COMPUTER ANALYSIS OF A CYCLIC SERVICE ALGORITHM WITH VARYING DURATIONS

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Abstract

For a loss queueing systems of servicing conflict flows within a class of cyclic algorithms with variable durations a mathematical model has been constructed in form of a Markov chain with incomes. A numerical search for an optimal control has been carried out.

1 Introduction

Different algorithms are used to control conflict flows, such as cyclic algorithms, time-sharing algorithms, algorithms with dynamic priorities, algorithms with prolongations, etc. In some real queueing systems it is important for an algorithm to be predictable in behavior from customers' point of view. For example participants of a road traffic expect traffic-lights operating in cycles and deviations from this scheme can be perceived as a malfunction of a traffic light and lead to accident situations.

Hence it is interesting to search for cyclic algorithms with variable durations which minimize certain objective functions. In the present paper we study mean sojourn time of all customers during a working cycle. A similar problem in the class of dynamic priorities was considered in [1].

2 Main results

Consider a loss queueing system with $m < \infty$ conflict flows Π_1, Π_2, …, Π_m. During a time slot Δ, $0 < \Delta < \infty$, a customer of a flow Π_j arrives with probability λ_j and doesn't arrive with probability $1 - \lambda_j$, $j = 1, 2, \ldots, m$. Customers of the flow Π_j are places in a bunker O_j with finite capacity $N_j < \infty$. Servicing device has $2m$ states $\Gamma^{(1)}$, $\Gamma^{(2)}$, …, $\Gamma^{(2m)}$. In a state $\Gamma^{(2j-1)}$ only customers of a flow Π_j are served, this is the reason of conflictness of flows; in a state $\Gamma^{(2j)}$ customers are not served. During a time slot Δ in a state $\Gamma^{(2j-1)}$ a service of a customer from a queue O_j is completed with probability β_j and the customer leaves the system, or with probability $1 - \beta_j$ the service is not completed. Servicing device shifts its states in a cyclic order: $\Gamma^{(1)} \to \Gamma^{(2)} \to \cdots \to \Gamma^{(2m)} \to \Gamma^{(1)} \to \cdots$. Servicing device can function in one of the n regimes. In a regime r, $r = 1, 2, \ldots, n$, duration of a state $\Gamma^{(s)}$, $s = 1, 2, \ldots, 2m$, is non-random and equals $T_{s,r}\Delta$. A time interval during which the states of the server go from $\Gamma^{(1)}$ to $\Gamma^{(2m)}$ inclusively, is called *a cycle*. A regime get selected in the beginning of a cycle, i.e. at time 0 and at epochs of servicing device state shifts from

$\Gamma^{(2m)}$ to $\Gamma^{(1)}$. Assuming that amounts of customers in the queues is described with a vector $(x_1, x_2, \ldots, x_m) \in \{0, 1, \ldots, N_1\} \times \{0, 1, \ldots, N_2\} \times \ldots \times \{0, 1, \ldots, N_m\} = X$, the regime $u(x) = r$ is chosen, where $u(\cdot)$ is a predefined mapping of an integer nonnegative lattice X into $\{1, 2, \ldots, n\}$.

We will observe the system at discrete time instants 0, Δ, 2Δ, ... Denote by $\kappa_{j,i}$ the amount of customers in the queue O_j at time $i\Delta$, $\eta_{j,i}$ the amount of customers of the flow Π_j arrived during a time interval $(i\Delta, (i+i)\Delta]$, $\xi_{j,i}$ the virtual amount of served customers from the queue O_j during the time interval $(i\Delta, (i+1)\Delta]$ under assumption of presence of customers in the queue, $\kappa_i = (\kappa_{1,i}, \kappa_{2,i}, \ldots, \kappa_{m,i})$. A recurrent relation $\kappa_{j,i+1} = \min\{N_j, \max\{0, \kappa_{j,i} + \eta_{j,i} - \xi_{j,i}\}\}$ holds, which characterizes formation of queue from the flow Π_j during the time interval $(i\Delta, (i+1)\Delta]$.

Define $T^{(r)} = T_{1,r} + T_{2,r} + \ldots + T_{2m,r}$. Put $\tau_i = 0$ and $\tau_{i+1} = \tau_i + T^{(r)}$ for $u(\kappa_{\tau_i}) = r$. A random variable $\eta_{j,i}$ takes value $\delta \in \{0, 1\}$ with probability $\lambda_j^\delta (1-\lambda_j)^{1-\delta}$ and doesn't depend on other random variables defined up to the moment $i\Delta$ inclusively. A random variable $\xi_{j,i}$ takes value $\delta \in \{0, 1\}$ with probability $\beta_j^\delta (1-\beta_j)^{1-\delta}$ when for an integer θ inequalities $\tau_\theta + T_{1,r} + T_{2,r} + \ldots + T_{2j-2} \leq i < \tau_\theta + T_{1,r} + T_{2,r} + \ldots + T_{2j-1}$ hold and $u(\kappa_{\tau_\theta}) = r$, in the rest of the cases $\xi_{j,i} = 0$.

