# ON RETRIAL QUEUES CONTROLLED BY THE THRESHOLD AND HYSTERESIS STRATEGIES 

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Retrial queues of the type $M_{Q} / M / m / \propto$ with controlled rate of input flow are considered. For calculation of stationary probabilities an approximate approach have been developed. Proposed calculating scheme was applyed to solve an optimization problem in a class of the threshold strategies.

Keywords: retrial queues, stationary distribution, threshold and hysteresis strategies, optimization problems.

We deal with a $m$-server retrial model of the type $M / M / m$ in which a rate of primary call flow $\lambda_{j}$ depends on the number $j$ - of customers in the orbit. The rates of repeated calls $v$ and of the service process $\mu$ are supposed to be constant. The effective approach to find a steady state distribution of the service process is proposed. It consists of two stages. At the first stage we construct the explicit vector-matrix form of the stationary distribution for the queue with finite orbit. At the second stage the stationary probabilities for the queue with infinite orbit are approximated by the formulae which are obtained at the first stage.

Let us define the basic model as the two-dimensional Markov chain with continuous time $Q(t)=\left(Q_{1}(t), Q_{2}(t)\right), t \geq 0$ in space $S=I \times J, I=\{0,1, \ldots, m\}, \quad J=\{0,1, \ldots\}, Q_{1}(t)-$ number of busy servers, $Q_{2}(t)$ - number of repeated call sources. Infinitesimal characteristics $q_{(i, j)\left(i, j^{\prime}\right)}$ of $Q(t)$ are given by the parameters $\lambda_{j}, \nu, \mu$ in the ordinary way (see [1], p. 95-96).

We point out the conditions of existence of the stationary regime in the following lemma.
Lemma 1. Let $\lambda=\limsup \lambda_{j}<\infty$. Then on condition that $\frac{\lambda}{m \mu}<1$ the chain $Q(t)$ is ergodic and its limit distribution coincides with the unique stationary distribution.

At first we consider the $M_{Q} / M / m / N$ - model where $N$ is a maximal possible number of sources of repeated calls. Infinitesimal characteristics $q_{(i, j)\left(i, j^{\prime}\right)}^{(N)}$ of $Q^{(N)}(t)=\left(Q_{1}^{(N)}(t), Q_{2}^{(N)}(t)\right)$ coincide with $q_{(i, j)\left(i^{\prime}, j^{\prime}\right)}$ under $i=0,1, \ldots, m-1, \quad j=0,1, \ldots, N$ and under $i=m, \quad j=0,1, \ldots, N-1$. If $i=m, \quad j=N$, then

$$
q_{(m, N)\left(i^{\prime}, j^{\prime}\right)}^{(N)}=\left\{\begin{array}{c}
m \mu, \text { if } \quad\left(i^{\prime}, j^{\prime}\right)=(m-1, N), \\
-m \mu, \quad \text { if } \quad\left(i^{\prime}, j^{\prime}\right)=(m, N), \\
0, \\
\text { otherwise. }
\end{array}\right.
$$

Via $\pi_{i j}^{(N)}, \quad(i, j) \in S^{(N)}=\{0,1, \ldots, m\} \times\{0,1, \ldots, N\}$ we denote the stationary probabilities of $Q^{(N)}(t)$. To formulate the basic result we introduce the notation:

$$
\text { for } j=0,1, \ldots, N-1, A(j)=\left\|a_{i k}(j)\right\|_{i, k=0}^{m-1} \text { is tridiagonal matrix with }
$$

$$
a_{i k}(j)=\left\{\begin{array}{c}
\lambda_{j}+i \mu+j v, \quad k=i, i=0,1, \ldots, m-1, \\
-\lambda_{j}, \quad k=i+1, i=0,1, \ldots, m-2, \quad B=\left\|b_{i k}\right\|_{i, k=0}^{m-1}, b_{i k}=\left\{\begin{array}{c}
1, k=i+1, i=0,1, \ldots m-2, \\
-i \mu, \quad k=i-1, i=1,2, \ldots m-1, \\
0, \text { otherwise }
\end{array}\right. \\
0, \text { otherwise },
\end{array}\right.
$$

$$
C=\left\|c_{i k}\right\|_{i, k=0}^{m-1}, c_{i k}=\left\{\begin{array}{c}
1, k=m-1, i=0,1, \ldots m-1 \\
0, \text { otherwise }
\end{array}\right.
$$

