{kp} x_{\tau\rho}^k = A^p - \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_N^k} \Lambda_{\tau\rho}^{kp} x_{\tau\rho}^k, p = \overline{1,q}. \tag{6.3.7}$$

Let's perform the analogous modifications to the system (6.1.3). We note that the equations in the system (6.1.3) are a special case of the system (6.1.2) with coefficients λ_{ij}^{kp}, equal to either 1 or 0. But using the properties of the sparse system (6.1.3) we can acquire the optimal computational procedures.

To do this, we will enumerate the arcs in the set U_0 in the arbitrary order and let $\xi = \xi(i,j)$ be the number of the arc $(i,j) \in U_0$, $\xi \in \{1,2,\ldots,|U_0|\}$. In other words, we enumerate the equations of the system (6.1.3), where each equation with number $\xi(i,j)$ corresponds to the arc $(i,j) \in U_0$. We denote

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau,\rho), k \in K_0(i,j), \\ 0, k \notin K_0(i,j), \end{cases} \tag{6.3.8}$$

$$(i,j) \in U_0, (\tau,\rho)^k \in U^k \setminus U_L^k,\ k \in K,$$

where $B_{\tau\rho}^k = U_L^k \cup (\tau,\rho)^k$ is a bicycle, entailed by the arc $(\tau,\rho)^k \in U^k \setminus U_L^k$. After we substitute the general solution (6.2.26) of the system (6.1.1) for each $k \in K$ into the equations (6.1.3):

$$\sum_{k\in K_0(i,j)} x_{ij}^k = \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U_L^k}} x_{ij}^k + \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U^k \setminus U_L^k}} x_{ij}^k =$$

$$= \sum_{\substack{k\in K_0(i,j),\\ (i,j)^k \in U_L^k}} \left[\sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \tilde{x}_{ij}^k \right] +$$

$$+ \sum_{\substack{k \in K_0(i,j),\\ (i,j)^k \in U^k \backslash U_L^k}} x_{ij}^k = z_{ij}, \quad (i,j) \in U_0. \tag{6.3.9}$$

We change the order of summation, and get:

$$\sum_{\substack{k \in K_0(i,j),\\ (i,j)^k \in U^k \backslash U_L^k}} x_{\tau\rho}^k \left[\sum_{(i,j)^k \in U_L^k} \delta_{ij}^k(\tau,\rho) \right] +$$

$$+ \sum_{\substack{k \in K_0(i,j),\\ (i,j)^k \in U^k \backslash U_L^k}} x_{ij}^k = z_{ij} - \sum_{\substack{k \in K_0(i,j),\\ (i,j)^k \in U_L^k}} \tilde{x}_{ij}^k, \ (i,j) \in U_0. \tag{6.3.10}$$

We note that for each arc $(\tau,\rho)^k \in U^k \backslash U_L^k$, $k \in K_0(i,j)$, which entails a bicycle $B_{\tau\rho}^k$, the sum $\sum\limits_{(i,j)^k \in B_{\tau\rho}^k} \delta_{ij}^k(\tau,\rho)$ is equal to $\delta_{ij}(B_{\tau\rho}^k)$:

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau,\rho), k \in K_0(i,j), \\ \\ 0, k \notin K_0(i,j), \end{cases} \quad (i,j) \in U_0. \tag{6.3.11}$$

We denote a right hand side in (6.3.10) through A_{ij}:

$$A_{ij} = z_{ij} - \sum_{k \in K_0(i,j)} \sum_{(i,j)^k \in U_L^k} \tilde{x}_{ij}^k, \ (i,j) \in U_0. \tag{6.3.12}$$

So, in accordance with (6.3.11), (6.3.12) the equations (6.3.10) take the following form:

$$\sum_{k \in K_0(i,j)} \sum_{(\tau,\rho)^k \in U^k \backslash U_L^k} \delta_{ij}(B_{\tau\rho}^k) x_{\tau\rho}^k = A_{ij}, \ (i,j) \in U_0. \tag{6.3.13}$$

In the equations of the system (6.3.13) we group the variables, which correspond to the sets $U_B^k, k \in K$, where $U_B = \{U_B^k \subseteq U^k \backslash U_L^k, k \in K\}$ is the set of bicyclic arcs, $|U_B| = q + |U_0|$:

$$\sum_{k \in K_0(i,j)} \sum_{(\tau,\rho)^k \in U_B^k} \delta_{ij}(B_{\tau\rho}^k) x_{\tau\rho}^k =$$

$$= A_{ij} - \sum_{k\in K_0(i,j)} \sum_{(\tau,\rho)^k\in U_N^k} \delta_{ij}(B_{\tau\rho}^k)x_{\tau\rho}^k, \quad (i,j)\in U_0. \tag{6.3.14}$$

So, the equations (6.3.7) and (6.3.14) can be presented in the following vector-matrix form:

$$Dx_B = \beta, \tag{6.3.15}$$

where

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},\ D_1 = (\Lambda_{\tau\rho}^{kp}, p=\overline{1,q}, t(\tau,\rho)^k = \overline{1,|U_B|}),$$

$$D_2 = (\delta_{ij}(B_{\tau\rho}^k), \xi(i,j) = \overline{1,|U_0|}, t(\tau,\rho)^k = \overline{1,|U_B|}),$$

$$\beta' = \left(\beta_p, p=\overline{1,q};\ \beta_{q+\xi(i,j)}, (i,j)\in U_0\right)',$$

$\xi = \xi(i,j)$ is the number of arc $(i,j)\in U_0$, $\xi\in\{1,2,\ldots,|U_0|\}$.

The desired unknowns $x_B = (x_{\tau\rho}^k, (\tau,\rho)^k \in U_B^k, k\in K)$ are ordered in accordance with the numeration of the set of bicyclic arcs:

$$t = t(\tau,\rho)^k, (\tau,\rho)^k\in U_B^k, k\in K, t\in\{1,2,\ldots,|U_B|\}.$$

Here

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho), \quad (\tau,\rho)^k\in U^k\backslash U_L^k \tag{6.3.16}$$

and for each arc $(\tau,\rho)^k\in U^k\backslash U_L^k, k\in K$:

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau,\rho), k\in K_0(i,j), \\ \\ 0, k\notin K_0(i,j), \end{cases} \quad (i,j)\in U_0, \tag{6.3.17}$$

$$\beta_p = A^p - \sum_{k\in K} \sum_{(\tau,\rho)^k\in U_N^k} \Lambda_{\tau\rho}^{kp} x_{\tau\rho}^k, p=\overline{1,q}, \tag{6.3.18}$$

$$\beta_{q+\xi(i,j)} = A_{ij} - \sum_{k\in K} \sum_{(\tau,\rho)^k\in U_N^k} \delta_{ij}(B_{\tau\rho}^k)x_{\tau\rho}^k, \quad (i,j)\in U_0, \tag{6.3.19}$$

$$A^p = \alpha_p - \sum_{k\in K} \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp}\tilde{x}_{ij}^k, p=\overline{1,q}, \tag{6.3.20}$$

$$A_{ij} = z_{ij} - \sum_{\substack{k \in K_0(i,j),\\ (i,j)^k \in U_L^k}} \tilde{x}_{ij}^k, \quad (i,j) \in U_0. \tag{6.3.21}$$

In the case of non-singular matrix D from the system (6.3.15) we compute x_B, the components of which correspond to the set U_B of the bicyclic arcs:

$$x_B = D^{-1}\beta. \tag{6.3.22}$$

R e m a r k 6.3.1. In the general case we cannot guarantee the non-singularity of the matrix D because of the arbitrary selection of the arcs in the set $U_B = \{U_B^k, k \in K\}$. In the case when $det\, D = 0$, we have to include other arcs into the set U_B to recompute the determinants of the matrix D and the vector β of the system (6.3.15) anew.

We denote $D^{-1} = (\nu_{ls}; l,s = \overline{1,|U_B|})$. We present (6.3.22) in the component form:

$$\begin{gathered} x_{\tau\rho}^k = \sum_{p=1}^{q} \nu_{t,p}\beta_p + \sum_{(i,j)\in U_0} \nu_{t,q+\xi(i,j)}\beta_{q+\xi(i,j)}, \\ t = t(\tau,\rho)^k,\ (\tau,\rho)^k \in U_B^k, k \in K. \end{gathered} \tag{6.3.23}$$

$$\begin{gathered} x_{ij}^k = \sum_{(\tau,\rho)^k \in U_N^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \psi_{ij}^k + \tilde{x}_{ij}^k, (i,j)^k \in U_L^k, k \in K, \\ x_{\tau\rho}^k \in \mathbf{R}, (\tau,\rho)^k \in U_N^k, \psi_{ij}^k = \sum_{(\tau,\rho)^k \in U_B^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho). \end{gathered} \tag{6.3.24}$$

R e m a r k 6.3.2. The components of the vector $\tilde{x}^k = (\tilde{x}_{ij}^k, (i,j)^k \in U^k)$ which is the partial solution of the system (6.1.1), are constructed in accordance with the rules of the remark 6.2.2.

For the considered system (6.1.4) – (6.1.5) the set of arcs $U_L^k, k \in K$, $K = \{1,2,3,4\}$ (see Figure 6.2) is the support of the multinetwork S for the system (6.1.4). Let's form the set $U_B = \bigcup_{k=1}^{4} U_B^k = \{(3,4)^1\}$ of the bicyclic arcs. The set U_B contains one element. We also form the set of arcs $U_N = \bigcup_{k=1}^{4} U_N^k$, $U_N = \{(2,3)^4\}$, which, just as U_B, contains one element: $(2,3)^4$. According to (6.3.4) we compute the determinants of the bicycles

$B^k_{\tau\rho}$, entailed by arcs $(\tau,\rho)^k \in U^k \backslash U^k_L$, for each $k \in K = \{1,2,3,4\}$ by respect to the equations (6.1.5):

$$\Lambda^{1,1}_{3,4} = -\frac{62}{9},\ \Lambda^{4,1}_{2,3} = -\frac{1319}{611}.$$

To compute the elements of the matrix D of the system (6.3.15) we have to enumerate the arcs in the set $U_B = \bigcup\limits_{k=1}^{4} U^k_B = \{(3,4)^1\}$. In this case the numeration is trivial: $t(3,4)^1 = 1$.

We form the matrix D. As $U_0 = \varnothing$, then $D = D_1$, where $D_1 = \left(\Lambda^{1,1}_{3,4}\right) = \left(-\frac{62}{9}\right)$. Therefore $D^{-1} = \left(-\frac{9}{62}\right)$.

$$\begin{gathered}
\tilde{x}^1_{1,4} - \tilde{x}^1_{2,1} = 0, \\
\tilde{x}^1_{2,1} + \tilde{x}^1_{2,3} + \tilde{x}^1_{2,4} = 15, \\
\tilde{x}^1_{3,4} - \frac{9}{10}\tilde{x}^1_{2,3} = \frac{16}{5}, \\
-\frac{3}{5}\tilde{x}^1_{1,4} - \frac{3}{10}\tilde{x}^1_{2,4} - \frac{3}{10}\tilde{x}^1_{3,4} = -\frac{63}{10}, \\
\tilde{x}^1_{3,4} = 0.
\end{gathered} \tag{6.3.25}$$

To form a vector β which is the right hand side of the system (6.3.15) we need to construct in partial solution of the system (6.1.4). For this we need to construct the solutions of the independent subsystems of the system (6.1.4) for each $k \in K = \{1,2,3,4\}$. A partial solution of the system (6.1.4) for each $k \in K$ we construct in accordance with the rules in the remark 6.2.2. Let's consider the algorithm of the construction of a partial solution $\tilde{x}^k = (\tilde{x}^k_{ij},(i,j)^k \in U^k)$ of the system (6.1.4) for $k = 1$. The components of the vector $\tilde{x}^1$ satisfy the sparse SLAE (6.3.25).

