ON FINDING THE FUNDAMENTAL MATRIX OF FINITE STATE HOMOGENEOUS MARKOV CHAINS IN SPECIAL CASE

A.N. Gaiduk
Belarusian State University
Minsk, Belarus
e-mail: GaidukAN@bsu.by

Abstract

For a finite state homogeneous Markov chain with circulant transition matrix that describes shift register that clocks 1,2 times with probabilities p, q we have found fundamental matrix. From fundamental matrix we derive hitting time matrix.

1 Introduction

Let $(X_i)_i \in N_0$ be a Markov Chain with finite state space $\{1, \ldots, n\}$ and circulant transition matrix $P = circ(p_1, p_2, \ldots, p_n)$. In this article we will consentrate on a special case $p_1 = 0, p_2 = p, p_3 = q, p_4 = 0 = \ldots = p_n = 0, p, q \geq 0, \qquad p + q = 1$.

Then transition matrix is given by

$$P = circ\{0, p, q, 0, \ldots, 0\}. \tag{1}$$

This model describes shift register that clocks 1,2 times with probabilities p, q respectively.

We denote by I identity matrix, following [1] we introduce the fundamental matrix Z of P by

$$Z = (I - (P - A))^{-1}, \tag{2}$$

where $A = circ\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$ — is the matrix of a stationary distribution of the Markov chain.

We denote by $\mathbf{1}$ the columnvector with all entries equal to 1 and note that then $\mathbf{1}\mathbf{1}^t$ is a matrix with all entries equal to 1.

From [1] we learn that the expected hitting time matrix M has the representation

$$M = (I - Z + \mathbf{1}\mathbf{1}^t Z_{dg})D, \tag{3}$$

where $D = circ\{n, 0, 0, \ldots, 0\}$.

We denote the entries of fundamental matrix by

$$Z = circ\{z_1, z_2, z_3, \ldots, z_n\}. \tag{4}$$

Thus we successively calculate

$$\begin{gathered} M = circ\{m_1, m_2, \ldots, m_n\} = \\ = circ\{n, n(z_1 - z_2), n(z_1 - z_3), \ldots, n(z_1 - z_n)\}. \end{gathered} \tag{5}$$

2 Fundamental matrix and expected hitting times

Theorem 1. *Let $P = circ\{0, p, q, 0, \ldots, 0\}$ be a transition matrix of homogeneous Markov chain. Then the fundamental matrix is a circulant matrix and for the entries of Z we have*

$$\begin{array}{lcl} z_n & = & \frac{1}{n}(1-(-\frac{1}{n}(\frac{(n-1)(n-2)}{2}) + \frac{1}{n}H(n) + \Delta\cdot(-h(n)+n-1))), \\ z_1 & = & z_n + \Delta, \\ z_j & = & -\frac{j-1}{n} + \frac{1}{n}h(j) + \Delta(\beta(j)+1) + z_n \qquad j = \overline{2, n-1}, \end{array} \tag{6}$$

where

$$\begin{array}{lcl} \Delta = z_1 - z_n & = & -\frac{n(1+q)+(-q)^n-1}{n(1+q)((-q)^n-1)}, \\ h(j) & = & \frac{-(-q)^j-q^2-2q+j(q+q^2)}{(1+q)^2}, \qquad j=\overline{2,n}, \\ \beta(j) & = & \frac{-q(1-(-1)^{j-1}q^{j-1})}{1+q}, \qquad j=\overline{2,n}, \\ H(n)=\sum_{k=3}^{n-1}h(k) & = & \frac{(6q+2q^3+6q^2-5nq-8nq^2-3nq^3+2n^2q^2+n^2q^3+n^2q+2(-1)^nq^n)}{2(1+q)^3}. \end{array}$$

Proof. Matrix Z can be derived from equation $ZZ^{-1} = I$:

$$\left\{\begin{array}{rcl} \frac{1}{n}(z_1+z_2+\ldots z_n)+z_1-qz_{n-1}-pz_n & = & 1 \\ \frac{1}{n}(z_1+z_2+\ldots z_n)+z_2-pz_1-qz_n & = & 0 \\ \ldots & & \\ \frac{1}{n}(z_1+z_2+\ldots z_n)+z_n-pz_{n-1}-qz_{n-2} & = & 0 \end{array}\right. . \tag{7}$$