Theorem 1. *Given the probability distribution of vector κ_0 sequence*

$$\{\kappa_{\tau_i}; i = 0, 1, \ldots\} \tag{1}$$

is a Markov chain.

Introduce numbers

$$p_{k,l}^{(j)} = \begin{cases} 1 - \lambda_j + \lambda\beta_j, & k = l = 0, \\ (1-\lambda_j)(1-\beta_j) + \lambda_j\beta_j, & k = l = 1, 2, \ldots, N_j - 1, \\ (1-\lambda_j)\beta_j, & k - 1 = l = 0, 1, \ldots, N_j - 1, \\ (1-\beta_j)\lambda_j, & k + 1 = l = 1, 2, \ldots, N_j, \\ 1 - \beta_j + \lambda_j\beta_j, & k = l = N_j, \\ 0, & k \neq l, k \neq l \pm 1, \end{cases}$$

$$q_{k,l}^{(j)} = \begin{cases} 1 - \lambda_j, & k = l = 0, 1, \ldots, N_j - 1, \\ \lambda_j, & k + 1 = l = 1, 2, \ldots, N_j, \\ 1, & k = l = N_j, \\ 0, & k \neq l, k \neq l - 1. \end{cases}$$

Denote by $P^{(j)}$, $Q^{(j)}$ the matrices of numbers $p_{k,l}^{(j)}$, $q_{k,l}^{(j)}$. Then matrices $P^{(j)}(a)$, $Q^{(j)}(a)$ equal ath power of matrices $P^{(j)}$ and $Q^{(j)}$ correspondingly. For $w = (w_1, w_2, \ldots, w_m) \in$ $\in X$, $x \in X$ and $u(x) = r$ one has

$$\Pr\{\kappa_{\tau_{i+1}} = w | \kappa_{\tau_i} = x\} =$$
$$= \prod_{j=1}^{m} \Big[\big(Q^{(j)}(T_{1,r} + T_{2,r} + \ldots + T_{2j-2,r}) P^{(j)}(T_{2j-1}) Q^{(j)}(T_{2j,r} + T_{2j+1,r} + \ldots + T_{2m,r}) \big)_{x_j, w_j} \Big], \tag{2}$$

where $(A)_{k,l}$ denotes the element in the k-th row and the l-th column of a matrix A. It follows from the form of transition probabilities (2) that all the states of the Markov chain (1) belong to a single ergodic set.

Denote by $\zeta_{j,i}$ the total sojourn time of all customers in the queue O_j during the time interval $(i\Delta, (i+1)\Delta]$. The mean sojourn time of all customers in the system per cycle under condition $\kappa_{\tau_i} = x$ equals

$$z_i(x) = \sum_{j=1}^{m} \sum_{t=0}^{T^{(r)}-1} \mathrm{E}(\zeta_{j,\tau_i} | \{\kappa_{\tau_s} = x\}), \qquad r = u(x).$$

To compute $z_i(x)$ put

$$h_{k,l}^{(j)}(\Delta) = \begin{cases} \lambda_j \beta_j \Delta / 4(1 - \lambda_j + \lambda_j \beta_j), & k = l = 0, \\ k\Delta, & k = l = 1, 2, \ldots, N_j, \\ k\Delta + \Delta/2, & k + 1 = l = 1, 2, \ldots, N_j, \\ k\Delta - \Delta/2, & k - 1 = l = 0, 1, \ldots, N_j - 1, \\ 0, & k \neq l, k \neq l \pm 1, \end{cases}$$

$$g_{k,l}^{(j)}(\Delta) = \begin{cases} k\Delta, & k = l = 1, 2, \ldots, N_j, \\ k\Delta + \Delta/2, & k + 1 = l = 1, 2, \ldots, N_j, \\ 0, & k \neq l, k \neq l - 1, k = l = 0. \end{cases}$$

