STATISTICAL FORECASTING BASED ON BLOOMFIELD EXPONENTIAL MODEL

V.A. VOLOSHKO*, YU.S. KHARIN**
Belarusian State University
Minsk, BELARUS
e-mail: *valeravoloshko@yandex.ru, **kharin@bsu.by

Abstract

Forecasting of stationary time series based on the Bloomfield model is considered. The mean-square risk of forecasting is analyzed for the situation with known parameters of the model.

1 Introduction

The accuracy of statistical inferences (estimates, decisions, forecasts) in parametric data analysis, as it is known [1, 2], depends on the ratio of the number p of model parameters and the observation length T: $\rho(p,T) = p/T$. If $p << T$, i.e. $\rho(p,T) \to 0$ at $T \to \infty$, then statistical inferences based on classical methods (least squares, maximum likelihood, Bayesian method) appear consistent and give an acceptable accuracy for practice. On the other hand, if the number of model parameters p is comparable with T, i.e. $\rho(p,T) \to c$, $0 < c < 1$, then classical methods appear inapplicable, and acceptable accuracy is reached only for some special cases. As a result, the problem of development and statistical analysis of small-parametric models, i.e. models with small $\rho(p,T)$, is very topical. In [3] Bloomfield proposed the so-called exponential model EXP(p) of order p for stationary time series and built some estimators of its parameters. This paper is devoted to using this model for statistical forecasting.

2 Bloomfield model and its properties

Introduce the notation: $\Pi = [-\pi,\pi]$; $e_n(\lambda) = e^{in\lambda}, \lambda \in \Pi, n \in \mathbb{Z}$; $E_\Pi\{f\} = (2\pi)^{-1} \times \times \int_\Pi f(\lambda)d\lambda$, $D_\Pi\{f\} = E_\Pi\{|f|^2\} - |E_\Pi\{f\}|^2$, $f : \Pi \to \mathbb{C}$.

Let $\{x_t\}, t \in \mathbb{Z}$, be a real valued stationary time series with zero mean $E\{x_t\} = 0$, a covariance function $\sigma_\tau = E\{x_t x_{t+\tau}\}$, the correspondent correlation function $\theta_\tau = \sigma_\tau/\sigma_0$, $\tau \in \mathbb{Z}$, and the spectral density function $S(\lambda) = \sum_{\tau\in\mathbb{Z}} \sigma_\tau e_\tau(\lambda)$, $L(\lambda) = \ln S(\lambda)$, $\lambda \in \Pi$. For brevity we call $S(\lambda)$ and $L(\lambda)$ the *spectrum* and the *log-spectrum* respectively.

Under the assumption $L(\cdot) \in L_2(\Pi)$, any time series x_t is uniquely representable by the following models of infinite order: the Bloomfield exponential model EXP(∞) [3]:

$$S(\lambda) = \exp\left\{l_0 + 2\mathrm{Re}\{l(e^{i\lambda})\}\right\},\ l(z) = \sum_{n\in\mathbb{N}} l_n z^n,\ l_0, l_1, \dots \in \mathbb{R}, \tag{1}$$

4 Risk of forecasting and its asymptotic analysis

Theorem 1. *Let the forecasted time series x_t have the spectrum $S(\lambda)$, the coefficients $\{a_n\}$ of the forecasting statistic (3) generate the spectrum $S_*(\lambda) = \sigma_*^2/|\alpha(e^{i\lambda})|^2$ with the transfer function $\alpha(\cdot)$ specified by the series: $\alpha(z) = 1 - \sum_{n=1}^{\infty} a_n z^n$, and $\sigma = \sigma_*$. Then the risk $r(S_*|S) = E\{(\hat{x}_t - x_t)^2\}$ of the forecasting statistic (3) satisfies the relation:*

$$\ln\{r(S_*|S)\} - l_0(S) = \ln\{E_\Pi\{S/S_*\}\} - E_\Pi\{\ln\{S/S_*\}\} \geq 0. \tag{6}$$

Theorem 1 characterizes an increment of the risk of forecasting caused by an error of the approximation of the spectrum $S(\lambda)$ by the function $S_*(\lambda)$. Note, that by the Jensen inequality the right side of (6) is always nonnegative, invariant to the scaling of each of the functions $S(\lambda), S_*(\lambda)$ by some positive factor and vanishes at $S_*(\lambda) = c \cdot S(\lambda)$, $c > 0$. Also note, that the minimal risk of forecasting $r(S|S) = \exp(l_0(S)) = \sigma^2$ is reached at $S_*(\cdot) = S(\cdot)$, that fits [1].

Corollary 1. *Let $S(\lambda) = S_*(\lambda) + \delta(\lambda)$, where $\delta(\lambda)$ is an approximation error. If $m = \sup_{\lambda \in \Pi}\{|\delta(\lambda)/S(\lambda)|\} \to 0$, then the following asymptotic expressions hold:*

$$r(S_*|S) = e^{l_0(S)} \cdot e^{0.5 D_\Pi\{\delta/S\}} + O(m^3) \geq e^{l_0(S)},$$
$$\Delta r = r(S_*|S) - r(S|S) = e^{l_0(S)} \cdot 0.5 D_\Pi\{\delta/S\} + O(m^3),$$
$$\kappa = \Delta r / r(S|S) = 0.5 D_\Pi\{S_*/S\} + O(m^3)$$

This result characterizes the risk deviation from the minimal one, when the deviation of the model $S_*(\lambda)$ from the true model $S(\lambda)$ is small.

Due to the scheme (5), for the EXP(p)-forecast of the depth $T \in \mathbb{N}$ we have $S_* = \alpha_T(\varepsilon_p(S))$. Denote the risk of this forecast: $r_{T,p} = r(\alpha_T(\varepsilon_p(S))|S)$, and investigate its asymptotics.

Theorem 2. *For $p \in \mathbb{N}$ and increasing depth $T \to +\infty$ the risk of the EXP(p)-forecast $r_{T,p}$ satisfies the asymptotics:*

$$r_{T,p} \to \exp\{l_0(S)\} \cdot E_\Pi\left\{\exp\left\{2\sum_{k=p+1}^{\infty} l_k \cos(k\lambda)\right\}\right\}. \tag{7}$$

Note, that the rate of decrease of the right side of (7) at $p \to \infty$ is directly related to the rate of decrease of coefficients l_n at $n \to \infty$. The following result formulate the conditions, when the sequence of the EXP(∞)-coefficients l_n decreases faster, than the sequence of the AR(∞)-coefficients b_n.

Theorem 3. *Let $l_c(\lambda) = l(e^{i\lambda}), \lambda \in \Pi$, be twice differentiable and $Re\{l_c(\lambda)\}$ have $M < +\infty$ minimums l_r^- at points $\lambda_1, \ldots, \lambda_M$, while the second derivatives $l_c''(\lambda_k) = R_k e^{i\varphi_k}, R_k > 0, |\varphi_k| < \pi/2, k = 1, \ldots, M$. Then the coefficients of the Wold autoregression $b_n(\gamma)$ for the spectrum $(S(\lambda))^\gamma$ have the following form at $\gamma \to +\infty$*