To compute the components $\tilde{x}^1_{i,j}$, $(i,j) \in U^1_L$ of the solution of the system (6.3.25) in $O(n)$ time in the worst case, where $n = |I^k|$, we use the same rooted structures, we used for the solution of the system (6.2.21) of the computation of the characteristic vector $\delta^k(\tau,\rho)$ for the fixed k. So,

$$\tilde{x}^1 = (\tilde{x}^1_{1,4}, \tilde{x}^1_{2,1}, \tilde{x}^1_{2,3},\ \tilde{x}^1_{2,4}, \tilde{x}^1_{3,4},\ ) = \left(\frac{22}{9}, \frac{22}{9},\ -\frac{32}{9}, \frac{145}{9}, 0\right).$$

By analogy $\tilde{x}^1$ we construct a partial solution of system (6.1.4) for $k = 2, 3, 4$:

$$\tilde{x}^2 = (\tilde{x}^2_{1,4}, \tilde{x}^2_{2,1}, \tilde{x}^2_{2,4}\,) = (1, 2, 2)\,,$$

$$\tilde{x}^3 = (\tilde{x}^3_{1,3}, \tilde{x}^3_{1,4}, \tilde{x}^3_{2,1},\ \tilde{x}^3_{2,4}\,) = (7, 3, 6, 8)\,,$$

$$\tilde{x}^4 = (\tilde{x}^4_{1,3}, \tilde{x}^4_{2,1}, \tilde{x}^4_{2,3},\ \tilde{x}^4_{2,4}, \tilde{x}^4_{3,4}\ ) = (6, 1, 0, 1, 1)\,.$$

After substitute to (6.3.20) of the components of the vectors $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4$ which is a partial solution of the system (6.1.4), we compute $A^1 = -\dfrac{310}{9}$.

Let's construct the vector β of the right hand side of the system (6.3.15):

$$\beta_1 = A^1 - \Lambda^{4,1}_{2,3}\, x^4_{2,3} = -\frac{310}{9} + \frac{1319}{611} x^4_{2,3}.$$

As the matrix D is non-singular, we can use the formula (6.3.22) to find the components of the vector $x_B = (x^k_{\tau\rho,(\tau,\rho)^k} \in U^k_B, k \in K)$:

$$x^1_{3,4} = -\frac{9}{62}\left(-\frac{310}{9} + \frac{1319}{611} x^4_{2,3}\right) = 5 - \frac{11871}{37882} x^4_{2,3}.$$

Finally, the general solution of the sparse linear underdetermined system (6.1.4) – (6.1.5) by respect to the support $U_S = U_L \cup U_B$, where the set U_L is presented in the Figure 6.2, the set U_B is comprised of the unique arc $(3,4)^1$, $U_B = \{(3,4)^1\}$, which looks like the following:

$$x^1_{1,4} = x^1_{3,4}\delta^1_{14}(3,4) + \tilde{x}^1_{1,4} = 3 - \frac{1319}{37882} x^4_{2,3},$$

$$x^1_{2,1} = x^1_{3,4}\delta^1_{21}(3,4) + \tilde{x}^1_{2,1} = 3 - \frac{1319}{37882} x^4_{2,3},$$

$$x^1_{2,3} = x^1_{3,4}\delta^1_{23}(3,4) + \tilde{x}^1_{2,3} = 2 - \frac{6595}{18941} x^4_{2,3},$$

$$x^1_{2,4} = x^1_{3,4}\delta^1_{24}(3,4) + \tilde{x}^1_{2,4} = 10 + \frac{14509}{37882} x^4_{2,3},$$

$$x^1_{3,4} = \nu_{1,1}\beta_1 = -\frac{9}{62}\left(-\frac{310}{9} + \frac{1319}{611} x^4_{2,3}\right) = 5 - \frac{11871}{37882} x^4_{2,3},$$

$$x^2_{1,4} = 1,\ x^2_{2,1} = 2,\ x^2_{2,4} = 2,$$

$$x^3_{1,3} = 7,\ x^3_{1,4} = 3,\ x^3_{2,1} = 6,\ x^3_{2,4} = 8,$$

$$x^4_{1,3} = x^4_{2,3}\delta^4_{1,3}(2,3) + \tilde{x}^4_{1,3} = 6 + \frac{3}{611}x^4_{2,3},$$

$$x^4_{2,1} = x^4_{2,3}\delta^4_{2,1}(2,3) + \tilde{x}^4_{2,1} = 1 + \frac{10}{611}x^4_{2,3},$$

$$x^4_{2,4} = x^4_{2,3}\delta^4_{2,4}(2,3) + \tilde{x}^4_{2,4} = 1 - \frac{621}{611}x^4_{2,3},$$

$$x^4_{3,4} = x^4_{2,3}\delta^4_{3,4}(2,3) + \tilde{x}^4_{3,4} = 1 + \frac{552}{611}x^4_{2,3},$$

$x^4_{2,3}$ is the independent variable, $x^4_{2,3} \in \mathbf{R}$.

6.4. Example of decomposition of sparse systems

Let's consider the example (6.4.1) – (6.4.3) of the construction of the solution of the system of type (6.1.1) - (6.1.3) for the generalized multinetwork $S = (I,U)$, $K = \{1,2,3,4,5\}$ (see Figure 6.6) The multinetwork $S = (I,U)$ is represented as union $|K| = 5$ of the networks S^k, $S^k = (I^k,U^k), k \in K$, where

$$I = \{1,2,3,4,5,6\}, U = \{(1,3),(1,4),(1,5),(1,6),(2,1),$$

$$(2,6),(3,2)(3,4),(3,6),(4,6),(5,2),(5,4),(6,5)\}.$$

The system (6.4.1) – (6.4.3) has the following form:

$$x^1_{1,3} - 0.6x^1_{2,1} = 3.4$$

$$x^1_{2,1} = 1$$

$$x^1_{3,4} + x^1_{3,6} - x^1_{1,3} = 12$$

$$x^1_{4,6} - 0.7x^1_{3,4} - 0.8x^1_{5,4} = -6.1$$

$$x^1_{5,4} - 0.1x^1_{6,5} = 5.2$$

$$x^1_{6,5} - x^1_{3,6} - 0.2x^1_{4,6} = 0$$

$$
\begin{gathered}
x_{1,3}^2 + x_{1,4}^2 + x_{1,5}^2 + x_{1,6}^2 = 24 \\
-0.9x_{3,2}^2 = -8.1 \\
x_{3,2}^2 + x_{3,4}^2 - 0.3x_{1,3}^2 = 7.6 \\
x_{4,6}^2 - x_{1,4}^2 - 0.9x_{3,4}^2 = -0.9 \\
-0.5x_{1,5}^2 = -2.5 \\
-0.9x_{1,6}^2 - 0.7x_{4,6}^2 = -8.7
\end{gathered}
$$

$$
\begin{gathered}
x_{1,3}^3 + x_{1,5}^3 - 0.5x_{2,1}^3 = -0.5 \\
x_{2,1}^3 - x_{5,2}^3 = 8 \\
-0.5x_{1,3}^3 = -1 \\
x_{4,6}^3 - 0.6x_{5,4}^3 = 6.6 \\
x_{5,2}^3 + x_{5,4}^3 - 0.5x_{1,5}^3 - 0.6x_{6,5}^3 = -0.2 \\
x_{6,5}^3 - 0.9x_{4,6}^3 = -1.1
\end{gathered}
\tag{6.4.1}
$$

$$
\begin{gathered}
x_{1,4}^4 + x_{1,5}^4 + x_{1,6}^4 - 0.7x_{2,1}^4 = 1.5 \\
x_{2,1}^4 - 0.8x_{3,2}^4 - x_{5,2}^4 = -4.8 \\
x_{3,2}^4 + x_{3,4}^4 + x_{3,6}^4 = 4 \\
x_{4,6}^4 - 0.1x_{1,4}^4 - 0.5x_{3,4}^4 - 0.3x_{5,4}^4 = 0.1 \\
x_{5,2}^4 + x_{5,4}^4 - 0.7x_{1,5}^4 = 10.3 \\
-x_{1,6}^4 - 0.6x_{3,6}^4 - 0.5x_{4,6}^4 = -2.6
\end{gathered}
$$

$$
\begin{gathered}
x_{1,5}^5 - 0.1x_{2,1}^5 = 1 \\
x_{2,1}^5 + x_{2,6}^5 - 0.1x_{3,2}^5 = 6.8 \\
x_{3,2}^5 + x_{3,4}^5 = 7 \\
x_{4,6}^5 - 0.6x_{3,4}^5 = 6 \\
-x_{1,5}^5 - 0.1x_{6,5}^5 = -1.1 \\
x_{6,5}^5 - 0.9x_{2,6}^5 - 0.3x_{4,6}^5 = -8
\end{gathered}
$$

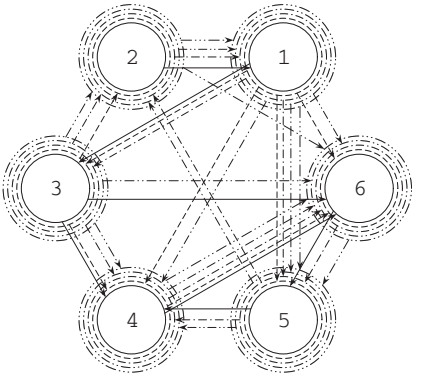

Fig. 6.6. The union of networks $S^k = (I^k,U^k), k \in K = \{1,2,3,4,5\}$

$$\begin{gathered}
2x^1_{1,3} + 3x^2_{1,3} + 8x^3_{1,3} + x^2_{1,4} + 8x^4_{1,4} + x^2_{1,5}+ \\
+10x^3_{1,5} + 7x^4_{1,5} + 2x^5_{1,5} + 3x^2_{1,6} + 8x^4_{1,6} + 6x^1_{2,1}+ \\
+5x^3_{2,1} + 5x^4_{2,1} + 2x^5_{2,6} + 4x^2_{3,2} + 8x^4_{3,2} + 6x^5_{3,2} + 5x^1_{3,4} + 8x^2_{3,4}+ \\
+4x^4_{3,4} + 7x^5_{3,4} + 6x^1_{3,6} + 5x^4_{3,6} + 2x^1_{4,6} + 7x^2_{4,6} + 7x^3_{4,6} + 9x^4_{4,6}+ \\
+6x^5_{4,6} + 4x^3_{5,2} + 5x^4_{5,2} + 3x^1_{5,4} + 6x^3_{5,4} + 6x^4_{5,4} + 8x^1_{6,5}+ \\
+3x^3_{6,5} + 2x^5_{6,5} = 801 \\
9x^1_{1,3} + 7x^2_{1,3} + 10x^3_{1,3} + x^2_{1,4} + 4x^4_{1,4} + 7x^2_{1,5}+ \\
+6x^3_{1,5} + 2x^4_{1,5} + 6x^5_{1,5} + 4x^2_{1,6} + 10x^4_{1,6}+ \\
+5x^1_{2,1} + 10x^3_{2,1} + 2x^4_{2,1} + 5x^5_{2,1} + 4x^5_{2,6}+ \\
+5x^2_{3,2} + 7x^4_{3,2} + 6x^5_{3,2} + 10x^1_{3,4} + x^2_{3,4} + x^4_{3,4} + 7x^5_{3,4} + 8x^1_{3,6}+ \\
+7x^4_{3,6} + 8x^1_{4,6} + 2x^2_{4,6} + 8x^3_{4,6} + 5x^4_{4,6} + 2x^5_{4,6} + x^3_{5,2} + 5x^4_{5,2}+ \\
+4x^1_{5,4} + 7x^3_{5,4} + 8x^4_{5,4} + 6x^1_{6,5} + 9x^3_{6,5} + 9x^5_{6,5} = 989
\end{gathered} \tag{6.4.2}$$

$$\begin{gathered}
x^2_{1,6} + x^4_{1,6} = 6 \\
x^2_{3,4} + x^4_{3,4} + x^5_{3,4} = 8.
\end{gathered} \tag{6.4.3}$$

We select the support U_L of the network $S = (I,U)$ for the system (6.4.1). Let the support $U_L = \{U^k_L, k \in K = \{1,2,3,4,5\}\}$ consist of the only one connected component for each of the types of flows:

$$U^1_L = \{(1,3)^1,(2,1)^1,(3,4)^1,(4,6)^1,(5,4)^1,(6,5)^1\},$$

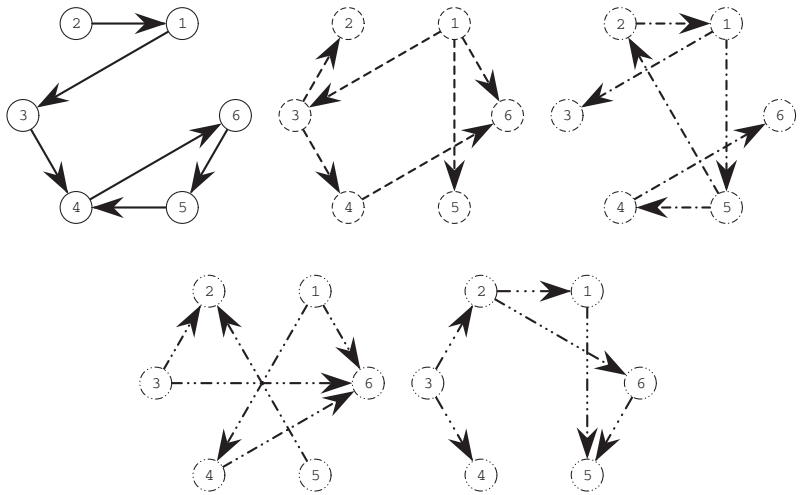

Fig. 6.7. The sets U_L^k for the networks S^k, $k \in K = \{1,2,3,4,5\}$

$$U_L^2 = \{(1,3)^2,(1,5)^2,(1,6)^2,(3,2)^2,(3,4)^2,(4,6)^2\},$$
$$U_L^3 = \{(1,3)^3,(1,5)^3,(2,1)^3,(4,6)^3,(5,2)^3,(5,4)^3\},$$
$$U_L^4 = \{(1,4)^4,(1,6)^4,(3,2)^4,(3,6)^4,(4,6)^4,(5,2)^4\},$$
$$U_L^5 = \{(1,5)^5,(2,1)^5,(2,6)^5,(3,2)^5,(3,4)^5,(6,5)^5\}.$$

The structures which represent the support U_L^k, $k \in K = \{1,2,3,4,5\}$ of the network $S = (I,U)$ for the sparse system (6.4.1), are presented in the Figure 6.7.