Let $\mathcal{T}^{(r)} = \{1, 2, \ldots, T^{(r)}\}$, $\mathcal{T}^{(j,r)} = \{T_{2j,r} + T_{2j+1,r} + \ldots + T_{2m,r} + 1, T_{2j,r} + T_{2j+1,r} + \ldots + T_{2m,r} + 2, \ldots, T_{2j-1,r} + T_{2j,r} + \ldots + T_{2m,r}\}$. Define recursively

$$H_l^{(j,r)}(0) = 0,$$

$$H_l^{(j,r)}(i) = \sum_{k=0}^{N_j} q_{l,k}^{(j)} (g_{l,k}^{(j)}(\Delta) + H_k^{(j,r)}(i-1))$$

for $i \in \mathcal{T}^{(r)} \setminus \mathcal{T}^{(j,r)}$,

$$H_l^{(j,r)}(i) = \sum_{k=0}^{N_j} p_{l,k}^{(j)} (h_{l,k}^{(j)}(\Delta) + H_k^{(j,r)}(i-1))$$

for $i \in \mathcal{T}^{(j,r)}$. One can see that

$$z_i(x) = \sum_{j=1}^{m} H_{x_j}^{(j,r)}(T^{(r)}), \qquad u(x) = r. \tag{3}$$

Relations (2), (3) define a Markov chain with incomes [2]. To determine the best rule $u(\cdot)$ it is possible to use the Howard's algorithm.

To carry out experiments a program in high-level programming language Octave [3] was written implementing the Howard's algorithm. Maps of optimal switching rules for

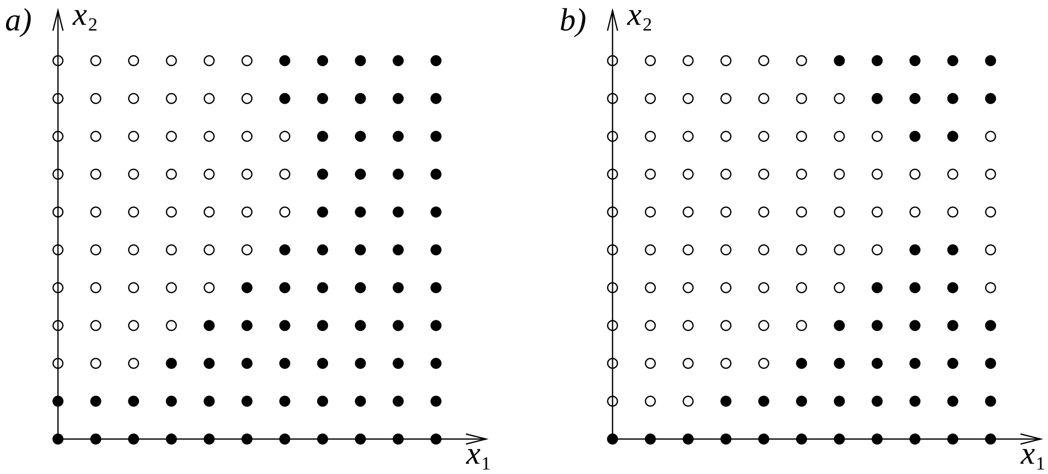

Figure 1: Optimal switching maps (parameters in the text)

two flows ($m = 2$), capacities $N_1 = N_2 = 10$ and two regimes ($n = 2$) are shown in fig. 1. A white circle denotes the first regime while a black circle denotes the second regime. Other parameters were as follows: $T_{1,1} = T_{3,1} = 8$, $T_{2,1} = T_{4,1} = T_{2,2} = T_{4,2} = 4$, $T_{1,2} = = 10$, $T_{3,2} = 6$, $\lambda_1 = 0.2$, $\lambda_2 = 0.15$. Fig. 1, a) corresponds to $\beta_1 = 0.6$, $\beta_2 = 0.65$, and fig. 1, b) assumes $\beta_1 = 0.6$, $\beta_2 = 0.55$. For each regime the cycle duration equals 24. The first flow has greater intensity then the second one. In the first regime the first flow is critical: the expected number of arrivals during a cycle $\lambda_1 T^{(1)} = 2.4$ equals the mean virtual number of served customers during one cycle (if the queue were infinite) $\beta_1 T_{1,1}$. In the second regime the first flow is not overloaded, $\lambda_1 T^{(1)} < \beta_1 T_{1,1}$. In fig. 1, a) the second flow is also not overloaded in both regimes. So the optimal switching rule $u(\cdot)$ recommends to turn on the regime which gives more time to process a longer queue. For fig. 1, b) the second queue in the second regime is overloaded, $\lambda_2 T^{(2)} < \beta_2 T_{2,3}$. Thus the regime that dedicates more time to the second queue is chosen more often.

References

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