COMPARISON OF TESTS FOR APPROACHING HYPOTHESES ON THE NONLINEAR REGRESSION PARAMETER

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1. Introduction

Let $\mathcal{R}^n$ be an n-dimensional Euclidean space and $\mathcal{B}^n$ the σ-algebra of its Borel subsets. We consider statistical experiments $\{\mathcal{R}^n, \mathcal{B}^n, \mathcal{P}_\theta^n : \theta \in \Theta\}$ generated by observations $X = (X_1, ..., X_n)$ of the form

$$X_j = g(j,\theta) + \varepsilon_j, j = 1, ..., n. \tag{1.1}$$

Here $g(j,\theta), j = 1, ..., n$, are non-random functions defined on Θ^c – the closure in $\mathcal{R}^1$ of an open set $\Theta \subset \mathcal{R}^1$ and $\{\varepsilon_j\}$ are i.i.d. random variables whose common distribution function $\mathcal{P}(x)$ is independent of θ and such that $\mathbb{E}\varepsilon_j = 0$, $\mathbb{E}\varepsilon_j^2 = \sigma^2 > 0$. We do not assume that $g(j,\theta)$ is a linear function of the parameter θ.

Define a *least squares estimator* of the unknown parameter $\theta \in \Theta^c$ from observation X of form (1.1) as a random variable $\hat\theta_n = \hat\theta_n(X_1, ..., X_n) \in \Theta^c$ such that $S(\hat\theta_n) = \inf_{\tau\in\Theta^c} \|X - g(\tau)\|^2$.

Let be $g_r = (d^r/d\theta^r)g$, $b_n^{(r)}(\theta) = n^{-1/2}\sum_{j=1}^n (X_j - g(j,\theta))g_r(j,\theta)$, $\Lambda_n = \Pi_n^{-1}(\theta)$, where $\Pi_n(\theta) = \sum_{j=1}^n g_1^2(j,\theta)$. Putting $u(\theta) = n^{1/2}(\hat\theta_n - \theta)$ and $Z_{1n} = \sigma^{-1}\sqrt{\Lambda_n}b_n^{(1)}$, $Z_{2n} = \Lambda_n b_n^{(2)}$ we consider the following functionals of $\hat\theta_n$ and θ:

$$\begin{aligned} Z_n(\hat\theta_n) &= n^{-1/2}Z_{2n}(\hat\theta_n) + Z_{1n}^2(\hat\theta_n), && (1.2)\\ T^{(0)}(\theta) &= \frac{1}{4n}\Lambda_n(\theta)\left(\frac{d}{d\theta}S(\theta)^2\right), && (1.3)\\ T^{(1)}(\hat\theta_n) &= \sigma^{-2}\{S(\theta) - S(\hat\theta_n)\}, && (1.4)\\ T^{(2)}(\hat\theta_n) &= \Pi_n(\hat\theta_n)u^2(\theta). && (1.5)\end{aligned}$$

In the paper [1] the problem of constructing of the preference regions for the tests $\psi_n^{(m)}$ with statistics (1.3)-(1.5) produced by the model (1.1) was investigated. In the paper [2] has been made an asymptotic comparison (up to $o(n^{-1})$) between Rao's test and locally most powerful unbiased (LMP) test under contiguous alternatives, $\theta_0 + \delta n^{-1/2}$, both tests having the same size α (up to $o(n^{-1})$). Now we solve a similar problem but in wider statement. This problem is following.

2. The problem statement

Fix a number $\delta \neq 0$ and consider a sequences l_n and $\psi_n^{(m)}$ of tests for testing a simple hypothesis $H_0 : \theta = \theta_0, \theta \in \Theta$, against a simple alternative $H_\delta : \theta = \theta_\delta = \theta_0 + \delta n^{-1/2}$ on the basis of observations $X = (X_1, ..., X_n)$. The question is as follows: provided that first kind errors are balanced in a certain manner, which of the tests l_n and $\psi_n^{(m)}$ generated by statistics (1.2) and (1.3)-(1.5) accordingly are more powerful asymptotically (as $n \to \infty$)? We get the answer for this question in the form of preference regions for the l_n and $\psi_n^{(m)}$ tests in the some invariant coordinate system.

