ANALYSIS OF HM-NETWORKS WITH ORBITRARY TOPOLOGY

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Abstract

In article finding of expected incomes of open queueing network with help of z-transformation method is observed. Partial differential equations are composed and solved with help of earlier developed algorithm for function of z-transformation. Expected incomes have form of multiple functional series and are coefficients of expansion of z-transformation function in series by power of $z_1, z_2, ..., z_n$.

1 Introduction

Let's consider open (Howard-Matalytski)-network with one-type messages that consists of n queueing systems S_1, S_2, ..., S_n and environment S_0. State of such network could be described by vector $k(t) = (k_1, k_2, ..., k_n, t)$, where k_i – number of messages in system S_i in the moment t, $i = \overline{1,n}$. The incoming flow arrives in the network with rate λ. Let's denote the service rate in system S_i as μ_i; p_{ij} – probability that the message after service in system S_i comes to S_j, $\sum\limits_{j=1}^{n} p_{ij} = 1$, $i = \overline{0,\, n}$.

Let's $v_i(k,t)$ – complete expected income which system S_i receives during time t, if it was in state k at the initial time moment; I_i – n-vector with zero components excepting of component with number i, which equals 1; $r_i(k)$ – income of the system S_i in unit time during time when network is in state k; $-R_{i0}(k - I_i)$ – income of the system S_i, when network changes its state k for $(k - I_i)$; $r_{0i}(k + I_i)$ – income of the system S_i, when network changes state k for $(k + I_i)$; $r_{ij}(k + I_i - I_j)$ – income of the state S_i, when network changes state k for $(k + I_i - I_j)$, i, $j = \overline{1,n}$.

2 Main result

Let's introduce z-transformation for expected income of the system S_i:

$$\varphi_i(z,t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_n=0}^{\infty} v_i(k_1, k_2, \dots, k_n, t) z_1^{k_1} z_2^{k_2} \cdot \dots \cdot z_n^{k_n} = \sum_{\substack{k_i = 0, \\ i = \overline{1,n}}}^{\infty} v_i(k,t) \prod_{l=1}^{n} z_l^{k_l},$$

$|z| < 1$. Then the next statement is true

Theorem 1. Function $\varphi_i(z,t)$ satisfies the relation

$$\frac{\partial \varphi_i(z,t)}{\partial t} = -\lambda \varphi_i(z,t) - \sum_{j=1}^{n} \mu_j z_j \frac{\partial \varphi_i(z,t)}{z_j} +$$

$$\sum_{j=1}^{n} \left\{ \frac{\lambda p_{0j}}{z_j} \left[\varphi_i(z,t) - \varphi_i^{\{j\}}(z,t) \right] + \mu_j p_{j0} z_j \left[z_j \frac{\partial \varphi_i(z,t)}{\partial z_j} + \varphi_i(z,t) \right] \right\} +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left\{ \frac{\mu_j p_{ji} z_j}{z_i} \left[z_j \frac{\partial \left(\varphi_i(z,t) - \varphi_i^{\{i\}}(z,t) \right)}{\partial z_j} + \varphi_i(z,t) - \varphi_i^{\{i\}}(z,t) \right] + \right.$$

$$\left. + \frac{\mu_i p_{ij} z_i}{z_j} \left[z_i \frac{\partial \left(\varphi_i(z,t) - \varphi_i^{\{j\}}(z,t) \right)}{\partial z_i} + \varphi_i(z,t) - \varphi_i^{\{j\}}(z,t) \right] \right\} +$$

$$+ \sum_{\substack{c,s=1, \\ c,s \neq i}}^{n} \frac{\mu_s p_{sc} z_s}{z_c} \left[z_s \frac{\partial \left(\varphi_s(z,t) - \varphi_s^{\{c\}}(z,t) \right)}{\partial z_s} + \varphi_s(z,t) - \varphi_s^{\{c\}}(z,t) \right]$$