For each $k \in K = \{1,2,3,4,5\}$ we construct the set of the characteristic vectors $\{\delta^k(\tau,\rho),(\tau,\rho)^k \in U^k \setminus U_L^k\}$ by respect to the selected support (see Figure 6.7) of the multinetwork $S = (I,U)$ for the system (6.4.1).

The set of the characteristic vectors by respect to the selected support U_L^1 of the network $S^1 = (I^1,U^1)$ is presented in Table 6.4.1.

The set of the characteristic vectors by respect to the selected support U_L^2 of the network $S^2 = (I^2,U^2)$ consists of the only one vector, the components of which are presented in Table 6.4.2.

Table 6.4.1

Characteristic vectors by respect to the selected support U_L^1

$(i,j)^1$	$(1,3)^1$	$(2,1)^1$	$(3,4)^1$	$(3,6)^1$	$(4,6)^1$	$(5,4)^1$	$(6,5)^1$
$\delta_{ij}^1(3,6)$	0	0	-1	1	$-\frac{155}{246}$	$\frac{43}{492}$	$\frac{215}{246}$

Table 6.4.2

Characteristic vectors by respect to the selected support U_L^2

$(i,j)^2$	$(1,3)^2$	$(1,4)^2$	$(1,5)^2$	$(1,6)^2$	$(3,2)^2$	$(3,4)^2$	$(4,6)^2$
$\delta_{ij}^2(1,4)$	$-\frac{200}{711}$	1	0	$-\frac{511}{611}$	0	$-\frac{20}{237}$	$\frac{73}{79}$

The set of the characteristic vectors by respect to the selected support U_L^3 of the network $S^3 = (I^3,U^3)$ also consists of the only one vector, the components of which are presented in Table 6.4.3.

Table 6.4.3

Characteristic vectors by respect to the selected support U_L^3

$(i,j)^3$	$(1,3)^3$	$(1,5)^3$	$(2,1)^3$	$(4,6)^3$	$(5,2)^3$	$(5,4)^3$	$(6,5)^3$
$\delta_{ij}^3(6,5)$	0	$-\frac{338}{405}$	$-\frac{676}{405}$	$\frac{10}{9}$	$-\frac{676}{405}$	$\frac{50}{27}$	1

The components of the characteristic vectors $\delta^4(1,5)$, $\delta^4(2,1)$, $\delta^4(3,4)$, $\delta^4(5,4)$, which correspond to the arcs $(1,4)^4$, $(1,5)^4$, $(1,6)^4$, $(2,1)^4$, $(3,2)^4$, are presented in Table 6.4.4.

The components of the characteristic vectors

$$\delta^4(1,5), \delta^4(2,1), \delta^4(3,4), \delta^4(5,4),$$

which correspond to the arcs $(3,4)^4, (3,6)^4, (4,6)^4, (5,2)^4, (5,4)^4$ are presented in Table 6.4.4.

Table 6.4.4

Components $\delta^4_{ij}(1,5)$, $\delta^4_{ij}(2,1)$, $\delta^4_{ij}(3,4)$, $\delta^4_{ij}(5,4)$

$(i,j)^4$	$(1,4)^4$	$(1,5)^4$	$(1,6)^4$	$(2,1)^4$	$(3,2)^4$
$\delta^4_{ij}(1,5)$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$-\frac{7}{8}$
$\delta^4_{ij}(2,1)$	$-\frac{1}{19}$	0	$\frac{143}{190}$	1	$\frac{5}{4}$
$\delta^4_{ij}(3,4)$	$-\frac{7}{19}$	0	$\frac{7}{19}$	0	0
$\delta^4_{ij}(5,4)$	$-\frac{12}{19}$	0	$\frac{12}{19}$	0	$\frac{5}{4}$

$(i,j)^4$	$(3,4)^4$	$(3,6)^4$	$(4,6)^4$	$(5,2)^4$	$(5,4)^4$
$\delta^4_{ij}(1,5)$	0	$\frac{7}{8}$	$-\frac{1}{20}$	$\frac{7}{10}$	0
$\delta^4_{ij}(2,1)$	0	-1	$\frac{44}{95}$	0	0
$\delta^4_{ij}(3,4)$	1	-1	$\frac{5}{37}$	0	0
$\delta^4_{ij}(5,4)$	0	$-\frac{5}{4}$	$\frac{9}{38}$	-1	1

Table 6.4.5

The characteristic vector $\delta^5(4,6)$

$(i,j)^5$	$(1,5)^5$	$(2,1)^5$	$(2,6)^5$	$(3,2)^5$	$(3,4)^5$	$(4,6)^5$	$(6,5)^5$
$\delta^5_{ij}(4,6)$	$-\frac{3}{20}$	$-\frac{3}{2}$	$\frac{4}{3}$	$-\frac{5}{3}$	$\frac{5}{3}$	1	$\frac{3}{2}$

The set of the characteristic vectors by respect to the support U_L^5 of the network $S^5 = (I^5,U^5)$ consists of the only one vector $\delta^5(4,6)$, the components of which are represented in Table 6.4.5.

Let's construct a partial solution of the system (6.4.1) for each $k \in K$ in accordance with the rules of the remark 6.2.2. A partial solution to the system (6.4.1) for $k = 1$ is presented in Table 6.4.6, for $k = 2$ in Table 6.4.7.

Table 6.4.6

The partial solution of the system (6.4.1) for $k = 1$

$(i,j)^1$	$(1,3)^1$	$(2,1)^1$	$(3,4)^1$	$(3,6)^1$	$(4,6)^1$	$(5,4)^1$	$(6,5)^1$
$\widetilde{x}_{ij}^1$	4	1	16	0	$\frac{2315}{246}$	$\frac{2651}{492}$	$\frac{463}{246}$

Table 6.4.7

The partial solution of the system (6.4.1) for $k = 2$

$(i,j)^2$	$(1,3)^2$	$(1,4)^2$	$(1,5)^2$	$(1,6)^2$	$(3,2)^2$	$(3,4)^2$	$(4,6)^2$
$\widetilde{x}_{ij}^2$	$\frac{2296}{237}$	0	5	$\frac{2207}{237}$	9	$\frac{119}{79}$	$\frac{36}{79}$

The partial solution of the system (6.4.1) for $k = 3$ and $k = 4$ is presented in Tables 6.4.8 and 6.4.9 accordingly. The partial solution of the

Table 6.4.8

The partial solution of the system (6.4.1) for $k = 3$

$(i,j)^3$	$(1,3)^3$	$(1,5)^3$	$(2,1)^3$	$(4,6)^3$	$(5,2)^3$	$(5,4)^3$	$(6,5)^3$
$\widetilde{x}_{ij}^3$	2	$\frac{3176}{405}$	$\frac{8377}{405}$	$\frac{11}{9}$	$\frac{5137}{405}$	$-\frac{242}{27}$	0

system (6.4.1) for $k = 5$ is presented in Table 6.4.10.

Table 6.4.9

The partial solution of the system (6.4.1) for $k = 4$

$(i,j)^4$	$(1,4)^4$	$(1,5)^4$	$(1,6)^4$	$(2,1)^4$	$(3,2)^4$	$(3,4)^4$
$\widetilde{x}^4_{ij}$	$\frac{219}{38}$	0	$-\frac{81}{19}$	0	$-\frac{55}{8}$	0

$(i,j)^4$	$(3,6)^4$	$(4,6)^4$	$(5,2)^4$	$(5,4)^4$
$\widetilde{x}^4_{ij}$	$\frac{87}{8}$	$\frac{257}{380}$	$\frac{103}{10}$	0

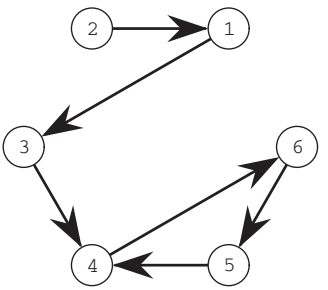

Fig. 6.8. The set of arcs U^1_L

As the matrix for the system (6.4.1) is block-diagonal, the general solution to the system (6.4.1) consists of the solutions of the independent subsystems for each $k \in K = \{1,2,3,4,5\}$. The general solution for each of the independent subsystems we construct in the form of sum of the general solution of the homogenous system, generated by the system (6.4.1) for the fixed $k \in K = \{1,2,3,4,5\}$, and the partial solution to the system for $k \in K$, which is computed according to the rules in the remark 6.2.2.

For the arcs of the set $U^1_L = \{(1,3)^1,(2,1)^1,(3,4)^1,(4,6)^1,(5,4)^1,(6,5)^1\}$ (see Figure 6.8) the general solution to the linear underdetermined system (6.4.1) looks like the following:

$$x^1_{1,3} = 4, x^1_{2,1} = 1,\ x^1_{34} = -x^1_{3,6} + 16,$$

Table 6.4.10

The partial solution of the system (6.4.1) for $k = 5$

$(i,j)^5$	$(1,5)^5$	$(2,1)^5$	$(2,6)^5$	$(3,2)^5$	$(3,4)^5$	$(4,6)^5$	$(6,5)^5$
$\widetilde{x}^5_{ij}$	$\frac{47}{20}$	$\frac{27}{2}$	-5	17	-10	0	$-\frac{25}{2}$

$$x^1_{4,6} = -\frac{155}{246}x^1_{3,6} + \frac{2315}{246},\ x^1_{5,4} = \frac{43}{492}x^1_{3,6} + \frac{2651}{492},$$
$$x^1_{6,5} = \frac{215}{246}x^1_{3,6} + \frac{463}{246}.$$

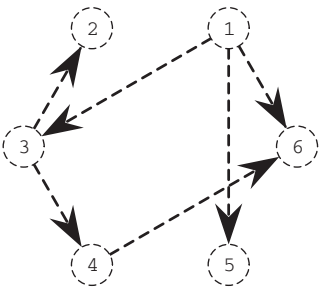

Fig. 6.9. The set of the arcs U^2_L

For the arcs of the set $U^2_L = \{(1,3)^2,(1,5)^2,(1,6)^2,(3,2)^2,(3,4)^2,(4,6)^2\}$ (see Figure 6.9) the general solution to the linear underdetermined system (6.4.1) looks like the following:

$$x^2_{1,3} = -\frac{200}{711}x^2_{1,4} + \frac{2296}{237},\ x^2_{1,5} = 5,$$

$$x^2_{1,6} = -\frac{511}{711}x^2_{1,4} + \frac{2207}{237},\ x^2_{3,2} = 9,$$

$$x^2_{3,4} = -\frac{20}{237}x^2_{1,4} + \frac{119}{79},\ x^2_{4,6} = \frac{73}{79}x^2_{1,4} + \frac{36}{79}.$$

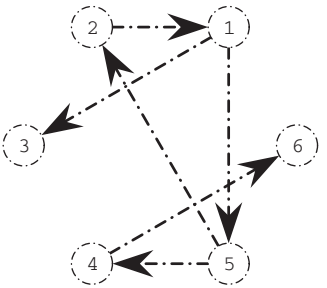

Fig. 6.10. The set of arcs U_L^3

For the arcs of the set $U_L^3 = \{(1,3)^3,(1,5)^3,(2,1)^3,(4,6)^3,(5,2)^3,(5,4)^3\}$ (see Figure 6.10) the general solution to the linear underdetermined system (6.4.1) looks like the following:

$$x_{1,3}^3 = 2, \quad x_{1,5}^3 = -\frac{338}{405}x_{6,5}^3 + \frac{3176}{405},$$

$$x_{2,1}^3 = -\frac{676}{405}x_{6,5}^3 + \frac{8377}{405},$$

$$x_{5,2}^3 = -\frac{676}{405}x_{6,5}^3 + \frac{5137}{405},$$

$$x_{4,6}^3 = \frac{10}{9}x_{6,5}^3 + \frac{11}{9},$$

$$x_{5,4}^3 = \frac{50}{27}x_{6,5}^3 - \frac{242}{27}.$$

For the arcs of the set $U_L^4 = \{(1,4)^4,(1,6)^4,(3,2)^4,(3,6)^4,(4,6)^4,(5,2)^4\}$ (see Figure 6.11) the general solution to the linear underdetermined system (6.4.1) looks like the following:

$$x_{1,4}^4 = -\frac{1}{19}x_{2,1}^4 - \frac{1}{2}x_{1,5}^4 - \frac{7}{19}x_{3,4}^4 - \frac{12}{19}x_{5,4}^4 + \frac{219}{38},$$

$$x_{1,6}^4 = \frac{143}{190}x_{2,1}^4 - \frac{1}{2}x_{1,5}^4 + \frac{7}{19}x_{3,4}^4 + \frac{12}{19}x_{5,4}^4 - \frac{81}{19},$$

