QUEUEING SYSTEM GI/M/1 WITH RANDOMIZED THRESHOLD ADMISSION CONTROL

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Queueing system type GI/M/1 with randomized threshold admission control is considered. The stationary distribution at arrival moments and at arbitrary moments is calculated. The Little's law is proved for this system.

Keywords: GI/M/1 queueing model, threshold admission control, Little’s law.

MATHEMATICAL MODEL

Queueing systems with variable operating conditions are adequate mathematical model of processes occurring in many components of modern communication networks and computer networks. Some results for systems with threshold admission control type M/G/1, GI/M/1 [3, 7], BMAP/G/1 [6], GI/PH/1 [8] and also for other systems [5, 9, 10] were obtained. In this paper, we will consider a queueing system type GI/M/1 with randomized threshold admission control; find the stationary probabilities and calculate some characteristics of this system.

Consider a single-server queueing system that can work in two modes. The system will switch to the first mode if the number of customers in this system is not larger than a fixed threshold j, else switch to the second mode in the opposite case. Assume that in the $\nu-$mode, the distribution function of inter-arrival times is $A_\nu(t)$ with intensity $\lambda_\nu = \left(\int\limits_0^\infty (1-A(t))dt\right)^{-1}$ and its Laplace-Stieltjes transform $A_\nu^*(s) = \int\limits_0^\infty e^{-st} dA(t)$ while service times have exponential distribution with parameter μ_ν. A customer arriving in the $\nu-$mode will be accepted with probability p_ν or rejected with probability $1-p_\nu$, $\nu = 1, 2$.

Let i_n denote the number of customers in the system immediately before the arrival moment of *n*-th customer. Clearly, $\{i_n\}_{n\ge 0}$ is a homogeneous irreducible Markov chain. Let $p_{ik} = P\{i_n = k \mid i_{n-1} = i\}$, $i, l \ge 0$ be the one-step transition the one-step transition probability of this Markov chain. By the complete probability formula, we easily obtain that,

$$p_{ik}=\begin{cases} p_1\phi^{(1)}_{i+1-k}+(1-p_1)\phi^{(1)}_{i-k}, 1\le k\le i\le j;\\ p_1\phi^{(1)}_0,\ 0\le i=k-1\le j;\\ p_2\phi^{(2)}_{i+1-k}+(1-p_2)\phi^{(2)}_{i-k}, 1\le k\le i, i\ge j+1;\\ p_2\phi^{(2)}_0, i=k-1\ge j+1;\\ 0, k>i+1\ge 1;\\ 1-\sum\limits_{l=1}^{\infty}p_{il}, k=0, \end{cases}$$

where $\phi^{(\nu)}_k=\int\limits_0^{\infty}e^{-\mu_\nu x}\frac{(\mu_\nu x)^k}{k!}dA_\nu(x)$ is the probability when during an inter-arrival time in the $\nu-$mode, there are k customers leaving from the system after completing their services, $k\ge 0$.

STATIONARY PROBABILITIES

Let $r_l=\lim\limits_{n\to\infty}P\{i_n=l\}, l\ge 0$ denote the stationary probability of the Markov chain $\{i_n\}_{n\ge 0}$. Applying the Chapman – Kolmogorov equation, we have the following recurrent relation

$$r_i=\sum_{l=i-1}^{j}\theta^{(1)}_{l+1-i}r_l+\sum_{l=j+1}^{\infty}\theta^{(2)}_{l+1-i}r_l,\ 1\le i\le j; \tag{1}$$

$$r_{j+1}=\theta^{(1)}_0 r_j+\sum_{l=j+1}^{\infty}\theta^{(2)}_{l-j}r_l; \tag{2}$$

$$r_i=\sum_{l=i-1}^{j}\theta^{(2)}_{l+1-i}r_l, j+2\le i<\infty. \tag{3}$$

where

$$\theta^{(\nu)}_k=\begin{cases} p_\nu\phi^{(\nu)}_k+(1-p_\nu)\phi^{(\nu)}_{k-1}, k\ge 1;\\ p_\nu\phi_0,\quad k=0, \end{cases}\qquad \nu=1,2.$$

Solving the system of recurrent relations (1)–(2)–(3) by applying of geometric method, we get the following result

Theorem 1. *The stationary probabilities of Markov chain* $\{i_n\}_{n\ge 0}$ *are given by the formula*

$$r_i=\begin{cases} C\delta^i, \qquad i\ge j+1;\\ C\sum\limits_{k=0}^{j-i}a_k\delta^{i+k},\ 0\le i\le j, \end{cases}$$

where

1. δ *is the unique solution in the interval* $(0,1)$ *of the equation*

$$\delta=(p_2+(1-p_2)\delta)A_2^*(\mu_2(1-\delta)), \tag{4}$$

with the satisfaction of ergodic condition of the Markov chain given by $\rho_2=\frac{\lambda_2}{\mu_2}<\frac{1}{p_2}.$