Via $D(N)$ we will designate a triangular matrix

$$
D(N)=\left(\begin{array}{cccccc}
\mu & 0 & 0 & \ldots & 0 & 0 \\
-\left(N \nu+\lambda_{N}\right) & 2 \mu & 0 & \ldots & 0 & 0 \\
-N \nu & -\left(N \nu+\lambda_{N}\right) & 3 \mu & \ldots & 0 & 0 \\
& & & \ldots & & \\
-N v & -N v & -N v & \ldots & -\left(N v+\lambda_{N}\right) & (m-1) \mu
\end{array}\right)
$$

Also it be necessary for us the following vectors:
$\pi^{(N)}(j)=\left(\pi_{0 j}^{(N)}, \pi_{1 j}^{(N)}, \ldots, \pi_{m-1 j}^{(N)}\right), \quad G^{(N)}(j)=\frac{\pi^{(N)}(j)}{\pi_{0 N}^{(N)}}=\left(G_{0 j}^{(N)}, G_{1 j}^{(N)}, \ldots, G_{m-1 j}^{(N)}\right), \overline{1}(m-1) \quad$ is $(m-1)$ - dimensional vector composed of $1, e_{i}(m-1)$ is $(m-1)$ - dimensional vector with $i$-th component is equal to 1 and the rest are equal to 0 . Via $\overline{1}, e_{i}$ we will designate the same vectors of dimensional $m$.

Theorem 1. If $\lambda_{j}>0, j=0,1, \ldots N$ then stationary probabilities $\pi_{i j}^{(N)}, \quad(i, j) \in S^{(N)}$ has the following form

$$
\begin{gathered}
\left(\pi_{1 N}^{(N)}, \pi_{2 N}^{(N)}, \ldots, \pi_{m-1 N}^{(N)}\right)^{\prime}=\pi_{0 N}^{(N)} D^{-1}(N)\left(N v \overline{1}(m-1)+\lambda_{N} e_{1}(m-1)\right), \\
\pi_{m N}^{(N)}=\frac{\pi_{0 N}^{(N)}}{m \mu} G^{\prime}(N)\left(N v \overline{1}+\lambda_{N} e_{m}\right), \pi_{j}^{(N)^{\prime}}=\frac{\pi_{0 N}^{(N)} N!v^{N-j}}{j!} G^{(N)^{\prime}}(N) T(N-1) \times \ldots \times T(j), \\
\pi_{m j}^{(N)}=\frac{\pi_{0 N}^{(N)} N!v^{N-j}}{\lambda_{j} j!} G^{(N)}(N) T(N-1) \times \ldots \times T(j+1) \overline{1}, \quad j=0,1, \ldots, N-1,
\end{gathered}
$$

where $\quad \pi_{o N}^{(N)}=\left\{G^{(N)}(N)\left(\overline{1}+N!\sum_{j=0}^{N-1} \frac{v^{N-j}}{j!} T(N-1) \times \ldots \times T(j+1)\left[T(j)+\frac{1}{\lambda_{j}}\right] \overline{1}+\right.\right.$

$$
\left.\left.+\frac{1}{m \mu}\left(N v \overline{1}+\lambda_{N} e_{m}\right)\right)\right\}^{-1}
$$

$$
G^{(N)}(N)=\binom{1}{D^{-1}(N)\left(N v \overline{1}(m-1)+\lambda_{N} e_{1}(m-1)\right)}, \quad T(j)=\left[B+\frac{m \mu}{\lambda_{j}} C\right] A^{-1}(j)
$$

Evidently these formulas is an effective recurrent procedure to calculate the stationary distribution.

When the conditions of Lemma 1 hold true and $N \rightarrow \infty$ the stationary probabilities $\pi_{i j}^{(N)}, \quad(i, j) \in S^{(N)}$ converge to the corresponding probabilities for the system $M_{Q} / M / m / \propto$.

Variable rate of the input flow allows to consider a queue controlled by different strategies. In the report we consider two variants: threshold and hysteresis strategies.

Let us consider the threshold strategy which realizes the following algorithm of the service process control: we set $\lambda_{j}=\lambda^{(1)}$ if $j=0,1, \ldots, h$ and $\lambda_{j}=\lambda^{(2)}$ if $j=h \dashv 1, \ldots$.