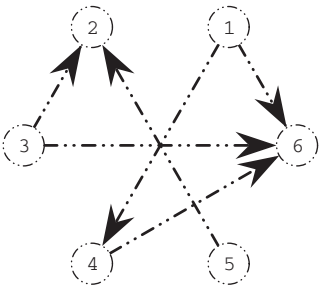

Fig. 6.11. The set of arcs U_L^4

$$x_{3,2}^4 = \frac{5}{4}x_{2,1}^4 - \frac{7}{8}x_{1,5}^4 + \frac{5}{4}x_{5,4}^4 - \frac{55}{8},$$

$$x_{3,6}^4 = -\frac{5}{4}x_{2,1}^4 + \frac{7}{8}x_{1,5}^4 - x_{3,4}^4 - \frac{5}{4}x_{5,4}^4 + \frac{87}{8},$$

$$x_{4,6}^4 = -\frac{1}{190}x_{2,1}^4 - \frac{1}{20}x_{1,5}^4 + \frac{44}{95}x_{3,4}^4 + \frac{9}{38}x_{5,4}^4 + \frac{257}{380},$$

$$x_{5,2}^4 = \frac{7}{10}x_{1,5}^4 - x_{5,4}^4 + \frac{103}{10}.$$

For the arcs of the set $U_L^5 = \{(1,5)^5,(2,1)^5,(2,6)^5,(3,2)^5,(3,4)^5,(6,5)^5\}$ (see Figure 6.12) the general solution to the linear underdetermined system (6.4.1) looks like the following:

$$x_{1,5}^5 = -\frac{3}{20}x_{4,6}^5 + \frac{47}{20},\ x_{2,1}^5 = -\frac{3}{2}x_{4,6}^5 + \frac{27}{2},$$
$$x_{2,6}^5 = \frac{4}{3}x_{4,6}^5 - 5,$$
$$x_{3,2}^5 = -\frac{5}{3}x_{4,6}^5 + 17,$$

$$x^5_{3,4} = \frac{5}{3}x^5_{4,6} - 10 \;\; x^5_{6,5} = \frac{3}{2}x^5_{4,6} - \frac{25}{2}.$$

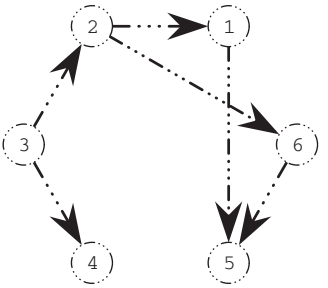

Fig. 6.12. The set of arcs U^5_L

According to (6.3.4) we compute the determinants of the bicycles $B^k_{\tau\rho}$, entailed by the arcs $(\tau,\rho)^k \in U^k \backslash U^k_L$, for each $k \in K = \{1,2,3,4,5\}$, by respect to the equations (6.4.2) with numbers $p = 1,2$ (see Table 6.4.11).

Table 6.4.11

Determinants of the bicycles $B^k_{\tau\rho}$, entailed by the arcs
$(\tau,\rho)^k \in U^k \backslash U^k_L, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(3,6)^1$	$(1,4)^2$	$(6,5)^3$	$(4,6)^5$	$(1,5)^4$	$(2,1)^4$	$(3,4)^4$	$(5,4)^4$
$\Lambda^{k1}_{\tau\rho}$	$\frac{1147}{164}$	$\frac{899}{237}$	$-\frac{599}{405}$	$\frac{391}{30}$	$-\frac{23}{40}$	$\frac{1087}{76}$	$\frac{301}{95}$	$\frac{523}{76}$
$\Lambda^{k2}_{\tau\rho}$	$-\frac{178}{123}$	$-\frac{493}{237}$	$\frac{3031}{405}$	$\frac{141}{10}$	$-\frac{7}{4}$	$\frac{353}{38}$	$-\frac{28}{19}$	$\frac{303}{38}$

According to the formula (6.3.8) we compute $\delta_{ij}(B^k_{\tau\rho})$, $(i,j) \in U_0$, $(\tau,\rho)^k \in U^k \setminus U^k_L$, $k \in K = \{1,2,3,4,5\}$ for the system (6.4.1) – (6.4.3), $U_0 = \{(1,6)(3,4)\}$, $K_0(1,6) = \{2,4\}$, $K_0(3,4) = \{2,4,5\}$ (see Table 6.4.12).

Table 6.4.12

Values $\delta_{ij}(B^k_{\tau\rho}), (i,j) \in U_0, (\tau,\rho)^k \in U^k \setminus U^k_L, k \in K = \{1,2,3,4,5\}$

$(\tau,\rho)^k$	$(3,6)^1$	$(1,4)^2$	$(6,5)^3$	$(4,6)^5$	$(1,5)^4$	$(2,1)^4$	$(3,4)^4$	$(5,4)^4$
$\delta_{16}(B^k_{\tau\rho})$	0	$-\frac{511}{711}$	0	0	$-\frac{1}{2}$	$\frac{143}{190}$	$\frac{7}{19}$	$\frac{12}{19}$
$\delta_{34}(B^k_{\tau\rho})$	0	$-\frac{20}{237}$	0	$\frac{5}{3}$	0	0	1	0

In the considered example (6.4.1) – (6.4.3) to construct the matrix D for the system (6.3.15) we enumerate the arcs of the set

$$U_B = \bigcup_{k=1}^{5} U^k_B, U_B = \{(3,6)^1,(1,4)^2,(6,5)^3,(4,6)^5\}$$

$$t(3,6)^1 = 1, t(1,4)^2 = 2, t(6,5)^3 = 3, t(4,6)^5 = 4.$$

Let's enumerate the set $U_0 = \{(1,6),(3,4)\}$: $\xi(1,6) = 1, \xi(3,4) = 2$. We construct the matrix $D_1 = (\Lambda^{kp}_{\tau\rho}, p = \overline{1,2}, t(\tau,\rho)^k = \overline{1,4})$ of the determinants of bicycles $B^k_{\tau\rho}$, entailed by the arcs $(\tau,\rho)^k \in U_B$, selecting the corresponding columns from the Table 6.4.11. By analogy selecting the corresponding columns from the Table 6.4.12, we form the matrix D_2. Then, we combine matrices D_1 and D_2, and get the matrix D of the system (6.3.15).

$$D = \begin{pmatrix} \frac{1147}{164} & \frac{899}{237} & -\frac{599}{405} & \frac{391}{30} \\ -\frac{178}{123} & -\frac{493}{237} & \frac{3031}{405} & \frac{141}{10} \\ 0 & -\frac{511}{711} & 0 & 0 \\ 0 & -\frac{20}{237} & 0 & \frac{5}{3} \end{pmatrix}, \ \det D \neq 0.$$

Let's compute the numbers A^1, A^2, $A_{1,6}$, $A_{3,4}$, using the formulas (6.3.20) and (6.3.21):

$$A^1 = \alpha_1 - 2\tilde{x}^1_{1,3} - 3\tilde{x}^2_{1,3} - 8\tilde{x}^3_{1,3} - \tilde{x}^2_{1,4} - \tilde{x}^2_{1,5} - 8\tilde{x}^4_{1,4} - 10\tilde{x}^3_{1,5} -$$

$$-7\tilde{x}^4_{1,5} - 2\tilde{x}^5_{1,5} - 3\tilde{x}^2_{1,6} - 8\tilde{x}^4_{1,6} - 6\tilde{x}^1_{2,1} - 5\tilde{x}^3_{2,1} - 5\tilde{x}^4_{2,1} - 2\tilde{x}^5_{2,6} -$$

$$-4\tilde{x}^2_{3,2} - 8\tilde{x}^4_{3,2} - 6\tilde{x}^5_{3,2} - 5\tilde{x}^1_{3,4} - 8\tilde{x}^2_{3,4} - 4\tilde{x}^4_{3,4} - 7\tilde{x}^5_{3,4} - 6\tilde{x}^1_{3,6} -$$

$$-5\tilde{x}^4_{3,6} - 2\tilde{x}^1_{4,6} - 7\tilde{x}^2_{4,6} - 7\tilde{x}^3_{4,6} - 9\tilde{x}^4_{4,6} - 6\tilde{x}^5_{4,6} - 5\tilde{x}^4_{5,2} - 4\tilde{x}^3_{5,2} -$$

$$-3\tilde{x}^1_{5,4} - 6\tilde{x}^3_{5,4} - 6\tilde{x}^4_{5,4} - 8\tilde{x}^1_{6,5} - 3\tilde{x}^3_{6,5} - 2\tilde{x}^5_{6,5} = \frac{53776624249}{53776624249},$$

$$A^2 = \frac{21372313241}{99696420}, \quad A_{1,6} = \frac{4282}{4503}, \quad A_{3,4} = \frac{1303}{79}.$$

Let's compute the vector β of the right hand side of (6.3.15):

$$\beta_1 = A^1 - \Lambda^{4,1}_{1,5}x^4_{1,5} - \Lambda^{4,1}_{21}x^4_{2,1} - \Lambda^{4,1}_{3,4}x^4_{3,4} - \Lambda^{4,1}_{5,4}x^4_{5,4} =$$
$$= A^1 + \frac{23}{40}x^4_{1,5} - \frac{1087}{76}x^4_{2,1} - \frac{301}{95}x^4_{3,4} - \frac{523}{76}x^4_{5,4},$$

$$\beta_2 = A^2 - \Lambda^{4,2}_{1,5}x^4_{1,5} - \Lambda^{4,2}_{2,1}x^4_{2,1} - \Lambda^{4,2}_{3,4}x^4_{3,4} - \Lambda^{4,2}_{5,4}x^4_{5,4} =$$
$$= A^2 + \frac{7}{4}x^4_{1,5} - \frac{353}{38}x^4_{2,1} = \frac{28}{19}x^4_{3,4} - \frac{303}{38}x^4_{5,4},$$

$$\beta_3 = A_{1,6} - \delta_{1,6}(B^4_{1,5})x^4_{1,5} - \delta_{1,6}(B^4_{2,1})x^4_{2,1} - \delta_{1,6}(B^4_{3,4})x^4_{3,4} - \delta_{1,6}(B^4_{5,4})x^4_{5,4} =$$
$$= A_{1,6} + \frac{1}{2}x^4_{1,5} - \frac{143}{190}x^4_{2,1} - \frac{7}{19}x^4_{3,4} - \frac{12}{19}x^4_{5,4},$$

$$\beta_4 = A_{3,4} - \delta_{3,4}(B^4_{1,5})x^4_{1,5} - \delta_{3,4}(B^4_{2,1})x^4_{2,1} - \delta_{3,4}(B^4_{3,4})x^4_{3,4} - \delta_{3,4}(B^4_{5,4})x^4_{5,4} =$$
$$= A_{3,4} - x^4_{3,4}.$$

As the matrix D is non-singular we use the formula (6.3.22) to find the solution $x_B = (x^k_{\tau,\rho},(\tau,\rho)^k \in U^k_B, k \in K) = (x^1_{3,6},x^2_{1,4},x^3_{6,5},x^5_{4,6})$ of the system (6.3.15).

6.5. Technology of implementation

Consider the example (6.5.1) – (6.5.3) constructing the solution of the sparse system for the generalized multinetwork $S = (I,U)$, $I = \{1,2,3,4\}$, $U = \{1,3),(1,4),(2,1),(2,3),(2,4),(3,4)\}$. Let $K = \{1,2,3\}$ are types of flows in the multinetwork $S = (I,U)$. The sets of arcs U^1, U^2, U^3 for the flows of first, second and third types include the following arcs, correspondingly:

$$U^1 = \{(1,3)^1,(1,4)^1,(2,1)^1,(2,3)^1,(2,4)^1,(3,4)^1\},\ U^2 = U^1,\ U^3 = U^1.$$

Multinetwork $S = (I,U)$ is shown in Figure 6.13 as a union of networks $S^k = (I^k, U^k)$, $k \in K = \{1,2,3\}$.