$$+ \sum_{\substack{k_l = 0 \\ l = \overline{1,n}}}^{\infty} \alpha_i(k) \prod_{l=1}^{n} z_l^{k_l}. \tag{1}$$

where

$$\varphi_i^{\{j\}}(z,t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j-1}=0}^{\infty} \sum_{k_{j+1}=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} v_i(k_1, \ldots, k_{j-1}, 0, k_{j+1}, \ldots k_n, t) \times \tag{2}$$

$$\times z_1^{k_1} \cdot \ldots \cdot z_{j-1}^{k_{j-1}} z_{j+1}^{k_{j+1}} \cdot \ldots \cdot z_n^{k_n} = \sum_{\substack{k_l = 0, \\ l = \overline{1,n},\, l \neq j}}^{\infty} v_i(k,t)\big|_{k_j=0} \prod_{\substack{l=1, \\ l \neq j}}^{n} z_l^{k_l}, \quad j = \overline{1,n},$$

$$\alpha_i(k) = \sum_{j=1}^{n} \left[\mu_j(k_j) u(k_j) p_{ji} r_{ij}(k + I_i - I_j) - \mu_i(k_i) u(k_i) p_{ij} r_{ij}(k - I_i + I_j) \right] +$$

$$+ \lambda\, p_{0i}\, r_{0i}(k + I_i) - \mu_i(k_i) u(k_i) p_{i0} R_{i0}(k - I_i) + r_i(k), \;\; i = \overline{1,\, n.} \tag{3}$$

Let $i_1 \neq i_2 \neq \ldots \neq i_j$. Let's introduce notation series connected with z-transformation of income of the system S_i:

$$\varphi_i^{\Omega}(z,t) = \varphi_i(z,\, t)\big|_{k_j=0,\, z_j=1,\, j=\overline{1,n}} = v_i(0, 0, \ldots, 0, t);$$

$$\varphi_i^{\Omega \setminus \{i_1\}}(z,t) = \varphi_i(z,\, t)\big|_{k_j=0,\, z_j=1,\, j=\overline{1,n},\, j \neq i_1} = \sum_{k_{i_1}=0}^{\infty} v_j(0, \ldots, 0, k_{i_1}, 0, \ldots, 0, t) z_{i_1}^{k_{i_1}}, \quad i_1 = \overline{1,n};$$

$$\varphi_i^{\Omega \setminus \{i_1, i_2\}}(z,t) = \varphi_i(z, t\,)\big|_{k_j=0,\, z_j=1,\, j=\overline{1,n},\, j \neq i_1, i_2} =$$

$$= \sum_{k_{i_1}=0}^{\infty} \sum_{k_{i_2}=0}^{\infty} v_i(0, \ldots, 0, k_{i_1}, 0, \ldots, 0, k_{i_2}, 0, \ldots, 0, t) z_{i_1}^{k_{i_1}} z_{i_2}^{k i_2}, \quad i_1, i_2 = \overline{1,n}, \quad i_1 \neq i_2;$$

. . .

$$\varphi_i^{\Omega\backslash\{i_1,i_2,\dots,i_l\}}(z,t) = \varphi_i(z\ ,t)|_{k_j=0,\, z_j=1,\, j=\overline{1,n},\, j\neq i_1,i_2,\dots,i_l} =$$

$$= \sum_{k_{i_1}=0}^{\infty}\sum_{k_{i_2}=0}^{\infty}\dots\sum_{k_{i_l}=0}^{\infty} v_i(0,\dots,0,k_{i_1},0,\dots,0,k_{i_2},0,\dots,0,k_{i_l},0,\dots,0,t) z_{i_1}^{k_{i_1}} z_{i_2}^{k_{i_2}}\cdot\ldots\cdot z_{i_j}^{k_{i_l}},$$

$$i_1,i_2,\dots,i_l=\overline{1,n},\quad i_1\neq i_2\neq\ldots\neq i_l;$$

. . .

$$\varphi_i^{\Omega\backslash\{i_1,i_2,\dots,i_{n-1}\}}(z,t) = \varphi_i(z,t\ )|_{k_j=0,\, z_j=1,\, j=\overline{1,n},\, j\neq i_1,i_2,\dots,i_{n-1}} = \varphi_i^{\{j\}}(z,t),\quad i=\overline{1,n},$$

$$i_1\neq i_2\neq\ldots\neq i_{n-1}.$$

And zeros between k_{i_j}in arguments of function v_i can be absent, it just is important that $i_1\neq i_2\neq\ldots\neq i_j$.