The explicit formulae for steady state probabilities enable to propose an effective algorithm for optimal decision making which consists in finding of optimal position for a level to maximize some objective functional. As such a function we took the following

$$
\begin{equation*}
W(h)=C_{1} S_{1}(h)-C_{2} S_{2}(h)-C_{3} S_{3}(h)-\max , \tag{1}
\end{equation*}
$$

where $S_{1}(h)$ is the number of calls have being served; $S_{2}(h)$ - the number of calls which become repeated calls; $S_{3}(h)$ - the number of switching of the input flow; $C_{1}$ - income associated with service of a call; $C_{2}$ - penalty connected with a refusal in service; $C_{3}$ - penalty connected with a switching of the input flow rate.

We seek the threshold $h$ which maximizes a mean income for the system operating in a stationary regime. In the conditions of the stationary regime existence (conditions of Lemma 1) functionals $S_{i}(h), i=1,2,3$ also exist and can be written through stationary probabilities $\pi_{i j}=\pi_{i j}(h), \quad(i, j) \in I \times J: \quad S_{1}(h)=\sum_{j=0}^{\infty} \sum_{i=1}^{m} i \mu \pi_{i j}(h)$,

$$
S_{2}(h)=\lambda^{(1)} \sum_{j=0}^{h} \pi_{m j}(h)+\lambda^{(2)} \sum_{j=h+1}^{\infty} \pi_{m j}(h), S_{3}(h)=\lambda^{(1)} \pi_{m h}(h)+v(h+1) \sum_{k=0}^{m-1} \pi_{k h+1}(h) .
$$

Thus to solve the problem (1) we must calculate the stationary distribution of the $M_{Q} / M / m / \propto$ - model.

A hysteresis strategy may be introduced by means of the two thresholds $0<h_{1} \leq h_{2}$. The rate of input flow changes from $\lambda^{(1)}$ to $\lambda^{(2)}$ if the number of repeated calls reaches the level $h_{2}$ from below and it changes from $\lambda^{(2)}$ to $\lambda^{(1)}$ if the number reaches $h_{1}$ from above. If the number of repeated calls is in the interval $\left[h_{1}, h_{2}\right)$ then the queue follows the mode in which it was at the previous moment of time. The similar controlled systems with single server have been considered in [2, 3].

We will consider hysteresis strategy on the example of the system $M_{\tilde{Q}} / M / m / \infty$. Let us define our model as the three-dimensional Markov chain with continuous time $\tilde{Q}(t)=\left(\tilde{Q}_{1}(t), \tilde{Q}_{2}(t), \tilde{Q}_{3}(t)\right), t \geq 0$ in space $\tilde{S}=\{0,1\} \times\{0,1,2, \ldots\} \times\{1,2\}, \tilde{Q}_{1}(t)$ - number
of busy servers, $\tilde{Q}_{2}(t)$ - number of repeated call sources, $\tilde{Q}_{3}(t)$ - regime in which the system operates in the moment $t$. If $\tilde{Q}_{3}(t)=1$ then the system operates in the first regime with the rate of input $\lambda^{(1)}$. If $\tilde{Q}_{3}(t)=2$ then the rate of input is $\lambda^{(2)}$. Infinitesimal characteristics $q_{\left.(i, j, r,)^{\prime}\right)}^{\left(i^{\prime}, r^{\prime}\right)}$ of $\tilde{Q}(t)$ may be written via the system parameters.

Under $\lambda^{(2)} / m \mu<1$ for $\tilde{Q}(t)$ there exists the stationary regime. As an approximation of $\tilde{Q}(t)$ we consider the Markov chain $\tilde{Q}^{(N)}(t)$ in by analogy with $Q^{(N)}(t)$.