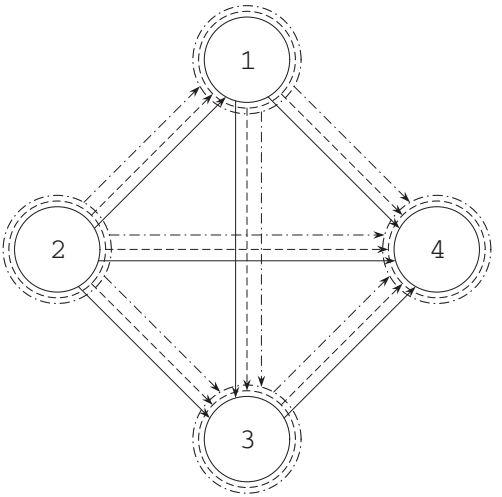

Fig. 6.13. Union of networks $S^k = (I^k,U^k)$, $k \in K = \{1,2,3\}$

$$
\begin{gathered}
x^1_{1,3} + x^1_{1,4} - x^1_{2,1} = 2 \\
x^1_{2,1} + x^1_{2,3} + x^1_{2,4} = 14 \\
x^1_{3,4} - 0.4x^1_{1,3} - 0.1x^1_{2,3} = 1.1 \\
-0.3x^1_{1,4} - 0.8x^1_{2,4} - 0.7x^1_{3,4} = -5.9 \\
\\
x^2_{1,3} + x^2_{1,4} - 0.1x^2_{2,1} = 2.3 \\
x^2_{2,1} + x^2_{2,3} + x^2_{2,4} = 21 \\
x^2_{3,4} - 0.1x^2_{1,3} - 0.2x^2_{2,3} = 6.8 \\
-0.3x^2_{1,4} - 0.8x^2_{2,4} - 0.8x^2_{3,4} = -10.7 \\
\\
x^3_{1,3} + x^3_{1,4} - 0.2x^3_{2,1} = 8.8 \\
x^3_{2,1} + x^3_{2,3} + x^3_{2,4} = 18 \\
x^3_{3,4} - 0.4x^3_{1,3} - 0.4x^3_{2,3} = -0.6 \\
-0.2x^3_{1,4} - 0.6x^3_{2,4} - 0.9x^3_{3,4} = -10.5
\end{gathered}
\tag{6.5.1}
$$

$$
\begin{gathered}
9x^1_{1,3} + x^2_{1,3} + 10x^3_{1,3} + 5x^1_{1,4} + 5x^2_{1,4} + 2x^3_{1,4} + 8x^1_{2,1} + \\
+7x^3_{2,1} + x^1_{2,3} + 6x^2_{2,3} + 7x^3_{2,3} + x^1_{2,4} + 7x^2_{2,4} + \\
+3x^1_{3,4} + 4x^2_{3,4} + 6x^3_{3,4} = 396 \\
9x^1_{1,3} + 2x^2_{1,3} + 4x^3_{1,3} + 6x^1_{1,4} + 2x^2_{1,4} + 9x^3_{1,4} + 6x^1_{2,1} + \\
+7x^2_{2,1} + 6x^3_{2,1} + x^1_{2,3} + 3x^2_{2,3} + 8x^1_{2,4} + 8x^2_{2,4} + \\
+9x^3_{2,4} + 2x^1_{3,4} + x^2_{3,4} + 10x^3_{3,4} = 434
\end{gathered}
\tag{6.5.2}
$$

$$
x^1_{2,1} + x^2_{2,1} + x^3_{2,1} = 23 \tag{6.5.3}
$$

We choose support U_L for the generalized multinetwork $S = (I,U)$ of the system (6.5.1). Let $U_L = \{U^k_L, k \in K = \{1,2,3\}\}$ consists of one connected component for each flow type:

$$
U^1_L = \{(1,3)^1,(1,4)^1,(2,4)^1,(3,4)^1\},
$$

$$
U^2_L = \{(1,3)^2,(1,4)^2,(2,4)^2,(3,4)^2\},
$$

$$
U^3_L = \{(1,3)^3,(1,4)^3,(2,4)^3,(3,4)^3\}.
$$

Structures, representing the support $U_L^k, k \in K = \{1,2,3\}$ of the generalized multinetwork S of the system (6.5.1) are shown in Figure 6.14.

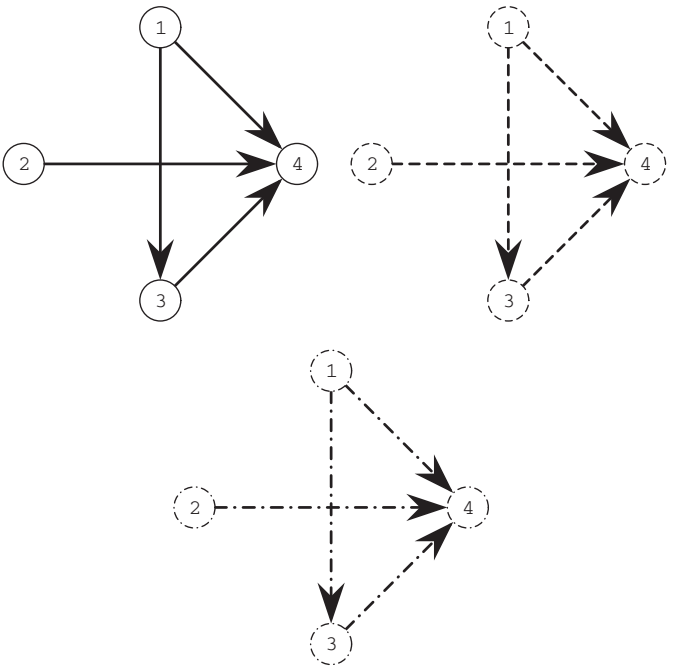

Fig. 6.14. Sets of arcs $U_L^k, k \in K = \{1,2,3\}$

We construct the general solution of the system (6.5.1) – (6.5.3) with respect to a support of the generalized multinetwork $S = (I,U)$ for the system (6.5.1) – (6.5.3) using the algorithms of decomposition of the support generalized multinetwork $S = (I,U)$. The support of the generalized multinetwork $S = (I,U)$ for the system (6.5.1) is presented in Figure 6.14. The set $U_B = \bigcup_{k=1}^{3} U_B^k$ of bicyclic arcs is:

$$U_B^1 = \{(2,1)^1\},\ U_B^2 = \{(2,1)^2\},\ U_B^3 = \{(2,1)^3\}.$$

To construct general solution of the system (6.5.1) – (6.5.3) we perform the following steps:

- *Step 1.* Select the data structures to represent the support of the generalized multinetwork of sparse system (6.5.1). Construct a system of characteristic vectors with respect to the support $U_L^k, k \in K = \{1,2,3\}$ of generalized multinetwork $S = (I,U)$ for system (6.5.1) (see Figure 6.14). The number of operations to compute each characteristic vector $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k)$ is $O(|U^k|)$ in the worst case, $(\tau,\rho)^k \in U^k \backslash U_L^k, k \in K = \{1,2,3\}$.

- *Step 2.* Compute the partial solution of the system (6.5.1) with block-diagonal matrix A. Compute the partial solution according to the rules of Remark 6.2.2. The number of operations for computation of the partial solution of the system (6.5.1) for each block of the size $|I^k| \times |U^k|$ of the network $S^k = (I^k,U^k)$ is $O(|U^k|)$ in the worst case, $k \in K = \{1,2,3\}$.

- *Step 3.* Form the matrix of determinants and the vector β of the right-hand side of the system (6.3.15). Using the principle of decomposition, compute the unknowns of the system (6.3.15), which correspond to the bicyclic arcs: $U_B = \bigcup_{k=1}^{3} U_B^k$, $U_B^1 = \{(2,1)^1\}$, $U_B^2 = \{(2,1)^2\}$, $U_B^3 = \{(2,1)^3\}$.

- *Step 4.* Compute the remaining unknowns of the system (6.5.1) – (6.5.3), which correspond to the arcs

$$U_L^1 = \{(1,3)^1,(1,4)^1,(2,4)^1,(3,4)^1\},$$

$$U_L^2 = \{(1,3)^2,(1,4)^2,(2,4)^2,(3,4)^2\},$$

$$U_L^3 = \{(1,3)^3,(1,4)^3,(2,4)^3,(3,4)^3\}.$$

  using the graph theoretical properties of the support of the generalized multinetwork $S = (I,U)$ for the system (6.5.1).

- *Step 5.* Verify the obtained rational solution of the underdetermined system of linear algebraic equations (6.5.1) – (6.5.3) using the built-in *Simplify* function of CAS *Wolfram Mathematica*.

As a result of *Step 1– Step 5* we obtain:

- Data structures to represent of the support $U_L^k, k \in K = \{1,2,3\}$ generalized multinetwork $S = (I,U)$ of system (6.5.1) (see Figure 6.14) are:
- the list $\{t[i],\ i \in I^k\}$ for the connected components (I^k,U_L^k) contains elements: $t = \{2,3,4,1\}$, for each $k \in K$. List $\{t[i],\ i \in I^k\}$ determines the order of solving the equations of sparse system of type (6.5.1);
- the list $p = \{-3,4,-4,-1\}$ defines for each node $i \in I^k$ value $|p[i]|$, which is the father of node i in root structure (minus sign identifies the root node root structure), for each $k \in K$;
- the list $d = \{-1,-1,-1,1\}$ defines for each node $i \in I^1$ direction of the arc in the root structure: if the node i is root, then $d[i] = 0$; if the arc $(|p[i]|,i) \in U^1$, then $d[i] = 1$; if the arc $(i,|p[i]|) \in U^1$, then the value $d[i] = -1$. For each connected component (I^2,U_L^2) and (I^3,U_L^3) we can use the data structure provided for the connected component (I^1,U_L^1).
- The system of characteristic vectors
$\delta^1(2,1),\ \delta^1(2,3),\ \delta^2(2,1),\ \delta^2(2,3),\ \delta^3(2,1),\ \delta^3(2,3)$ is:

$$\delta^1(2,1) = (\delta^1_{2,1}(2,1) \to 1,\ \ \delta^1_{2,3}(2,1) \to 0,\ \ \delta^1_{2,4}(2,1) \to -1,$$
$$\delta^1_{1,3}(2,1) \to -25,\ \ \delta^1_{1,4}(2,1) \to 26,\ \ \delta^1_{3,4}(2,1) \to -10);$$

$$\delta^1(2,3) = (\delta^1_{2,1}(2,3) \to 0,\ \ \delta^1_{2,3}(2,3) \to 1,\ \ \delta^1_{2,4}(2,3) \to -1,$$
$$\delta^1_{1,3}(2,3) \to -\frac{73}{2},\ \ \delta^1_{1,4}(2,3) \to \frac{73}{2},\ \ \delta^1_{3,4}(2,3) \to -\frac{29}{2});$$

$$\delta^2(2,1) = (\delta^2_{2,1}(2,1) \to 1,\ \ \delta^2_{2,3}(2,1) \to 0,\ \ \delta^2_{2,4}(2,1) \to -1,$$
$$\delta^2_{1,3}(2,1) \to -\frac{7}{2},\ \ \delta^2_{1,4}(2,1) \to \frac{18}{5},\ \ \delta^2_{3,4}(2,1) \to -\frac{7}{20});$$

$$\delta^2(2,3) = (\delta^2_{2,1}(2,3) \to 0,\ \ \delta^2_{2,3}(2,3) \to 1,\ \ \delta^2_{2,4}(2,3) \to -1,$$
$$\delta^2_{1,3}(2,3) \to -\frac{32}{11},\ \ \delta^2_{1,4}(2,3) \to \frac{32}{11},\ \ \delta^2_{3,4}(2,3) \to -\frac{1}{11});$$

$$\delta^3(2,1) = (\delta^3_{2,1}(2,1) \to 1,\ \ \delta^3_{2,3}(2,1) \to 0,\ \ \delta^3_{2,4}(2,1) \to -1,$$

$$\delta^3_{1,3}(2,1) \to \frac{7}{2},\ \ \delta^3_{1,4}(2,1) \to -\frac{33}{10},\ \ \delta^3_{3,4}(2,1) \to \frac{7}{5});$$

$$\delta^3(2,3) = (\delta^3_{2,1}(2,3) \to 0,\ \ \delta^3_{2,3}(2,3) \to 1,\ \ \delta^3_{2,4}(2,3) \to -1,$$
$$\delta^3_{1,3}(2,3) \to \frac{3}{2},\ \ \delta^3_{1,4}(2,3) \to -\frac{3}{2},\ \ \delta^3_{3,4}(2,3) \to 1).$$

