VARIANCE GAMMA PROCESS SIMULATION AND IT'S PARAMETERS ESTIMATION

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Abstract

Variance gamma process is a three parameter process. Variance gamma process is simulated as a gamma time-change Brownian motion and as a difference of two independent gamma processes. Estimations of simulated variance gamma process parameters are presented in this paper.

1 Introduction

This paper presents the estimations of simulated variance gamma process parameters. The variance gamma process is simulated as a gamma time-change Brownian motion (with constant drift and volatility) and as a difference of two independent gamma processes. The objective is to determine the way of simulation achieving less errors in estimates of the variance gamma process parameters. The algorithms of variance gamma process simulation are described in [1] and [4]. The variance gamma process properties are considered in [2] and [3]. The description of the variance gamma process is presented in section 2. Section 3 is devoted to the estimations of simulated variance gamma process parameters.

2 Variance gamma process

Definition 1. A random variable ϑ is governed by the variance gamma law if it's probability density function is given by

$$f_\vartheta(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} exp\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right) \frac{y^{\frac{1}{\nu}-1} exp\left(-\frac{g}{\nu}\right)}{\nu^{\frac{1}{\nu}}\Gamma\left(\frac{1}{\nu}\right)} dg, \tag{1.1}$$

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz x > 0, \sigma > 0, \nu > 0, \theta \in \mathbb{R}$$

A random variable ϑ with the variance gamma law is denoted as $\vartheta \sim V(\sigma, \nu, \theta)$.

Definition 2. A stochastic process $V = (V_t)_{t\geq 0}$ with parameters $\sigma > 0, \nu > 0, \theta \in \mathbb{R}$, defined on a probability space (Ω,F,P) is a variance gamma process if:

1) $V_0 = 0$ a.s.;

2) V has independent and stationary increments;

3) for $s \geq 0, t \geq 0$ V has variance gamma distributed increments with parameters $\sigma\sqrt{t} > 0, \nu/t > 0, t\theta > 0$

$$V_{t+s} - V_s = V_t - V_0 \sim V\left(\sigma\sqrt{t}, \nu/t, t\theta\right).$$

The variance gamma process can be defined as a gamma time-changed Brownian motion with drift. More precisely, let $G = (G_t)_{t\geq 0}$ be a gamma process with parameters $a = 1/\nu > 0$, $b = 1/\nu > 0$. Let $W = (W_t)_{t\geq 0}$ denote a Brownian motion with drift $\theta \in \mathbb{R}$ and volatility $\sigma \geq 0$. Then a variance gamma process $V = (V_t)_{t\geq 0}$ with parameters $\sigma > 0, \nu > 0, \theta \in \mathbb{R}$ can be defined as

$$V_t = \theta G_t + \sigma W_{G_t}.$$

Theorem1. The characteristic function of the variance gamma law is given by

$$\phi_{V_t}(u) = \left(\frac{1}{1 - i\nu\theta u + (\sigma^2\nu/2)\, u^2}\right)^{\frac{t}{\nu}}, \sigma > 0, \nu > 0, \theta \in \mathbb{R} \tag{1.2}$$

The variance gamma process has mean θt, variance $(\sigma^2+\nu\theta^2)t$, skewness $\theta\nu\,(3\sigma^2 + 2\nu\theta^2)/\left(t^{1/2}(\sigma^2 + \nu\theta^2)\right)^{3/2}$ and kurtosis $3\left(1 + 2\nu/t - \nu\theta^4/(t\,(\sigma^2 + \nu\theta^2)^2)\right)$.

Another way of defining a variance gamma process is by seeing it as the difference of two independent gamma processes. More precisely, a variance gamma process $\bar{V} = (\bar{V}_t)_{t\geq 0}$ with parameters $a > 0, b_1 > 0, b_2 > 0$ can be decomposed as

$$\bar{V}_t = G_t^1 - G_t^2,$$

where $G^1 = (G_t^1)_{t\geq 0}$ is a gamma process with parameters $a > 0, b_1 > 0$ and $G^2 = (G_t^2)_{t\geq 0}$ is a gamma process with parameters $a > 0, b_2 > 0$.

Theorem 2. The variance gamma process $\bar{V} = (\bar{V}_t)_{t\geq 0}$ with paremeters $a > 0, b_1 > 0, b_2 > 0$ are associated with the variance gamma process $V = (V_t)_{t\geq 0}$ with paremeters $\sigma > 0, \nu > 0, \theta \in \mathbb{R}$ by the following equations

$$a = 1/\nu > 0.$$

$$b_1 = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} > 0, \tag{1.3}$$

$$b_2 = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} > 0.$$

3 Simulation and parameters estimations

The variance gamma process simulation as a time-change Brownian motion and a variance gamma process simulation as difference of two independent gamma processes (represented in [1] and [4]) are implemented with in MATLAB® 7.6.0 (R2008a). One hundred variance gamma processes$V^i = ({V^i}_t)_{t=1,2,...,T}, T = 10000, i = 1, 2, ..., T$ are simulated. Let

$$m = MV, d = DV, s = \frac{M\left[(V-m)^3\right]}{d^{\frac{3}{2}}}, k = \frac{M\left[(V-m)^4\right]}{d^2}$$