Since $p = 1-q$ from (7) we derive:

$$\left\{\begin{array}{rcl} z_1+z_2+\ldots z_n & = & 1 \\ \frac{1}{n}+(z_2-z_1)+q(z_1-z_n) & = & 0 \\ \frac{1}{n}+(z_3-z_2)+q(z_2-z_1) & = & 0 \\ \frac{1}{n}+(z_4-z_3)+q(z_3-z_2) & = & 0 \\ \frac{1}{n}+(z_5-z_4)+q(z_4-z_3) & = & 0 \\ \ldots & & \\ \frac{1}{n}+(z_n-z_{n-1})+q(z_{n-1}-z_{n-2}) & = & 0 \end{array}\right. . \tag{8}$$

We can express $z_2 - z_1$ from the second equation through $z_1 - z_n$ and substitute to the third eqation. Similiar we express $z_3 - z_2$, through $z_1 - z_n$. And we derive at:

$$\left\{\begin{array}{rcl} z_1+z_2+\ldots z_n & = & 1 \\ (z_2-z_1) & = & -\frac{1}{n}-q(z_1-z_n) \\ (z_3-z_2) & = & -\frac{1}{n}+\frac{1}{n}(q)+q^2(z_1-z_n) \\ (z_4-z_3) & = & -\frac{1}{n}+\frac{1}{n}(q-q^2)-q^3(z_1-z_n) \\ (z_5-z_4) & = & -\frac{1}{n}+\frac{1}{n}(q-q^2+q^3)+q^4(z_1-z_n) \\ \ldots & & \\ (z_n-z_{n-1}) & = & -\frac{1}{n}+\frac{1}{n}(q-q^2+q^3-q^4+\ldots+(-1)^{n-1}q^{n-2})+(-1)^{n-1}q^{n-1}(z_1-z_n) \end{array}\right. . \tag{9}$$

Here we introduce some more notations:

$$\Delta = z_1 - z_n,$$

$$\beta(j) = (-q + q^2 + \ldots + (-1)^{j-1}q^{j-1}) = \frac{-q(1-(-1)^{j-1}q^{j-1})}{1+q}, \qquad j = \overline{2,n} \tag{10}$$

$$h(n) = ((n-2)q + (n-3)\cdot(-1)^3q^2 + (n-4)\cdot(-1)^4q^3 + \ldots + 2\cdot(-1)^{n-2}q^{n-3} + (-1)^{n-1}q^{n-2}). \tag{11}$$

Using definition of $h(n)$ we calculate:

$h(n) = ((n-2)q + (n-3)\cdot(-1)^3q^2 + (n-4)\cdot(-1)^4q^3 + \ldots + (n-(n-2))\cdot(-1)^{n-2}q^{n-3} + (n-(n-1))(-1)^{n-1}q^{n-2}) = n(q - q^2 + \ldots + (-1)^{n-1}q^{n-2}) + (-2q + 3q^2 - 4q^3 + \ldots + (n-1)(-1)^n q^{n-2})$.

We simplify the expression in the right side:

$n(q - q^2 + \ldots + (-1)^{n-1}q^{n-2}) = nq(1 - q + \ldots + (-1)^{n-3}q^{n-3}) = nq\frac{1-(-1)^{n-2}q^{n-2}}{1+q} = -n\beta(n-1)$.

Since

$(-2q + 3q^2 - 4q^3 + \ldots + (n-1)(-1)^n q^{n-2}) = (-q^2 + q^3 - q^4 + \ldots + (-1)^n q^{n-1}))' = (-q^2(1-q+\ldots+(-1)^{n-3}q^{n-3}))' = (-q^2\frac{1-(-1)^{n-2}q^{n-2}}{1+q})' = -2q\frac{1-(-1)^{n-2}q^{n-2}}{1+q} - q^2(\frac{-(n-2)(-1)^{n-2}q^{n-3}}{1+q} - \frac{1-(-1)^{n-2}q^{n-2}}{(1+q)^2}) = \beta(n-1)(\frac{q}{1+q} + 2) - -q^2(\frac{-(n-2)(-1)^{n-2}q^{n-3}}{1+q} - \frac{1-(-1)^{n-2}q^{n-2}}{(1+q)^2})$.