- Partial solution of system (6.5.1) (result of *Step 2*) is:

$$\widetilde{x}^1 = (\widetilde{x}^1_{2,1} \to 0,\ \ \widetilde{x}^1_{2,3} \to 0,\ \ \widetilde{x}^1_{2,4} \to 14,\ \ \widetilde{x}^1_{1,3} \to \frac{667}{2},$$
$$\widetilde{x}^1_{1,4} \to -\frac{663}{2},\ \ \widetilde{x}^1_{3,4} \to \frac{269}{2});$$

$$\widetilde{x}^2 = (\widetilde{x}^2_{2,1} \to 0,\ \ \widetilde{x}^2_{2,3} \to 0,\ \ \widetilde{x}^2_{2,4} \to 21,\ \ \widetilde{x}^2_{1,3} \to \frac{1223}{22},$$
$$\widetilde{x}^2_{1,4} \to -\frac{2931}{55},\ \ \widetilde{x}^2_{3,4} \to \frac{2719}{22});$$

$$\widetilde{x}^3 = (\widetilde{x}^3_{2,1} \to 0,\ \ \widetilde{x}^3_{2,3} \to 0,\ \ \widetilde{x}^3_{2,4} \to 18,\ \ \widetilde{x}^3_{1,3} \to -\frac{19}{2},$$
$$\widetilde{x}^3_{1,4} \to \frac{183}{10},\ \ \widetilde{x}^3_{3,4} \to -\frac{22}{5}).$$

- Unknown $x_B = (x^1_{2,1}, x^2_{2,1}, x^3_{2,1})$ of system (6.5.1) – (6.5.3), corresponding to the set U_B of bicyclic arcs $U_B = \bigcup\limits_{k=1}^{3} U^k_B$, $U^1_B = \{(2,1)^1\}$, $U^2_B = \{(2,1)^2\}$, $U^3_B = \{(2,1)^3\}$ satisfy the system of linear equations:

$$\begin{pmatrix} -118 & \frac{61}{10} & \frac{219}{5} \\ -91 & -\frac{23}{20} & -\frac{47}{10} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1_{2,1} \\ x^2_{2,1} \\ x^3_{2,1} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\frac{13929}{11} + \frac{379}{2} y^1_{2,3} - \frac{113}{11} y^2_{2,3} - 25 y^3_{2,3} \\ -\frac{61123}{44} + \frac{291}{2} y^1_{2,3} + \frac{56}{11} y^2_{2,3} + \frac{13}{2} y^3_{2,3} \\ 23 \end{pmatrix}.$$

6.6. Implementation in Wolfram Mathematica

In the code in Listing 3 we present implementation in CAS *Wolfram Mathematica* of decomposition algorithms for solution of the sparse underdetermined system (6.5.1) – (6.5.3). The support of the generalized multinetwork of sparse system (6.5.1) is presented in Figure 6.14. The computation is performed in a rational arithmetic.

Listing 3

$$\mathbf{system} = \{\mathbf{x1}_{1,3} + \mathbf{x1}_{1,4} - \mathbf{x1}_{2,1} == 2,$$

$$\mathbf{x1}_{2,1} + \mathbf{x1}_{2,3} + \mathbf{x1}_{2,4} == 14,$$

$$\mathbf{x1}_{3,4} - \frac{2}{5}\mathbf{x1}_{1,3} - \frac{1}{10}\mathbf{x1}_{2,3} == \frac{11}{10},$$

$$-\frac{3}{10}\mathbf{x1}_{1,4} - \frac{4}{5}\mathbf{x1}_{2,4} - \frac{7}{10}\mathbf{x1}_{3,4} == -\frac{59}{10},$$

$$\mathbf{x2}_{1,3} + \mathbf{x2}_{1,4} - \frac{1}{10}\mathbf{x2}_{2,1} == \frac{23}{10},$$

$$\mathbf{x2}_{2,1} + \mathbf{x2}_{2,3} + \mathbf{x2}_{2,4} == 21,$$

$$\mathbf{x2}_{3,4} - \frac{1}{10}\mathbf{x2}_{1,3} - \frac{1}{5}\mathbf{x2}_{2,3} == \frac{34}{5},$$

$$-\frac{3}{10}\mathbf{x2}_{1,4} - \frac{4}{5}\mathbf{x2}_{2,4} - \frac{4}{5}\mathbf{x2}_{3,4} == -\frac{107}{10},$$

$$x3_{1,3} + x3_{1,4} - \frac{1}{5}x3_{2,1} == \frac{44}{5},$$

$$x3_{2,1} + x3_{2,3} + x3_{2,4} == 18,$$

$$x3_{3,4} - \frac{2}{5}x3_{1,3} - \frac{2}{5}x3_{2,3} == -\frac{3}{5},$$

$$-\frac{1}{5}x3_{1,4} - \frac{3}{5}x3_{2,4} - \frac{9}{10}x3_{3,4} == -\frac{105}{10},$$

$$9x1_{1,3} + x2_{1,3} + 10x3_{1,3} + 5x1_{1,4} + 5x2_{1,4} + 2x3_{1,4} + 8x1_{2,1} + 7x3_{2,1} +$$
$$x1_{2,3} + 6x2_{2,3} + 7x3_{2,3} + x1_{2,4} + 7x2_{2,4} + 3x1_{3,4} + 4x2_{3,4} + 6x3_{3,4} == 396,$$

$$9x1_{1,3} + 2x2_{1,3} + 4x3_{1,3} + 6x1_{1,4} + 2x2_{1,4} + 9x3_{1,4} + 6x1_{2,1} + 7x2_{2,1} +$$
$$6x3_{2,1} + x1_{2,3} + 3x2_{2,3} + 8x1_{2,4} + 8x2_{2,4} + 9x3_{2,4} + 2x1_{3,4} +$$
$$x2_{3,4} + 10x3_{3,4} == 434,$$

$$x1_{2,1} + x2_{2,1} + x3_{2,1} == 23\};$$

$$\text{system1} = \{x1_{1,3} + x1_{1,4} - x1_{2,1} == 0,$$

$$x1_{2,1} + x1_{2,3} + x1_{2,4} == 0,$$

$$x1_{3,4} - \frac{2}{5}x1_{1,3} - \frac{1}{10}x1_{2,3} == 0,$$

$$-\frac{3}{10}x1_{1,4} - \frac{4}{5}x1_{2,4} - \frac{7}{10}x1_{3,4} == 0\Big\};$$

$t = \{2,3,4,1\};$
$p = \{-3,4,-4,-1\};$
$d = \{-1,-1,-1,1\};$

$\text{system1a} = \text{system1};$
$\text{system1a}[[1]] = \text{system1a}[[1]]/.\{x1_{2,1} \to 1\};$
$\text{system1a}[[2]] = \text{system1a}[[2]]/.\{x1_{2,1} \to 1\};$
$\text{system1a}[[2]] = \text{system1a}[[2]]/.\{x1_{2,3} \to 0\};$
$\text{system1a}[[3]] = \text{system1a}[[3]]/.\{x1_{2,3} \to 0\};$

```
δ1_{2,1} = {x1_{2,1} → 1,x1_{2,3} → 0};

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1a[[t[[i]]]],x1_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system1a[[t[[i]]]],x1_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]];

δ1_{2,1} = Join [δ1_{2,1},δ];
If[i ≠ 3,
system1a[[Abs[p[[t[[i]]]]]]] = system1a[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) − 1;
δ = {x1_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)};
},
{
φ0 = x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1a[[t[[4]]]]/.
(δ/.{x1_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x1_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ1_{2,1} = Join [δ1_{2,1}/.δ,δ];

}];
}];
system1b = system1;
system1b[[1]] = system1b[[1]]/.{x1_{2,1} → 0};
system1b[[2]] = system1b[[2]]/.{x1_{2,1} → 0};
system1b[[2]] = system1b[[2]]/.{x1_{2,3} → 1};
```

```
system1b[[3]] = system1b[[3]]/. {x1_{2,3} → 1};

δ1_{2,3} = {x1_{2,1} → 0,x1_{2,3} → 1};

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1b[[t[[i]]]],x1_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system1b[[t[[i]]]],x1_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]] ];
δ1_{2,3} = Join [δ1_{2,3},δ];
If[i ≠ 3,
system1b[[Abs[p[[t[[i]]]]]]] = system1b[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1b[[t[[4]]]]/.
(δ/. {x1_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1b[[t[[4]]]]/.
(δ/. {x1_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) − 1;
δ = {x1_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)};
},
{
φ0 = x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1b[[t[[4]]]]/.
(δ/. {x1_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1b[[t[[4]]]]/.
(δ/. {x1_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x1_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ1_{2,3} = Join [δ1_{2,3}/.δ,δ];

}];
}];
system2 = {x2_{1,3} + x2_{1,4} − 1/10 x2_{2,1} == 0,

x2_{2,1} + x2_{2,3} + x2_{2,4} == 0,
```

$$x2_{3,4} - \frac{1}{10}x2_{1,3} - \frac{1}{5}x2_{2,3} == 0,$$
$$\left.-\frac{3}{10}x2_{1,4} - \frac{4}{5}x2_{2,4} - \frac{4}{5}x2_{3,4} == 0\right\};$$

```
t = {2,3,4,1};
p = {-3,4, - 4, - 1};
d = {-1, - 1, - 1,1};

system2a = system2;
system2a[[1]] = system2a[[1]]/. {x2_{2,1} → 1} ;
system2a[[2]] = system2a[[2]]/. {x2_{2,1} → 1} ;
system2a[[2]] = system2a[[2]]/. {x2_{2,3} → 0} ;
system2a[[3]] = system2a[[3]]/. {x2_{2,3} → 0} ;

δ2_{2,1} = {x2_{2,1} → 1,x2_{2,3} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system2a[[t[[i]]]],x2_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system2a[[t[[i]]]],x2_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]] ;
δ2_{2,1} = Join [δ2_{2,1},δ] ;
If[i ≠ 3,
system2a[[Abs[p[[t[[i]]]]]]] = system2a[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x2_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system2a[[t[[4]]]]/.
(δ/. {x2_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x2_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x2_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system2a[[t[[4]]]]/.
(δ/. {x2_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x2_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) - 1;
δ = {x2_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0-φ1)} ;
},
{
φ0 = x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2a[[t[[4]]]]/.
```

```
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 0}),x2_{t[[4]],Abs[p[[t[[4]]]]]}][[1]];
φ1 = (x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2a[[t[[4]]]]/.
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 1}),x2_{t[[4]],Abs[p[[t[[4]]]]]}][[1]]) − 1;
δ = {x2_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ2_{2,1} = Join [δ2_{2,1}/.δ,δ];
}];
}];

system2b = system2;
system2b[[1]] = system2b[[1]]/.{x2_{2,1} → 0};
system2b[[2]] = system2b[[2]]/.{x2_{2,1} → 0};
system2b[[2]] = system2b[[2]]/.{x2_{2,3} → 1};
system2b[[3]] = system2b[[3]]/.{x2_{2,3} → 1};

δ2_{2,3} = {x2_{2,1} → 0,x2_{2,3} → 1};

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system2b[[t[[i]]]],x2_{Abs[p[[t[[i]]]]],t[[i]]}][[1]],
δ = Solve [system2b[[t[[i]]]],x2_{t[[i]],Abs[p[[t[[i]]]]]}][[1]]];
δ2_{2,3} = Join [δ2_{2,3},δ];
If[i ≠ 3,
system2b[[Abs[p[[t[[i]]]]]]] = system2b[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x2_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system2b[[t[[4]]]]/.
(δ/.{x2_{Abs[p[[t[[4]]]]],t[[4]]} → 0}),x2_{Abs[p[[t[[4]]]]],t[[4]]}][[1]];
φ1 = (x2_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system2b[[t[[4]]]]/.
(δ/.{x2_{Abs[p[[t[[4]]]]],t[[4]]} → 1}),x2_{Abs[p[[t[[4]]]]],t[[4]]}][[1]]) − 1;
δ = {x2_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)};
},
{
φ0 = x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2b[[t[[4]]]]/.
```

```
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x2_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2b[[t[[4]]]]/.
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x2_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x2_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ2_{2,3} = Join [δ2_{2,3}/.δ,δ] ;

}];
}];
system3 = {x3_{1,3} + x3_{1,4} − 1/5 x3_{2,1} == 0,

x3_{2,1} + x3_{2,3} + x3_{2,4} == 0,

x3_{3,4} − 2/5 x3_{1,3} − 2/5 x3_{2,3} == 0,
−1/5 x3_{1,4} − 3/5 x3_{2,4} − 9/10 x3_{3,4} == 0};


t = {2,3,4,1};
p = {−3,4, − 4, − 1};
d = {−1, − 1, − 1,1};

system3a = system3;
system3a[[1]] = system3a[[1]]/. {x3_{2,1} → 1} ;
system3a[[2]] = system3a[[2]]/. {x3_{2,1} → 1} ;
system3a[[2]] = system3a[[2]]/. {x3_{2,3} → 0} ;
system3a[[3]] = system3a[[3]]/. {x3_{2,3} → 0} ;

δ3_{2,1} = {x3_{2,1} → 1,x3_{2,3} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3a[[t[[i]]]],x3_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system3a[[t[[i]]]],x3_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]] ;
δ3_{2,1} = Join [δ3_{2,1},δ] ;
```

```
If[i ≠ 3,
system3a[[Abs[p[[t[[i]]]]]]] = system3a[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x3_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system3a[[t[[4]]]]/.
(δ/.{x3_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x3_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x3_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system3a[[t[[4]]]]/.
(δ/.{x3_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x3_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) − 1;
δ = {x3_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)};
},
{
φ0 = x3_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system3a[[t[[4]]]]/.
(δ/.{x3_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x3_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x3_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system3a[[t[[4]]]]/.
(δ/.{x3_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x3_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x3_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ3_{2,1} = Join [δ3_{2,1}/.δ,δ] ;
}];
}];

system3b = system3;
system3b[[1]] = system3b[[1]]/. {x3_{2,1} → 0} ;
system3b[[2]] = system3b[[2]]/. {x3_{2,1} → 0} ;
system3b[[2]] = system3b[[2]]/. {x3_{2,3} → 1} ;
system3b[[3]] = system3b[[3]]/. {x3_{2,3} → 1} ;

δ3_{2,3} = {x3_{2,1} → 0,x3_{2,3} → 1} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3b[[t[[i]]]],x3_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system3b[[t[[i]]]],x3_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]] ;
δ3_{2,3} = Join [δ3_{2,3},δ] ;
```

```
If[i ≠ 3,
system3b[[Abs[p[[t[[i]]]]]]] = system3b[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x3_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system3b[[t[[4]]]]/.
(δ/.{x3_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x3_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x3_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system3b[[t[[4]]]]/.
(δ/.{x3_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x3_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) − 1;
δ = {x3_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)};
},
{
φ0 = x3_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system3b[[t[[4]]]]/.
(δ/.{x3_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x3_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x3_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system3b[[t[[4]]]]/.
(δ/.{x3_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x3_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x3_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];
δ3_{2,3} = Join [δ3_{2,3}/.δ,δ] ;
}];
}];
```