2. *Coefficients* $a_k, k=0,1,...$ *are calculated by their generating function defined by*

$$\alpha(z)=\sum_{k=0}^{\infty}a_k z^k=\frac{z-(p_2+(1-p_2)z)A_2^*(\mu_2(1-z))}{z-(p_1+(1-p_1)z)A_1^*(\mu_1(1-z))},\ \ |z|<1, \tag{5}$$

3.
$$C=\frac{1}{\sum_{l=0}^{j}\sum_{k=0}^{j-l}a_k\delta^{k+l}+\sum_{l=j+1}^{\infty}\delta^l}=\frac{1-\delta}{\delta^{j+1}(1+b_j)}, \tag{6}$$

where coefficients $b_k=\frac{1-\delta}{\delta^{j+1}}\sum_{l=0}^{j}\sum_{k=0}^{j-l}a_k\delta^{k+l}, k=0,1,2,...$ *are calculated by their generating function defined by*

$$\beta(z)=\sum_{k=0}^{\infty}b_k z^k=\frac{(1-\delta)\alpha(z)}{(1-z)(\delta-z)},\ \ |z|<\delta.$$

Corollary 1. 1. *The loss probability of an arbitrary customer is calculated as*

$$P_{\text{reject}}=V(0)=(1-p_1)\sum_{m=0}^{j}r_m+(1-p_2)\sum_{m=j+1}^{\infty}r_m=1-p_1+\frac{p_1-p_2}{1+b_j}.$$

2. *The mean number of customers in the system immediately before the arrival moment of a customer is calculated as*

$$L_{\text{arrival}}=\sum_{k=1}^{\infty}kr_k=\frac{1}{(1+b_j)(1-\delta)}+\frac{c_j+j}{1+b_j},$$

where coefficients $c_k=\frac{1-\delta}{\delta^{j+1}}\sum_{l=0}^{j}\sum_{k=0}^{j-l}l\delta^{k+l}a_k, k=0,1,2,...$ *are calculated by their generating function*

$$\gamma(z)=\sum_{k=0}^{\infty}c_k z^k=\frac{(1-\delta)z\alpha(z)}{(1-z)^2(\delta-z)},\ |z|<\delta.$$

Let $\tilde{i}_t$ denote the number of customers in the system at an arbitrary moment t and τ_n is the arrival moment of the *n*-th customer. Let $\pi_l=\lim_{t\to\infty}P\{\tilde{i}_t=l\}$ be the stationary probability of the random process $\{\tilde{i}_t\}_{t\ge0}$. We can easily verify that, process $\{\tilde{i}_t\}_{t\ge0}$ is a semi-generative process with embedded Markov renewal chain $\{i_n,\tau_n\}_{n\ge0}$, i. e. satisfying following properties:

a) For each $n\in\mathbb{N}_0$, τ_n – is a stopping time for $\{\tilde{i}_t\}_{t\ge0}$ and i_n is deterministic function of $\{\tilde{i}_u, u\le\tau_n\}$;

b) $\{i_n\}_{n\ge0}$ is a process on a countable state space E such that

$$P\{i_{n+1}=j,\tau_{n+1}-\tau_n\le t\,|\,i_0,...,i_n,\tau_0,...,\tau_n\}=P\{i_{n+1}=j,\tau_{n+1}-\tau_n\le t\,|\,i_n\},\forall n\in\mathbb{N}_0, j\in E$$

and probability $P\{i_{n+1}=j,\tau_{n+1}-\tau_n\le t\,|\,i_n=i\}=F_{ij}(t), i,j\in E$ is independent of n;

c) $P\{\tilde{i}_{\tau_n+t_1}=j_1,...,\tilde{i}_{\tau_n+t_k}=j_k\,|\,\tilde{i}_s, s\le\tau_n, i_n=i\}=P\{\tilde{i}_{t_1}=j_1,...,\tilde{i}_{t_k}=j_k\,|\,i_0=i\}$.

Theorem 2. *Let* $\{\tilde{i}_t\}_{t\ge0}$ *be a semi-generative process with irreducible and positive recurrent embedded Markov renewal chain* $(i_n,\tau_n)_{n\ge0}$ *with stationary probabilities* $\{r_l\}_{l\in E}$. *Assume that* $m=\sum_{k\in E}E(\tau_1\,|\,i_0=k)r_k<\infty$, *then*

$$\pi_l = \lim_{t\to\infty} P\{\tilde{i}_t = l\} = \frac{1}{m}\sum_{k\in E} r_k \int_0^\infty P\{\tau_1 > t, \tilde{i}_t = l \mid i_0 = k\}dt, \forall l \in E.$$

See the proof in [1, c. 211–213], or [2, c. 144–146].

