# INTERSECTIONS OF SOLVABLE HALL SUBGROUPS IN FINITE GROUPS

**E.P. Vdovin**[1], **V.I.Zenkov**[2],

[1] Sobolev Institute of Mathematics SB RAS,
4 Acad. Koptyg av., 630090 Novosibirsk, Russia
vdovin@math.nsc.ru

[2] Institute of Mathematics and Mechanics UrB RAS,
16, Kovalevskoi str. GSP-384 4, 620219 Ekaterinburg, Russia

The term "group" always means a finite group. In what follows $\pi$ is a set of primes, $\pi'$ is its complement in the set of all primes, and $\pi(n)$ is the set of prime divisors of a rational integer $n$. A rational integer $n$ is called a $\pi$-number if $\pi(n) \subseteq \pi$. For a group $G$ we set $\pi(G)$ to be equal to $\pi(|G|)$. A subgroup $H$ of $G$ is called a $\pi$-Hall subgroup if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. A maximal normal $\pi$-subgroup of $G$ is denoted by $O_\pi(G)$.

By a classical result, for primes $p$ and $q$ a group $G$ of order $p^\alpha q^\beta$ is solvable and $|O_p(G)| > p^\alpha/q^\beta$, except some explicitly determined cases. However, there was a gap in the list of these cases that was closed by V.S. Monakhov in 1975. V.S.Monakhov extended this Burnside theorem to solvable groups of order $p^a m$ with $(p, m) = 1$.

In 1967 D.S. Passman proved that a $p$-solvable finite group $G$ has three Sylow $p$-subgroups $P_1, P_2, P_3$ such that $P_1 \cap P_2 \cap P_3 = O_p(G)$. In particular, this means that if the order of $G$ is equal to $p^\alpha m$, where $(p, m) = 1$, then $|O_p(G)| > p^\alpha/m^2$. Using the classification of finite simple groups, in 1996 V.I. Zenkov proved that the same statement holds for an arbitrary finite group. In 2005 S. Dolfi proved that if $\pi$ is a set of odd primes and $G$ is a $\pi$-solvable group, then for every $\pi$-Hall subgroup $H$ of $G$ there exist $x$ and $y \in G$ such that $H \cap H^x \cap H^y = O_\pi(G)$. In 2007 S. Dolfi and, independently, E.P. Vdovin obtained a similar result for arbitrary $\pi$.

There exist finite groups $G$ where the intersection of every 4 conjugate solvable $\pi$-Hall subgroups is greater than $O_\pi(G)$. So we need to take at least 5 conjugate solvable $\pi$-Hall subgroups to get the intersection equal to $O_\pi(G)$.

For a prime $p \geqslant 7$ the symmetric group $S_p$ contains a non-solvable $p'$-Hall subgroup $S_{p-1}$. The intersection of $p-2$ distinct $p'$-Hall subgroups is equal to the stabilizer of $p-2$ points, and so has order 2. Hence this intersection is not equal to $\{e\} = O_{p'}(S_p)$. This example shows that one cannot find a fixed number $f$ such that for an arbitrary finite group $G$ with $\pi$-Hall subgroups the intersection of some $f$ distinct conjugate $\pi$-Hall subgroups of $G$ is equal to $O_\pi(G)$.

The aim of our talk is to investigate the following problems.

**Problem 1.** Let $H$ be a solvable $\pi$-Hall subgroup of a finite group $G$. Do there exist elements $x, y, z, t \in G$ such that $H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G)$?

**Problem 2.** Let $H$ be a $\pi$-Hall subgroup of a finite group $G$ and $p$ be a minimal element of $\pi'$. Does there exist a function $f(p)$ such that for some elements $x_1, \dots, x_{f(p)} \in G$ we have $\bigcap_{i=1}^{f(p)} H^{x_i} = O_\pi(G)$?

In particular, the following theorem is proved.

**Theorem 1.** *Any minimal (subject to inclusion) counterexample to Problem 1 is almost simple.*

Note that a complete classification of Hall subgroups in finite simple groups has been recently obtained by E.P. Vdovin and D.O. Revin. We hope to solve Problem 1 using this classification and the theorem.

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