$$\text{system1} = \{x1_{1,3} + x1_{1,4} - x1_{2,1} == 2,$$

$$x1_{2,1} + x1_{2,3} + x1_{2,4} == 14,$$

$$x1_{3,4} - \frac{2}{5}x1_{1,3} - \frac{1}{10}x1_{2,3} == \frac{11}{10},$$

$$-\frac{3}{10}x1_{1,4} - \frac{4}{5}x1_{2,4} - \frac{7}{10}x1_{3,4} == -\frac{59}{10}\Big\};$$

```
t = {2,3,4,1};
p = {−3,4, − 4, − 1};
d = {−1, − 1, − 1,1};

system1[[1]] = system1[[1]]/. {x1_{2,1} → 0} ;
```

```
system1[[2]] = system1[[2]]/. {x1_{2,1} → 0} ;
system1[[2]] = system1[[2]]/. {x1_{2,3} → 0} ;
system1[[3]] = system1[[3]]/. {x1_{2,3} → 0} ;

δ1 = {x1_{2,1} → 0,x1_{2,3} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system1[[t[[i]]]],x1_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system1[[t[[i]]]],x1_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]]] ;
δ1 = Join [δ1,δ] ;
If[i ≠ 3,
system1[[Abs[p[[t[[i]]]]]]] = system1[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1[[t[[4]]]]/.
(δ/. {x1_{Abs[p[[t[[4]]]]],t[[4]]} → 0}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]];
φ1 = (x1_{Abs[p[[t[[4]]]]],t[[4]]}/.Solve[system1[[t[[4]]]]/.
(δ/. {x1_{Abs[p[[t[[4]]]]],t[[4]]} → 1}) ,x1_{Abs[p[[t[[4]]]]],t[[4]]}] [[1]]) − 1;
δ = {x1_{Abs[p[[t[[4]]]]],t[[4]]} → φ0/(φ0−φ1)} ;
},
{
φ0 = x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1[[t[[4]]]]/.
(δ/. {x1_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x1_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system1[[t[[4]]]]/.
(δ/. {x1_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x1_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x1_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)} ;
}];

δ1 = Join [δ1/.δ,δ] ;
}];
}];
system2 = {x2_{1,3} + x2_{1,4} − 1/10 x2_{2,1} == 23/10,
```

$$\mathrm{x2}_{2,1} + \mathrm{x2}_{2,3} + \mathrm{x2}_{2,4} == 21,$$

$$\mathrm{x2}_{3,4} - \frac{1}{10}\mathrm{x2}_{1,3} - \frac{1}{5}\mathrm{x2}_{2,3} == \frac{34}{5},$$

$$\left.-\frac{3}{10}\mathrm{x2}_{1,4} - \frac{4}{5}\mathrm{x2}_{2,4} - \frac{4}{5}\mathrm{x2}_{3,4} == -\frac{107}{10}\right\};$$

$$t = \{2,3,4,1\};$$
$$p = \{-3,4,-4,-1\};$$
$$d = \{-1,-1,-1,1\};$$

$$\mathrm{system2}[[1]] = \mathrm{system2}[[1]]/.\{\mathrm{x2}_{2,1} \to 0\};$$
$$\mathrm{system2}[[2]] = \mathrm{system2}[[2]]/.\{\mathrm{x2}_{2,1} \to 0\};$$
$$\mathrm{system2}[[2]] = \mathrm{system2}[[2]]/.\{\mathrm{x2}_{2,3} \to 0\};$$
$$\mathrm{system2}[[3]] = \mathrm{system2}[[3]]/.\{\mathrm{x2}_{2,3} \to 0\};$$

$$\overline{\delta 2} = \{\mathrm{x2}_{2,1} \to 0, \mathrm{x2}_{2,3} \to 0\};$$

$$\mathrm{For}[i = 1, i \leqslant 3, ++i,$$
$$\{$$
$$\mathrm{If}[d[[t[[i]]]] == 1,$$
$$\delta = \mathrm{Solve}\left[\mathrm{system2}[[t[[i]]]], \mathrm{x2}_{\mathrm{Abs}[p[[t[[i]]]]],t[[i]]}\right][[1]],$$
$$\delta = \mathrm{Solve}\left[\mathrm{system2}[[t[[i]]]], \mathrm{x2}_{t[[i]],\mathrm{Abs}[p[[t[[i]]]]]}\right][[1]];$$
$$\overline{\delta 2} = \mathrm{Join}\left[\overline{\delta 2}, \delta\right];$$
$$\mathrm{If}[i \neq 3,$$
$$\mathrm{system2}[[\mathrm{Abs}[p[[t[[i]]]]]]] = \mathrm{system2}[[\mathrm{Abs}[p[[t[[i]]]]]]]/.\delta,$$
$$\{$$
$$\mathrm{If}[d[[t[[4]]]] == 1,$$
$$\{$$
$$\varphi_0 = \mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]}/.\mathrm{Solve}[\mathrm{system2}[[t[[4]]]]/.$$
$$\left(\delta/.\left\{\mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]} \to 0\right\}\right), \mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]}\Big][[1]];$$
$$\varphi_1 = \left(\mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]}/.\mathrm{Solve}[\mathrm{system2}[[t[[4]]]]/.\right.$$
$$\left.\left(\delta/.\left\{\mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]} \to 1\right\}\right), \mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]}\Big][[1]]\right) - 1;$$
$$\delta = \left\{\mathrm{x2}_{\mathrm{Abs}[p[[t[[4]]]]],t[[4]]} \to \frac{\varphi_0}{\varphi_0 - \varphi_1}\right\};$$
$$\},$$
$$\{$$

```
φ0 = x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2[[t[[4]]]]/.
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 0}) ,x2_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]];
φ1 = (x2_{t[[4]],Abs[p[[t[[4]]]]]}/.Solve[system2[[t[[4]]]]/.
(δ/.{x2_{t[[4]],Abs[p[[t[[4]]]]]} → 1}) ,x2_{t[[4]],Abs[p[[t[[4]]]]]}] [[1]]) − 1;
δ = {x2_{t[[4]],Abs[p[[t[[4]]]]]} → φ0/(φ0−φ1)};
}];

δ̄2 = Join [δ̄2/.δ,δ] ;

}];
}];
system3 = {x3_{1,3} + x3_{1,4} − 1/5 x3_{2,1} == 44/5,

x3_{2,1} + x3_{2,3} + x3_{2,4} == 18,

x3_{3,4} − 2/5 x3_{1,3} − 2/5 x3_{2,3} == −3/5,
−1/5 x3_{1,4} − 3/5 x3_{2,4} − 9/10 x3_{3,4} == −105/10};

t = {2,3,4,1};
p = {−3,4, − 4, − 1};
d = {−1, − 1, − 1,1};

system3[[1]] = system3[[1]]/. {x3_{2,1} → 0} ;
system3[[2]] = system3[[2]]/. {x3_{2,1} → 0} ;
system3[[2]] = system3[[2]]/. {x3_{2,3} → 0} ;
system3[[3]] = system3[[3]]/. {x3_{2,3} → 0} ;

δ̄3 = {x3_{2,1} → 0,x3_{2,3} → 0} ;

For[i = 1,i ⩽ 3,++i,
{
If[d[[t[[i]]]] == 1,
δ = Solve [system3[[t[[i]]]],x3_{Abs[p[[t[[i]]]]],t[[i]]}] [[1]],
δ = Solve [system3[[t[[i]]]],x3_{t[[i]],Abs[p[[t[[i]]]]]}] [[1]] ;
```

```
δ̄3 = Join [δ̄3,δ] ;
If[i ≠ 3,
system3[[Abs[p[[t[[i]]]]]]] = system3[[Abs[p[[t[[i]]]]]]]/.δ,
{
If[d[[t[[4]]]] == 1,
{
φ0 = x3Abs[p[[t[[4]]]]],t[[4]]/.Solve[system3[[t[[4]]]]/.
(δ/.{x3Abs[p[[t[[4]]]]],t[[4]] → 0}) ,x3Abs[p[[t[[4]]]]],t[[4]]] [[1]];
φ1 = (x3Abs[p[[t[[4]]]]],t[[4]]/.Solve[system3[[t[[4]]]]/.
(δ/.{x3Abs[p[[t[[4]]]]],t[[4]] → 1}) ,x3Abs[p[[t[[4]]]]],t[[4]]] [[1]]) − 1;
δ = {x3Abs[p[[t[[4]]]]],t[[4]] → φ0/(φ0−φ1)};
},
{
φ0 = x3t[[4]],Abs[p[[t[[4]]]]]/.Solve[system3[[t[[4]]]]/.
(δ/.{x3t[[4]],Abs[p[[t[[4]]]]] → 0}) ,x3t[[4]],Abs[p[[t[[4]]]]]] [[1]];
φ1 = (x3t[[4]],Abs[p[[t[[4]]]]]/.Solve[system3[[t[[4]]]]/.
(δ/.{x3t[[4]],Abs[p[[t[[4]]]]] → 1}) ,x3t[[4]],Abs[p[[t[[4]]]]]] [[1]]) − 1;
δ = {x3t[[4]],Abs[p[[t[[4]]]]] → φ0/(φ0−φ1)};
}];

δ̄3 = Join [δ̄3/.δ,δ] ;

}];
}];
R11_{2,1} = 9x1_{1,3} + 5x1_{1,4} + 8x1_{2,1} + x1_{2,4} + 3x1_{3,4}/.δ1_{2,1};
R21_{2,1} = 9x1_{1,3} + 6x1_{1,4} + 6x1_{2,1} + 8x1_{2,4} + 2x1_{3,4}/.δ1_{2,1};
R31_{2,1} = 0x1_{1,3} + 0x1_{1,4} + x1_{2,1} + 0x1_{2,4} + 0x1_{3,4}/.δ1_{2,1};

R11_{2,3} = 9x1_{1,3} + 5x1_{1,4} + x1_{2,3} + x1_{2,4} + 3x1_{3,4}/.δ1_{2,3};
R21_{2,3} = 9x1_{1,3} + 6x1_{1,4} + x1_{2,3} + 8x1_{2,4} + 2x1_{3,4}/.δ1_{2,3};
R31_{2,3} = 0x1_{1,3} + 0x1_{1,4} + 0x1_{2,3} + 0x1_{2,4} + 0x1_{3,4}/.δ1_{2,3};

R12_{2,1} = x2_{1,3} + 5x2_{1,4} + 0x2_{2,1} + 7x2_{2,4} + 4x2_{3,4}/.δ2_{2,1};
R22_{2,1} = 2x2_{1,3} + 2x2_{1,4} + 7x2_{2,1} + 8x2_{2,4} + x2_{3,4}/.δ2_{2,1};
R32_{2,1} = 0x2_{1,3} + 0x2_{1,4} + x2_{2,1} + 0x2_{2,4} + 0x2_{3,4}/.δ2_{2,1};
```