3 Algorithm of income finding

0-th step: it is necessary to define $\varphi_i^{\Omega_i}(z,t)=v_i(0,0,\dots,0,t)$;

1-st step: exchanging $\varphi_i(z,t)$ in (1) for $\varphi_i^{\Omega_i\backslash\{i_1\}}(z,t)$ we obtain z-transformation of system S_i income at the moment t, if it was in state $k=(0,0,...,k_{i_1},\,0,...,0)$ at the initial time; subject to $k_l=0,\ \ l\neq i_1$ relation (1) will rewrite as:

$$\frac{\partial\varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial t} = -\lambda\varphi_i^{\Omega\backslash\{i_1\}}(z,t) - \mu_{i_1}z_{i_1}\frac{\partial\varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial z_{i_1}} + \frac{\lambda p_{0i_1}}{z_{i_1}}\left(\varphi_i^{\Omega\backslash\{i_1\}}(z,t) - v_i(0,0,....,0,\,t)\right) +$$

$$+\mu_{i_1}p_{i_10}\left(z_{i_1}^2\frac{\partial\varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial z_{i_1}} + z_{i_1}\varphi_i^{\Omega\backslash\{i_1\}}(z,t)\right) + \sum_{\substack{k_j=0,\\ j=\overline{1,n}}}^{\infty}\alpha(k)\prod_{l=1}^{n}z_l^{k_l} \tag{4}$$

Let consider expression (3). Let's suppose that incomes $r_i(k)$, $r_{ij}(k)$, $r_{0i}(k)$, $R_{i0}(k)$ don't depend from network state k, i.e. $r_i(k)=r_i$, $r_{ij}(k)=r_{ij}$, $r_{0i}(k)=r_{0i}$, $R_{i0}(k)=R_{i0}$. Since

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\prod_{l=1}^{n}z_l^{k_l} = \prod_{l=1}^{n}\frac{1}{1-z_l}\quad\text{and}\quad \sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}k_j\prod_{l=1}^{n}z_l^{k_l} = z_j\frac{\partial\left(\prod_{l=1}^{n}\frac{1}{1-z_l}\right)}{\partial z_j} = \frac{z_j}{1-z_j}\prod_{l=1}^{n}\frac{1}{1-z_l},$$

then from (3) it follows that in this case

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\alpha_i(k)\prod_{l=1}^{n}z_l^{k_l} = \left[\sum_{j=1}^{n}\left(\mu_j p_{ji}r_{ij}\frac{z_j}{1-z_j} - \mu_i p_{ij}r_{ji}\frac{z_i}{1-z_i}\right) + \lambda p_{0i}r_{0i} - \mu_i p_{i0}R_{i0}\frac{z_i}{1-z_i} + \right.$$

$$\left. + \left(1-\lambda\, p_{0i} - \mu_i\frac{z_i}{1-z_i} - \mu_j\frac{z_j}{1-z_j}\right)r_i\right]\prod_{l=1}^{n}\frac{1}{1-z_l} = R(z)\prod_{l=1}^{n}\frac{1}{1-z_l}$$

