APPROXIMATE FORMULAE FOR EXPECTATION OF FUNCTIONALS CONTAINING STOCHASTIC INTEGRALS

A.D. Egorov
National Academy of Sciences, Institute of Mathematics
Minsk, BELARUS
e-mail: egorov@im.bas-net.by

Abstract

Some simple approximate formulae for mathematical expectations of random nonlinear functionals are considered. The functionals are defined on trajectories of process given by stochastic integral and solutions of linear Skorohod's equations.

1 Introduction

Evaluation of mathematical expectations of functionals defined on trajectories of random processes is a subject of interest of many researches [1,2]. In this report we consider some approach to constructing of approximate formulae for mathematical expectation of processes given by Skorohod stochastic integral, solution of stochastic differential equation Skorohod type and by one class of multiple stochastic integral. Our approach is motivated by similar approximations of functional integrals [2]. Necessary information on Skorohod integral and equations can be found in [3].

Let $W = \{W_s,\ s \in [0,1]\}$ be a standard Wiener process on the canonical Wiener space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_s,\ s \in [0,1]\}$ be the filtration generated by W. The Skorohod integral $\delta(u) \equiv \int_0^1 u_s \delta W_s$ is the adjoint operator from $L_2([0,1]\times\Omega, dt\times P,)$ to $L_2(\Omega, P)$ defined by the duality relationship

$$E[F\delta(u)] = E\Big[\int_0^1 u_s D_s F dt\Big].$$

In this equation D_s denotes Malliavin derivative defined on the set $\mathcal{S}$ of smooth Wiener functionals of the form $F = f(W_{t_1}, \ldots, W_{t_n})$ as

$$D_t f(W_{t_1}, \ldots, W_{t_n}) = \sum_{i=1}^n \frac{\partial}{\partial x_i} F(W_{t_1}, \ldots, W_{t_n}) 1_{[0,t_i]}(t)$$

and then extended to the closure of $\mathcal{S}$ with respect to the norm $\|F\|_{1,2} = (E[|F|^2] + E[\|DF\|_{L_2[0,1]}]^2)^{1/2}$. The operator δ is defined on subset of functions from $L_2([0,1] \times \Omega, dt \times P,)$ satisfying to the condition $E\Big[\int_0^1 u_s D_s F ds\Big] \le \mathrm{const}\|F\|_{L_2(\Omega)}$.

We will use the next representation for $u_t \in L^2([0,1]\times\Omega)$

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)),$$

where $f_n(t_1,\dots,t_n,t)\in L_2([0,1]^n)$ is symmetric in all variables. In this case

$$\delta(u_t)=\sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n(\cdot,t)),$$

where $\widetilde{f}_n(\cdot,t))$ is symmetrization of $f(\cdot,t)$. If $u_t,v_t\in Dom\delta$ then

$$E[\delta(u)\delta(v)]=\int\limits_T E[u_tv_t]dt+\int\limits_T\int\limits_T E[D_s(u_t)D_t(v_s)]dtds$$

under condition that the integrals in right parts are exist.

2 Approximate formulae

At first we give approximate formula for evaluation of mathematical expectation of functionals depending on trajectories of random process defined by the Skorohod integrals

$$X_t=\delta(u1_{[0,t]})\equiv\int_0^t u_s\delta W_s,$$

under condition that this integral exist for every $t\in[0,t]$. In what follows we mean that nonlinear functional $F(X_{(\cdot)})$ depends on trajectories of $X_{(\cdot)}$ but it not depends separately on $\omega\in\Omega$.

The formula is exact for polynomial functionals of X_t of second degree and have the form:

$$F[F(X_{(\cdot)})]\approx F(0)(1-A)+\frac{1}{2}\int\limits_{-1}^{1}\widehat{p}_1(v)F(\rho(\cdot,v))dv+$$

$$+\frac{1}{4}\int\limits_{-1}^{1}\int\limits_{-1}^{1}\widehat{p}_2(v_1,v_2)F(c_1\rho(\cdot,v_1)+c_2\rho(\cdot,v_2))dv_1dv_2,$$

where $\widehat{p}_1(v)$ and $\widehat{p}_2(v_1,v_2)$ are symmetrical continuations on $[-1,1]$ and $[-1,1]\times[-1,1]$ of the functions $p_1(v)=E[u_v^2]$ and $p_2(v_1,v_2)=E[D_{v_1}(u_{v_2})D_{v_2}(u_{v_1})]$, $\rho(t,v)=1_{[0,t]}(|v|)sign(v)$; constants A,c_1,c_2 can be evaluated.

As an example, let $u_s=\exp\{\int_0^1 a(s,\tau)dW_\tau\}$ then $p_1(s)=\exp\{2\int_0^1(a(s,\tau))^2d\tau\}$, $p_2(s_1,s_2)=a(s_2,s_1)a(s_1,s_2)\exp\{\frac{1}{2}\int_0^1(a(s_1,\tau)+a(s_2,\tau))^2d\tau\}$.