mean, variance, skewness and kurtosis of a variance gamma process; $\hat{m}_n^i, \hat{d}_n^i, \hat{s}_n^i, \hat{k}_n^i, i = 1, 2, ..., 100$ estimates of simulated variance gamma process i which are calculated in MATLAB® 7.6.0 (R2008a) using the MATLAB's functions ***mean, var, skewness, kurtosis***

$$\hat{m}_n^i = \tfrac{1}{n}\sum_{t=1}^{n} V_j^i, \quad \hat{d}_n^i = \tfrac{1}{n-1}\sum_{t=1}^{n}\left(V_j^i - \hat{m}^i\right)^2,$$

$$\hat{s}_n^i = \frac{1}{n}\sum_{t=1}^{n}\frac{\left(V_j^i - \hat{m}^i\right)^3}{\left(\sqrt{\hat{d}^i}\right)^3}, \quad \hat{k}_n^i = \frac{1}{n}\sum_{t=1}^{n}\frac{\left(V_j^i - \hat{m}^i\right)^4}{\left(\sqrt{\hat{d}^i}\right)^4},$$

$$n = T-1, V_j^i = V_{j\Delta t}^i - V_{(j-1)\Delta t}^i, j = 1, 2, ..., n, i = 1, 2, ..., 100.$$

Calculate the errors in estimates of $\hat{m}_n^i, \hat{d}_n^i, \hat{s}_n^i, \hat{k}_n^i, i = 1, 2, ..., 100$

$$\varepsilon_m^i = \left|m - \hat{m}_n^i\right|, \varepsilon_d^i = \left|d - \hat{d}_n^i\right|, \varepsilon_s^i = \left|s - \hat{s}_n^i\right|, \varepsilon_k^i = \left|k - \hat{k}_n^i\right|, i = 1, 2, ..., 100, n \geq 0.$$

Parameters estimates $\sigma^i > 0, \nu^i > 0, \theta^i \in \mathbb{R}$ of simulated as a time-change Brownian motion process $V^i = (V^i{}_t)_{t=1,2,\ldots,T}, T = 10000, i = 1, 2, ..., T$ in condition that $\hat{d}_n^i \geq \frac{\left(\hat{s}_n^i \hat{m}_n^i\right)^2}{4}$ are the following

$$\hat{\sigma}_n^i = \sqrt{\frac{-\hat{d}_n^i + \sqrt{9\left(\hat{d}_n^i\right)^2 - 4\hat{s}_n^i\hat{m}_n^i\left(\hat{d}_n^i\right)^{3/2}}}{2}}, \hat{\nu}_n^i = \frac{\hat{d}_n^i - (\hat{\sigma}_n^i)^2}{(\hat{m}_n^i)^2}, \hat{\theta}_n^i = \hat{m}_n^i, i = 1, 2, ..., 100. (1.4)$$

When $\theta^i = 0$, parameters estimates are the following

$$\hat{\sigma}_n^i = \sqrt{-\hat{d}_n^i}, \hat{\nu}_n^i = \hat{k}_n^i/6 - 1/2, i = 1, 2, ..., 100.$$

Calculate the errors in estimates of $\hat{\sigma}_n^i, \hat{\nu}_n^i, \hat{\theta}_n^i \in \mathbb{R}$

$$\varepsilon_\sigma^i = \left|\sigma - \hat{\sigma}_n^i\right|, \varepsilon_\nu^i = \left|\nu - \hat{\nu}_n^i\right|, \varepsilon_\theta^i = \left|\theta - \hat{\theta}_n^i\right|, i = 1, 2, ..., 100, n \geq 0.$$

Parameters estimates $a > 0, b_1 > 0, b_2 > 0$ of simulated as the difference of two independent gamma processes $\bar{V}^i = (\bar{V}^i{}_t)_{t=1,2,\ldots,T}, T = 10000, i = 1, 2, ..., T$ are deduced from (1.3) and (1.4)

$$\hat{a}_n^i = \frac{(\hat{m}_n^i)^2}{\hat{d}_n^i - (\hat{\sigma}_n^i)^2},$$

$$\hat{b}_{1n}^i = \frac{2\hat{m}_n^i}{\sqrt{\left(\hat{d}_n^i\right)^2 - (\hat{\sigma}_n^i)^4} + \hat{d}_n^i - (\hat{\sigma}_n^i)^2},$$

$$\hat{b}_{2n}^i = \frac{2\hat{m}_n^i}{\sqrt{\left(\hat{d}_n^i\right)^2 - (\hat{\sigma}_n^i)^4} - \hat{d}_n^i + (\hat{\sigma}_n^i)^2},$$

$i = 1, 2, ..., 100, n \geq 0$.

Calculate the errors in estimates of $\hat{a}_n^i, \hat{b}_{1n}^i, \hat{b}_{2n}^i$

$$\varepsilon_a^i = |a - \hat{a}_n^i|,\ \varepsilon_{b_1}^i = \left|b_1 - \hat{b}_{1n}^i\right|,\ \varepsilon_{b_2}^i = \left|b_2 - \hat{b}_{2n}^i\right|, i = 1, 2, ..., 100, n \geq 0.$$

According to achieved the errors in estimates parameters of simulated variance gamma process the conclusion is the follow:

1. Mean and standart deviation of errors in estimstes parameters don't depend on the variance gamma process simulation method;

2. Increasing of processes values during simulation leads to decreasing of the errors in estimates of simulated variance gamma process parameters.

References

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