Let us introduce the following notations: $e_{i}(m)=\left(\delta_{i o}, \delta_{i 1}, \ldots, \delta_{i m-1}\right)^{\prime}, \delta_{i j}$ is Kronecker delta,

$$
e(m)=(1,1, \ldots, 1)^{\prime}, \quad \pi_{j}^{(r)}(N)=\left(\pi_{0 j}^{(r)}(N), \ldots, \pi_{m-1 j}^{(r)}(N)\right)^{\prime}
$$

Let $A_{r j}=\left\|a_{i k}^{j}(r)\right\|_{i, k=0}^{m-1}, j=0,1, \ldots, N-1$ be tridiagonal matrix with

$$
a_{i k}^{j}(r)=\left\{\begin{array}{c}
\lambda_{r}+i \mu+j v, k=i, i=0,1, \ldots, m-1, \\
-\lambda_{r}, k=i-1, i=1,2, \ldots, m-2, \\
-(i+1) \mu, k=i+1, i=0,1, \ldots, m-1, \\
0, \text { otherwise }
\end{array}\right.
$$

$$
B_{r j}=\left\|b_{i k}^{j}(r)\right\|_{i, k=0}^{m-1} ; b_{i k}^{j}(r)=\left\{\begin{array}{c}
\begin{array}{c}
(j+1) v, k=i-1, i \neq m-1 \\
\frac{(j+1) v m \mu}{\lambda_{r}}, k \neq m-2, i=m-1 \\
\frac{(j+1) v\left(\lambda_{r}+m \mu\right)}{\lambda_{r}}, k=m-2, i=m-1, \\
0, \text { otherwise, }
\end{array}
\end{array}\right.
$$

$$
\begin{gathered}
C_{r j}=\left\|c_{i k}^{j}(r)\right\|_{i, k=0}^{m-1}, c_{i k}^{j}(r)=\left\{\begin{array}{l}
\frac{h_{1} v m \mu}{\lambda^{(r)}}, i=m-1, \quad D=\left\|d_{i k}\right\|_{i, k=0}^{m-1}, \quad d_{i k}=\left\{\begin{array}{c}
1, k=0, i=0, \\
0, \text { otherwise, },
\end{array}\right. \\
a_{i-1 k}^{N}(2), \text { otherwise }
\end{array}\right. \\
\Delta_{1 j}^{(1)}=\left(\prod_{i=j}^{h_{1}-1} A_{1 i}^{-1} B_{1 i}\right)\left(\left[\left(\prod_{i=h_{1}}^{h_{2}-2} A_{1 i}^{-1} B_{1 i}\right) A_{1 l_{2}-1}^{-1} C_{1 k_{2}-1}+\sum_{k=h_{1}}^{h_{2}-2}\left(\prod_{i=h_{1}}^{k-1} A_{1 i}^{-1} B_{1 i}\right) A_{1 k}^{-1} C_{1 k}\right]+E\right) \times \\
\times\left[E+\sum_{k=h_{1}}^{k_{2}-1}\left(\prod_{i=h_{1}}^{k-1} A_{2 i}^{-1} B_{2 i}\right) A_{2 k}^{-1} C_{2 k}\right]^{-1}\left(\prod_{i=h_{1}}^{N-1} A_{2 i}^{-1} B_{2 i}\right) D^{-1} e_{0}(m), j=0,1, \ldots, h_{1}-1, \\
\Delta_{2 j}^{(1)}=\left[\left(\prod_{i=j}^{h_{2}-2} A_{1 i}^{-1} B_{1 i}\right) A_{1 h_{2}-1}^{-1} C_{1 h_{2}-1}+\sum_{k=j}^{h_{2}-2}\left(\prod_{i=j}^{k-1} A_{1 i}^{-1} B_{1 i}\right) A_{1 k}^{-1} C_{1 k}\right] \times \\
\quad \times\left[E+\sum_{k=h_{1}}\left(\prod_{i=h_{1}}^{h_{2}-1} A_{2 i}^{-1} B_{2 i}\right) A_{2 k}^{-1} C_{2 k}\right]^{-1}\left(\prod_{i=h_{1}}^{N-1} A_{2 i}^{-1} B_{2 i}\right) D^{-1} e_{0}(m), j=h_{1}, \ldots, h_{2}-1,
\end{gathered}
$$