Next we consider the stochastic differential equation with the Skorohod integral

$$X_t=X_0+\int\limits_0^t X_s\delta W_s,\quad X_0=f(W_1).$$

The solution of this equation is not adapted to filtration generated by W because of initial condition and it is given by formulae [3]: $X_t=\exp(W_t-\frac{1}{2}t)f(W_1-t)$. We consider the case $f(x)=e^x$, then the solution is

$$X_t=\exp\Big\{W_t+W_1-\frac{1}{2}(3t+1)\Big\}e^{\frac{1}{2}}=e^{\frac{1}{2}}:\exp\{W_t+W_1\}:,$$

where Wick's ordering is denoted by dots. In oder to construct the formulae for evaluating the mathematical expectation of functionals exact for polynomial $n-$th degree we will use the formulae [2]

$$E\Big[\prod_{j=1}^{k} : \exp\{\int_0^1 \xi_j(s)dW_s\} : \Big] = \exp\Big\{\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} K(\xi_i,\xi_j)\Big\},$$

where $K(\xi_i,\xi_j)\Big\} = \int_0^1\int_0^1 \min(\tau_1,\tau_2)d\xi_i(\tau_1)d\xi_j(\tau_2)$. We put $\xi_j(s) = 1_{t_j}(s) + 1_1(s)$, where $1_t(s) = 0$ for $s < t$ and $1_t(s) = 1$ for $s \geq t$, so we get $\int_0^1 \xi_j(s)dW_s = W_t - t_j + W_1$, $K(\xi_i,\xi_j) = \min(t_i,t_j) + t_i + t_j + 1$ and

$$E\Big[\prod_{j=1}^{n} X_{t_j}\Big] = e^{\frac{n}{2}} \exp\Big\{\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} [\min(t,s) + t_i + t_j + 1]\Big\}.$$

The formula exact for polynomials of arbitrary fixed degree have the form:

$$E[F(X_{(\cdot)})] \approx (1 - A^{n(n-1)/2})F(0) +$$

$$+ \int\limits_{R^{n(n-1)/2}} F\Big(\sum_{j=1}^{n} a_j \prod_{l=1,l\neq j}^{n} \rho(u_{jl},\cdot)\Big)\nu^{n(n-1)/2}(du),$$

where $\rho(u.t) = e^t\rho_1(u,t)$, $d\nu(u) = e^1 d\nu_1(u)$, $\int\limits_R \rho_1(u,t)\rho_1(u,s)d\nu(u) = \exp\{\min(t,s)\}$; parameters A, a_j, $j = 1,\dots,n$ are given in [2].

Now let us consider Skorohod equation

$$X_t = G_t + \int\limits_0^t \alpha(s)X_s\delta W_s,$$

where $\alpha(s)$ - determinate function, $G_t = \sum_{n=1}^{N} I_n(g_n(\cdot,t))$. The solution of this equation is $X_t = \sum_{n=1}^{\infty} I_n(f_n(\cdot,t))$, where $f_n(\cdot,t)$ can be found recursively in explicit form [5]. Using relations $E[X_t] = g_0(t)$ and

$$E[X_{t_1}X_{t_2}] = \sum_{n=1}^{\infty}\int_{[0,1]^n} f_n(\tau_1,\dots,\tau_n,t_1)f_n(\tau_1,\dots,\tau_n,t_2)d\tau_1\dots d\tau_n,$$

it is easy to prove that the next approximate formula is exact for random polynomial functional second degree:

$$E[F(X.)] \approx F(0)(1-\Lambda) + \frac{1}{2}(F(g_0(\cdot)) - F(-g_0(\cdot))) +$$

$$+ \sum_{n=1}^{\infty}\lambda_n \int\limits_0^1 \overset{(n)}{\cdots} \int\limits_0^1 \triangle F(\lambda_n^{-\frac{1}{2}} f_n(\tau_1,\dots,\tau_n,\cdot))d\tau_1\dots\tau_n,$$

where $\triangle F(f) = \frac{1}{2}(F(f) + F(-f))$, $\Lambda = \sum_{n=1}^{\infty}\lambda_n < \infty$.

Such approach can be applied to the case of functional of random process given by some multiple stochastic integral. Its application is based on representation these integrals by Wick's power and exponent[2]:

$$\int\limits_0^t \int\limits_0^{t_n} \cdots \int\limits_0^{t_2} dW_{t_1} \ldots dW_{t_n} = \frac{1}{n!} : W_t^n :, \quad X_t = \int_0^t : W_s^{n-1}(s) : dW_s = \frac{1}{n} : W_t^n :,$$

$$Y_t = \int_0^t : \exp\{W_s\} : dW_s =: \exp\{W_t\} : -1.$$

Since the moments of X_t and Y_t are polynomials with respect to $: W_t^n :$ and $: \exp\{W_t\} :$ correspondingly, we use the results due to Petrov V.A. [4] (see also [2]) for Wiener integrals in order to receive the formulae for evaluation of expectations

$$E\Big[F\Big(\int_0^{(\cdot)} : W_s^n(s) : dW_s\Big)\Big], \quad E\Big[F\Big(\int_0^{(\cdot)} : \exp\{W_s\} : dW_s\Big)\Big].$$

Composed approximate formulae will be considered using the above formulas.

This research was supported by Belarusian Republican Foundation for Fundamental Research (project F08-023).

References

[1] Kloeden P.E., Platen E.P. (1995). *Numerical Solution of Stochastic Differential Equations*. Springer.

[2] Egorov A.D., Sobolevsky P.I., Yanovich L.A. (1993). *Functional Integrals. Approximate evaluation and Applications*. Kluver Academic Publishers, Dordrecht.

[3] Nualart D. (1995). *Malliavin Calculus and Related Topics*. Springer-Verlag, New York.

[4] Petrov V.A. (1984). Nekotorye priblizhennye formuly dlya vychisleniya kontinual'nych integralov ot funktsionalov, zavisyashich ot ekcponent. *Izv. AN BSSR. Ser. fiz.-mat. nauk*. No. 1. pp. 9-12.

[5] Jin Ma, Protter P., San Martin J. (1998). Anticipating integrals for a class of martingales. *Bernoulli*. Vol. 4, No. 1. pp. 81-114.