As $k_j=0$ when $j\neq i_1$, then

$$\sum_{\substack{k_j=0,\\ j=\overline{1,n}}}^{\infty} \prod_{l=1}^{n} z_l^{k_l} = \sum_{k_{i_1}=0}^{\infty} z_{i_1}^{k_{i_1}} = \frac{1}{1-z_{i_1}} \qquad \sum_{k_{i_1}=0}^{\infty} k_{i_1} z_{i_1}^{k_{i_1}} = z_{i_1} \frac{\partial\left(\frac{1}{1-z_{i_1}}\right)}{\partial z_{i_1}} = \frac{z_{i_1}}{(1-z_{i_1})^2},$$

and in this case

$$R(z) = \left(\mu_{i_1}(p_{i_1 i} r_{i i_1} - r_i(k)) \frac{z_{i_1}}{1-z_{i_1}} + \lambda p_{0i}(r_{0i} - r_i) + r_i\right) \frac{1}{1-z_{i_1}}, \; i_1 \neq i \tag{5}$$

$$R(z) = -\mu_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij} r_{ij} + p_{i0} R_{i0} + r_i \right) \frac{z_i}{(1-z_i)^2} + \frac{\lambda p_{0i}\,(r_{0i} - r_i) + r_i}{1-z_i}, \; i_1 = i. \tag{6}$$

In consideration of (5), (6) relation (4) for z-transformation $\varphi_n^{\Omega\backslash\{i_1\}}(z,t)$ is:

$$\frac{\partial \varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial t} = -\lambda \varphi_i^{\Omega\backslash\{i_1\}}(z,t) - \mu_{i_1} z_{i_1} \frac{\partial \varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial z_{i_1}} + \frac{\lambda p_{0i_1}}{z_{i_1}} \left(\varphi_i^{\Omega\backslash\{i_1\}}(z,t) - \right.$$

$$\left. - v_i(0,0,...,0,\,t)\right) + + \mu_{i_1} p_{i_1 0} \left(z_{i_1}^2 \frac{\partial \varphi_i^{\Omega\backslash\{i_1\}}(z,t)}{\partial z_{i_1}} + z_{i_1} \varphi_i^{\Omega\backslash\{i_1\}}(z,t) \right) +$$

$$\left(-\mu_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij} r_{ij} + p_{i0} R_{i0} + r_i \right) \frac{z_i}{1-z_i} + \lambda_i\,(r_{0i} - r_i) + r_i \right) \frac{1}{1-z_i},$$

making expansion of function $\varphi_i^{\Omega\backslash\{i_1\}}(z,t)$ in series by power $z_{i_1}^{k_{i_1}}$, it possible to find the coefficients in these expansions $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,t)$, $i_1 = \overline{1,n}$;

2-nd step: z-transformation $\varphi_i^{\Omega\backslash\{i_1,i_2\}}(z,t)$ finds. We recount relation (1) under condition $k_l = 0$, $l \neq i_1, i_2$; making expansion of found functions in series by powers $z_{i_1}^{k_{i_1}} z_{i_2}^{k_{i_2}}$, we obtain coefficients in these expansions $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,t)$;

thus at the j-th step we find z-transformation $\varphi_i^{\Omega\backslash\{i_1,i_2,\ldots,i_j\}}(z,t)$, and $i_1, i_2, \ldots, i_j = \overline{1,n}$, $i_1 \neq i_2 \neq \ldots \neq i$; coefficients of expansion of $\varphi_i^{\Omega\backslash\{i_1,i_2,\ldots,i_j\}}(z,t)$ in series by powers $z_{i_1}^{k_{i_1}} z_{i_2}^{k_{i_2}} \cdot \ldots \cdot z_{i_j}^{k_{i_j}}$ will give us incomes $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,k_{i_j},0,\ldots,0,t)$;

carry on making such steps, at the $(n-1)$-th step we will find $\varphi_i^{\{j\}}(z,t)$, $j = \overline{1,n}$, and at the n-th step – z-transformation $\varphi_i(z,t)$, that satisfy relation (1); .making expansion of it in series by powers $z_1^{k_1} z_2^{k_2} \cdot \ldots \cdot z_n^{k_n}$, it is possible to find the income.

By the instrumentality of this algorithm it is possible to calculate expected incomes of systems of the arbitrary configuration network.