And we successively calculate $h(n)$:

$$h(n) = \frac{-(-q)^n + q^2(n-1) + q(n-2)}{(1+q)^2}. \tag{12}$$

Summing all equations in (9), except the first one, we obtain:

$$z_n - z_1 = -\frac{n-1}{n} + \frac{1}{n}h(n) + \beta(n)(z_1 - z_n).$$

And

$$\Delta = (z_1 - z_n) = \frac{\frac{1}{n}(n-1-h(n))}{\beta(n)+1} = -\frac{n(1+q) + (-q)^n - 1}{n(1+q)((-q)^n - 1)}. \tag{13}$$

The system (9) has the representation:

$$\begin{cases} & z_1 + z_2 + \ldots z_n = 1 \\ z_1 = & \Delta + z_n \\ z_2 = & -\frac{1}{n} - q\Delta + \Delta + z_n \\ z_3 = & -\frac{2}{n} + \frac{1}{n}(q) + q^2\Delta - q\Delta + \Delta + z_n \\ z_4 = & -\frac{3}{n} + \frac{1}{n}(2q - q^2) - q^3\Delta + q^2\Delta - q\Delta + \Delta + z_n \\ \ldots & \\ z_{n-1} = & -\frac{n-2}{n} + \frac{1}{n}((n-3)q - (n-4)q^2 + (n-5)q^3 - (n-6)q^4 + \ldots + (-1)^{n-2}q^{n-3}) + \\ + & (-1)^{n-2}q^{n-2}\Delta + (-1)^{n-3}q^{n-3}\Delta + \ldots + \Delta + z_n \end{cases}. \tag{14}$$

We denote by $H(n)$ the sum of $\sum_{k=3}^{n-1} h(k)$:

$$H(n) = \sum_{k=3}^{n-1} h(k) = \frac{q(6-5n+n^2) + q^2(6-8n+2n^2) + q^3(2-3n+n^2) + 2(-q)^n}{2(1+q)^3}. \tag{15}$$

By summing all equations (except first one) (14), we can find:

$$z_1 + \ldots + z_{n-1} = -\frac{1}{n}(\frac{(n-1)(n-2)}{2}) + \frac{1}{n}H(n) + \Delta \cdot (-h(n) + n - 1) + (n-1)z_n.$$

By substitution to (14) we arive at:

$$z_n = \frac{1}{n}(1 - (-\frac{1}{n}(\frac{(n-1)(n-2)}{2}) + \frac{1}{n}H(n) + \Delta \cdot (-h(n) + n - 1))), \tag{16}$$

where $\beta(n), h(n)$,Δ are defined in (10), (12), and (13) correspondingly.
From (14) we arive at:

$$\begin{array}{rcl} z_1 & = & z_n + \Delta, \\ z_j & = & -\frac{j-1}{n} + \frac{1}{n}h(j) + \Delta(\beta(j) + 1) + z_n \qquad j = \overline{2, n-1}, \\ z_n & = & \frac{1}{n}(1 - (-\frac{1}{n}(\frac{(n-1)(n-2)}{2}) + \frac{1}{n}H(n) + \Delta \cdot (-h(n) + n - 1))). \end{array} \tag{17}$$

□

Corollary 1. *Let $P = circ\{0, p, q, 0, \ldots, 0\}$ be a transition matrix of homogeneous Markov chain. The matrix of expected hitting time is a circulant matrix and for the entries of M we have*

$$\begin{array}{rcl} m_1 & = & n, \\ m_j & = & j - 1 - h(j) - n\Delta\beta(j), \qquad j = \overline{2, n-1}, \\ m_n & = & n\Delta. \end{array} \tag{18}$$

References

[1] John G. Kemeny, J. Laurie Snell, *Finite Markov Chains*, D. Van Nostrand, 1960.