$$\mathrm{R12}_{2,3} = \mathrm{x2}_{1,3} + 5\mathrm{x2}_{1,4} + 6\mathrm{x2}_{2,3} + 7\mathrm{x2}_{2,4} + 4\mathrm{x2}_{3,4}/.\delta 2_{2,3};$$
$$\mathrm{R22}_{2,3} = 2\mathrm{x2}_{1,3} + 2\mathrm{x2}_{1,4} + 3\mathrm{x2}_{2,3} + 8\mathrm{x2}_{2,4} + \mathrm{x2}_{3,4}/.\delta 2_{2,3};$$
$$\mathrm{R32}_{2,3} = 0\mathrm{x2}_{1,3} + 0\mathrm{x2}_{1,4} + 0\mathrm{x2}_{2,3} + 0\mathrm{x2}_{2,4} + 0\mathrm{x2}_{3,4}/.\delta 2_{2,3};$$

$$\mathrm{R13}_{2,1} = 10\mathrm{x3}_{1,3} + 2\mathrm{x3}_{1,4} + 7\mathrm{x3}_{2,1} + 0\mathrm{x3}_{2,4} + 6\mathrm{x3}_{3,4}/.\delta 3_{2,1};$$
$$\mathrm{R23}_{2,1} = 4\mathrm{x3}_{1,3} + 9\mathrm{x3}_{1,4} + 6\mathrm{x3}_{2,1} + 9\mathrm{x3}_{2,4} + 10\mathrm{x3}_{3,4}/.\delta 3_{2,1};$$

$$\mathrm{DD} = \begin{pmatrix} \mathrm{R11}_{2,1} & \mathrm{R12}_{2,1} & \mathrm{R13}_{2,1} \\ \mathrm{R21}_{2,1} & \mathrm{R22}_{2,1} & \mathrm{R23}_{2,1} \\ \mathrm{R31}_{2,1} & \mathrm{R32}_{2,1} & \mathrm{R33}_{2,1} \end{pmatrix};$$

$$A_1 = 396 - (9\mathrm{x1}_{1,3} + \mathrm{x2}_{1,3} + 10\mathrm{x3}_{1,3} + 5\mathrm{x1}_{1,4} + 5\mathrm{x2}_{1,4} + 2\mathrm{x3}_{1,4} +$$
$$8\mathrm{x1}_{2,1} + 7\mathrm{x3}_{2,1} + \mathrm{x1}_{2,3} + 6\mathrm{x2}_{2,3} + 7\mathrm{x3}_{2,3} + \mathrm{x1}_{2,4} + 7\mathrm{x2}_{2,4} +$$
$$3\mathrm{x1}_{3,4} + 4\mathrm{x2}_{3,4} + 6\mathrm{x3}_{3,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$

$$A_2 = 434 - (9\mathrm{x1}_{1,3} + 2\mathrm{x2}_{1,3} + 4\mathrm{x3}_{1,3} + 6\mathrm{x1}_{1,4} + 2\mathrm{x2}_{1,4} + 9\mathrm{x3}_{1,4} + 6\mathrm{x1}_{2,1} +$$
$$7\mathrm{x2}_{2,1} + 6\mathrm{x3}_{2,1} + \mathrm{x1}_{2,3} + 3\mathrm{x2}_{2,3} + 8\mathrm{x1}_{2,4} + 8\mathrm{x2}_{2,4} + 9\mathrm{x3}_{2,4} +$$
$$2\mathrm{x1}_{3,4} + \mathrm{x2}_{3,4} + 10\mathrm{x3}_{3,4})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$

$$A_3 = 23 - (\mathrm{x1}_{2,1} + \mathrm{x2}_{2,1} + \mathrm{x3}_{2,1})/.\overline{\delta 1}/.\overline{\delta 2}/.\overline{\delta 3};$$

$$\beta_1 = A_1 - \left(\mathrm{R11}_{2,3}\mathrm{y1}_{2,3} + \mathrm{R12}_{2,3}\mathrm{y2}_{2,3} + \mathrm{R13}_{2,3}\mathrm{y3}_{2,3}\right);$$
$$\beta_2 = A_2 - \left(\mathrm{R21}_{2,3}\mathrm{y1}_{2,3} + \mathrm{R22}_{2,3}\mathrm{y2}_{2,3} + \mathrm{R23}_{2,3}\mathrm{y3}_{2,3}\right);$$
$$\beta_3 = A_3 - \left(\mathrm{R31}_{2,3}\mathrm{y1}_{2,3} + \mathrm{R32}_{2,3}\mathrm{y2}_{2,3} + \mathrm{R33}_{2,3}\mathrm{y3}_{2,3}\right);$$

$$Y = \mathrm{Simplify}\left[\mathrm{Inverse[DD]}.\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right];$$
$$\mathrm{rule1} = \{\mathrm{x1}_{2,1} \to Y[[1]][[1]], \mathrm{x2}_{2,1} \to Y[[2]][[1]],$$
$$\mathrm{x3}_{2,1} \to Y[[3]][[1]], \mathrm{x1}_{2,3} \to \mathrm{y1}_{2,3}, \mathrm{x2}_{2,3} \to \mathrm{y2}_{2,3}, \mathrm{x3}_{2,3} \to \mathrm{y3}_{2,3}\};$$

$$\mathrm{rule2} = \{\mathrm{x1}_{1,3} \to \mathrm{x1}_{2,1}\left(\mathrm{x1}_{1,3}/.\delta 1_{2,1}\right) + \mathrm{x1}_{2,3}\left(\mathrm{x1}_{1,3}/.\delta 1_{2,3}\right) + \left(\mathrm{x1}_{1,3}/.\overline{\delta 1}\right),$$
$$\mathrm{x1}_{1,4} \to \mathrm{x1}_{2,1}\left(\mathrm{x1}_{1,4}/.\delta 1_{2,1}\right) + \mathrm{x1}_{2,3}\left(\mathrm{x1}_{1,4}/.\delta 1_{2,3}\right) + \left(\mathrm{x1}_{1,4}/.\overline{\delta 1}\right),$$
$$\mathrm{x1}_{2,4} \to \mathrm{x1}_{2,1}\left(\mathrm{x1}_{2,4}/.\delta 1_{2,1}\right) + \mathrm{x1}_{2,3}\left(\mathrm{x1}_{2,4}/.\delta 1_{2,3}\right) + \left(\mathrm{x1}_{2,4}/.\overline{\delta 1}\right),$$
$$\mathrm{x1}_{3,4} \to \mathrm{x1}_{2,1}\left(\mathrm{x1}_{3,4}/.\delta 1_{2,1}\right) + \mathrm{x1}_{2,3}\left(\mathrm{x1}_{3,4}/.\delta 1_{2,3}\right) + \left(\mathrm{x1}_{3,4}/.\overline{\delta 1}\right),$$
$$\mathrm{x2}_{1,3} \to \mathrm{x2}_{2,1}\left(\mathrm{x2}_{1,3}/.\delta 2_{2,1}\right) + \mathrm{x2}_{2,3}\left(\mathrm{x2}_{1,3}/.\delta 2_{2,3}\right) + \left(\mathrm{x2}_{1,3}/.\overline{\delta 2}\right),$$
$$\mathrm{x2}_{1,4} \to \mathrm{x2}_{2,1}\left(\mathrm{x2}_{1,4}/.\delta 2_{2,1}\right) + \mathrm{x2}_{2,3}\left(\mathrm{x2}_{1,4}/.\delta 2_{2,3}\right) + \left(\mathrm{x2}_{1,4}/.\overline{\delta 2}\right),$$

```
x2_{2,4} → x2_{2,1} (x2_{2,4}/.δ2_{2,1}) + x2_{2,3} (x2_{2,4}/.δ2_{2,3}) + (x2_{2,4}/.δ̄2̄) ,
x2_{3,4} → x2_{2,1} (x2_{3,4}/.δ2_{2,1}) + x2_{2,3} (x2_{3,4}/.δ2_{2,3}) + (x2_{3,4}/.δ̄2̄) ,
x3_{1,3} → x3_{2,1} (x3_{1,3}/.δ3_{2,1}) + x3_{2,3} (x3_{1,3}/.δ3_{2,3}) + (x3_{1,3}/.δ̄3̄) ,
x3_{1,4} → x3_{2,1} (x3_{1,4}/.δ3_{2,1}) + x3_{2,3} (x3_{1,4}/.δ3_{2,3}) + (x3_{1,4}/.δ̄3̄) ,
x3_{2,4} → x3_{2,1} (x3_{2,4}/.δ3_{2,1}) + x3_{2,3} (x3_{2,4}/.δ3_{2,3}) + (x3_{2,4}/.δ̄3̄) ,
x3_{3,4} → x3_{2,1} (x3_{3,4}/.δ3_{2,1}) + x3_{2,3} (x3_{3,4}/.δ3_{2,3}) + (x3_{3,4}/.δ̄3̄)} ;

solution = Simplify[Join[rule1,rule2/.rule1]];
Print[Simplify[system/.solution]];
Print[solution];

{True,True,True,True,True,True,True,
True,True,True,True,True,True,True,True}
```

General solution of the underdetermined system of linear algebraic equations (6.5.1) – (6.5.3) with respect to a supporting set of arcs $U_L^k \cup U_B^k$, $k \in K = \{1,2,3\}$ (see Figure 6.14) is:

$$x_{2,1}^1 = \frac{24801685 - 2709553x_{2,3}^1 - 68402x_{2,3}^2 - 68772x_{2,3}^3}{1684276};$$

$$x_{2,1}^2 = \frac{-2432629 + 1581371x_{2,3}^1 + 376254x_{2,3}^2 + 706024x_{2,3}^3}{842138};$$

$$x_{2,1}^3 = \frac{18801921 - 453189x_{2,3}^1 - 684106x_{2,3}^2 - 1343276x_{2,3}^3}{1684276};$$

$$x_{1,3}^1 = \frac{-58336079 + 6262751x_{2,3}^1 + 1710050x_{2,3}^2 + 1719300x_{2,3}^3}{1684276};$$

$$x_{1,4}^1 = \frac{21626579 - 2243076x_{2,3}^1 - 444613x_{2,3}^2 - 447018x_{2,3}^3}{421069};$$

$$x_{2,4}^1 = \frac{-1221821 + 1025277x_{2,3}^1 + 68402x_{2,3}^2 + 68772x_{2,3}^3}{1684276};$$

$$x_{3,4}^1 = \frac{-5370432 + 668382x_{2,3}^1 + 171005x_{2,3}^2 + 171930x_{2,3}^3}{421069};$$

$$x_{1,3}^2 = \frac{110658837 - 11069597x_{2,3}^1 - 7533490x_{2,3}^2 - 4942168x_{2,3}^3}{1684276};$$

$$x_{1,4}^2 = \frac{-134089410 + 14232339x_{2,3}^1 + 9510926x_{2,3}^2 + 6354216x_{2,3}^3}{2105345};$$

$$x_{2,4}^2 = \frac{20117527 - 1581371x_{2,3}^1 - 1218392x_{2,3}^2 - 706024x_{2,3}^3}{842138};$$

$$x_{3,4}^2 = \frac{225189605 - 11069597x_{2,3}^1 - 4164938x_{2,3}^2 - 4942168x_{2,3}^3}{16842760};$$

$$x_{1,3}^3 = \frac{99612203 - 3172323x_{2,3}^1 - 4788742x_{2,3}^2 - 4350104x_{2,3}^3}{3368552};$$

$$x_{1,4}^3 = \frac{3(-9461845 + 453189x_{2,3}^1 + 684106x_{2,3}^2 + 577696x_{2,3}^3)}{1531160};$$

$$x_{2,4}^3 = \frac{11515047 + 453189x_{2,3}^1 + 684106x_{2,3}^2 - 341000x_{2,3}^3}{1684276};$$

$$x_{3,4}^3 = \frac{94559375 - 3172323x_{2,3}^1 - 4788742x_{2,3}^2 - 981552x_{2,3}^3}{8421380}.$$

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