Applying the fact in Theorem 2, using method intergral by parts and taking the generating functions, we get the following result

Theorem 3. *The stationary probabilities of the process* $\{i_t, t \ge 0\}$ *are given by the formula*

$$\pi_l = \begin{cases} p_2 \dfrac{C}{\mu_2 m}\delta^{l-1}, l \ge j+2; \\ \dfrac{C}{\mu_2 m}\delta^{l-1}\left(p_2 + \omega_{j-l+1}\right), 1 \le l \le j+1; \\ 1 - \dfrac{C}{\mu_2 m}\left(\dfrac{p_2}{1-\delta} + \sum\limits_{l=1}^{j+1}\delta^{l-1}\omega_{j-l+1}\right), l = 0, \end{cases}$$

where coefficients $\omega_n, n = 0,1,2,..., j$, *are calculated by their generating function*

$$\Omega(z) = \sum_{k=0}^{\infty}\omega_k z^k = \frac{\mu_2}{\mu_1}\kappa_1(z)\alpha(\delta z) - \kappa_2(z), \ |z| < 1,$$

$$\kappa_\nu(z) = \frac{(p_\nu + (1-p_\nu)\delta z)\left(1 - A_\nu^*(\mu_\nu(1-\delta z))\right)}{(1-\delta z)(1-z)}, \nu = 1,2.$$

If $\mu_1 = \mu_2$, *we can simplify the function to the form*

$$\Omega(z) = \frac{p_1\alpha(\delta z) - p_2}{1-z}, \ |z| < 1.$$

Corollary 2. *The mean number of customers in the system at an arbitrary moment is calculated as*

$$L = \frac{C}{\mu m}\left(\frac{p_2}{(1-\delta)^2} + \sum_{l=1}^{j+1} l\delta^{l-1}\omega_{j-l+1}\right).$$

SOME CHARACTERISTICS OF THE SYSTEM

In the next sequence, we assume that in two working mode of the system, the difference of two mode is only between the distribution of inter-arrival times and the accepted probability p_1, p_2, while service rates of two service mode are identical, i. e. $\mu_1 = \mu_2 = \mu$.

A customer arriving into the system when there are m other customer, may be loss with probability $1 - p_\nu$ or accepted to the system with probability p_ν ($\nu = 1$, if $m \le j$, $\nu = 2$ if $m \ge j+1$) and he will wait until all m customers, who had arrived into the system before him, completing their services. Their waiting times have Erlangian distribution $E_m(\mu)$. Applying formula of completed probability, we obtain the stationary distribution of waiting time of a customer arriving into the system in the form

$$W(t) = r_0 + \sum_{m=1}^{j} r_m\left(1 - p_1 + p_1\int_0^t \frac{\mu(\mu x)^{m-1}}{(m-1)!}e^{-\mu x}dx\right) + \sum_{m=j+1}^{\infty} r_m\left(1 - p_2 + p_2\int_0^t \frac{\mu(\mu x)^{m-1}}{(m-1)!}e^{-\mu x}dx\right).$$

Similarly, function of stationary distribution of sojourn time of a customer arriving into the system has the following form

$$V(t)=\sum_{m=0}^{j}r_m\left(1-p_1+p_1\int_0^t\frac{\mu(\mu x)^m}{(m)!}e^{-\mu x}dx\right)+\sum_{m=j+1}^{\infty}r_m\left(1-p_2+p_2\int_0^t\frac{\mu(\mu x)^m}{(m)!}e^{-\mu x}dx\right).$$

Laplace transforms of functions $W(t),V(t)$ are calculated as

$$W^*(s)=p_1\sum_{m=0}^{j}r_m\left(\frac{\mu}{\mu+s}\right)^m+p_2\sum_{m=j+1}^{\infty}r_m\left(\frac{\mu}{\mu+s}\right)^m,\qquad V^*(s)=W^*(s)\frac{\mu}{\mu+s}.$$

Theorem 4. 1. *The mean waiting time is calculated as*

$$\bar{W}=\frac{1}{\mu}\left(\frac{p_2}{(1-\delta)\left(1+b_j\right)}+\frac{p_1c_j+p_2j}{1+b_j}\right),$$

2. *The mean sojourn time is calculated as*

$$\bar{V}=\frac{1}{\mu}\left(\frac{p_2}{(1-\delta)\left(1+b_j\right)}+\frac{p_1\left(b_j+c_j\right)+p_2(j+1)}{1+b_j}\right).$$

Taking generating functions of $\frac{\mu}{C}\bar{V}$ and $\frac{\mu m}{C}L$, we have

$$\sum_{j=1}^{\infty}\frac{\mu}{C}\bar{V}z^j=\sum_{j=1}^{\infty}\frac{\mu m}{C}Lz^j=\frac{p_2z(1-\delta z)^2+(1-\delta)^2z(p_1\alpha(\delta z)-p_2)}{(1-\delta)^2(1-z)(1-\delta z)^2}$$

Consequently, we obtain that.

Theorem 5. *Little's Law holds true for the abovementioned system, i. e.* $\bar{V}=mL$.

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