$$
\begin{gathered}
\Delta_{1 j}^{(2)}=\left(E-\left[\sum_{k=j}^{h_{2}-1}\left(\prod_{i=j}^{k-1} A_{2 i}^{-1} B_{2 i}\right) A_{2 k}^{-1} C_{2 k}\right]\left[E+\sum_{k=h_{1}}^{k_{2}-1}\left(\prod_{i=h_{1}}^{k-1} A_{2 i}^{-1} B_{2 i}\right) A_{2 k}^{-1} C_{2 k}\right]^{-1}\left(\prod_{i=j}^{N-1} A_{2 i}^{-1} B_{2 i}\right) D^{-1} e_{0}(m),\right. \\
j=h_{1}, \ldots, h_{2}-1, \quad \Delta_{2 j}^{(2)}=\left(\prod_{i=j}^{N-1} A_{2 i}^{-1} B_{2 i}\right) D^{-1} e_{0}(m), j=h_{2}, \ldots, N-1 . \\
\Delta_{j}^{(1)}=\left\{\begin{array}{l}
\Delta_{1 j}^{(1)}, 0 \leq j \leq h_{1}-1, \\
\Delta_{2 j}^{(1)}, h_{1} \leq j \leq h_{2}-1,
\end{array} \Delta_{j}^{(2)}=\left\{\begin{array}{l}
\Delta_{1 j}^{(2)}, h_{1} \leq j \leq h_{2}-1, \\
\Delta_{2 j}^{(2)}, h_{2} \leq j \leq N-1,
\end{array} \gamma_{1}(j)= \begin{cases}1, j>h_{1}-2, \\
0, & j \leq h_{1}-2,\end{cases} \right.\right. \\
\gamma_{2}(j)= \begin{cases}1, & j<h_{2}, \\
0, & j \geq h_{2} .\end{cases}
\end{gathered}
$$

Theorem 2. The stationary probabilities of $\tilde{Q}^{(N)}(t)$ may be represented in the form

$$
\begin{gathered}
\pi_{j}^{(1)}(N)=\Delta_{j}^{(1)} \pi_{0 N}^{(2)}(N), j=0, \ldots, h_{2}-1, \quad \pi_{j}^{(2)}(N)=\Delta_{j}^{(2)} \pi_{0 N}^{(2)}(N), j=h_{1}, \ldots, N-1, \\
\pi_{N}^{(2)}(N)=D^{-1} e_{0}(m) \pi_{0 N}^{(2)}(N), \\
\pi_{m j}^{(1)}(N)=\frac{v}{\lambda^{(1)} e(m)^{\prime}\left((j+1) \Delta_{j+1}^{(1)}+\gamma_{1}(j) h_{1} \Delta_{h_{1}}^{(2)}\right) \pi_{0 N}^{(2)}(N), j=0, \ldots, h_{2}-1,} \\
\left.\pi_{m j}^{(2)}(N)=\frac{v}{\lambda^{(2)} e(m)^{\prime}\left((j+1) \Delta_{j+1}^{(2)}-\gamma_{2}(j) h_{1} \Delta_{h_{1}}^{(2)}\right) \pi_{0 N}^{(2)}(N), j=h_{1}, \ldots, N-1,} \begin{array}{r}
\pi_{m N}^{(2)}(N)=\frac{\left(\lambda^{(2)}+N v+(m-1) \mu\right) e_{m-1}^{\prime}(m)-\lambda^{(2)} e_{m-2}^{\prime}(m)}{m \mu} \times D^{-1} e_{0}(m) \pi_{0 N}^{(2)}(N), \\
\pi_{0 N}^{(2)}(N)=\left\{\sum_{j=0}^{h_{2-1} e^{\prime}(m) \Delta_{j}^{(1)} \lambda^{(1)}+v e^{\prime}(m)\left((j+1) \Delta_{j+1}^{(1)}+\gamma_{1}(j) h_{1} \Delta_{h_{1}}^{(2)}\right)} \lambda^{(1)}+\right. \\
\quad+\sum_{j=h_{1}}^{N-1} \frac{e^{\prime}(m) \Delta_{j}^{(2)} \lambda^{(2)}+v e^{\prime}(m)\left((j+1) \Delta_{j+1}^{(2)}-\gamma_{2}(j) h_{1} \Delta_{h_{1}}^{(2)}\right)}{\lambda^{(2)}}+ \\
\left.+\frac{e^{\prime}(m) m \mu+\left(\lambda^{(2)}+N v+(m-1) \mu\right) e_{m-1}^{\prime}(m)-\lambda^{(2)} e_{m-2}^{\prime}(m)}{m \mu} D^{-1} e_{0}(m)\right\}
\end{array}\right] .
\end{gathered}
$$

Theorem 2 may be applied to solve the optimization problems related to (1).

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