3. The l_n - test and its $W_n^{\pm}$ - statistics

Let be $L(x,\theta) = -\frac{1}{2\sigma^2}S(\theta)$. We define the l_n test with the critical region

$$X_{\alpha n} = \{Z_n \geq l_{0n} + 2l_{1n}Z_{1n}\}, \tag{3.1}$$

where Z_n is defined according to (1.2). We suppose $l_n \in \Psi_\alpha$, where Ψ_α is a set of any unbiased tests with a size α of a critical region (3.1), i.e. the parameters $l_{0n}, l_{1n} \in \mathcal{R}^1$ is founded from condition

$$\alpha = \mathbb{E}_{\theta_0} l_n = \int_{X_{\alpha n}} l_n(x)L(x,\theta)dx, \quad \int_{R^n} \frac{d}{d\theta} L(x,\theta_0)l_n(x)dx = 0. \tag{3.2}$$

Put $z = u_{1-\alpha/2}$ the quantile of a Gaussian distribution and let be

$$l_{0n} = z^2 + n^{-1/2}a_1 + n^{-1}a_2 + o(n^{-1}), \quad l_{1n} = n^{-1/2}a_3 + n^{-1}a_4 + o(n^{-1}), \tag{3.3}$$

where the coefficients $a_1, \ldots, a_4$ are defined from conditions (3.2). Taking into account (3.3) we realize the critical region (3.1) in the form

$$X_{\alpha n} = \{(Z_{1n} - l_{1n})^2 \geq d_n^2(1 - n^{-1/2}d_n^{-2}Z_{2n}) + o(n^{-1})\}, \tag{3.4}$$

where $d_n = \{z^2 + n^{-1/2}a_1 + n^{-1}(a_2 + a_3^2)\}^{1/2}$. To obtain asymptotic expansion of the probability of the first kind error and the power of the modified l_n test, we need a stochastic expansion of the statistic Z_n. Thereto we pass from one-sided region $X_{\alpha n}$ on to two-sided region $X_{\alpha n} = X_{\alpha n}^+ \cup X_{\alpha n}^-$, where $\mathbb{P}_{\theta_0}\{X_n^{\pm}\} = \frac{\alpha}{2} + o(n^{-1})$. We denote $c_1 = \frac{1}{2z}$, $c_2 = \frac{1}{8z}$, $c_3 = -\frac{a_1}{4z^3}$.

Theorem 1. *Under the assumption of Grigoriev and Ivanov* (1995) *two-sided region* $X_{\alpha n} = X_{\alpha n}^+ \cup X_{\alpha n}^-$ *is defined as*

$$X_{\alpha n}^+ = \{W_n^+ > z_\alpha^+\}, \quad X_{\alpha n}^- = \{W_n^- < z_\alpha^-\}, \tag{3.5}$$

where

1) $W_n^{\pm} = Z_{1n} \pm c_1 Z_{2n} n^{-1/2} \pm (c_2 Z_{2n}^2 + c_3 Z_{2n})n^{-1}$;
2) $z_\alpha^{\pm} = \pm z + z_1^{\pm} n^{-1/2} + z_2^{\pm} n^{-1}$ and $z_1^{\pm} = a_3 \pm c_1 a_1$, $z_2^{\pm} = a_4 \mp c_2 a_1^2 \pm c_1(a_2 + a_3^2)$.

Let's notice, that the coefficients a_j remain yet not certain. We calculate them later when we receive asymptotic expansions of sizes and powers of modified tests.

4. Asymptotic expansions of sizes and powers of modified tests l_n

To compare the powers of the modified tests l_n and $\psi_n^{(m)}$ we need to balance their first kind errors. The asymptotic expansions of sizes and powers of tests $\psi_n^{(m)}$ are obtained in [1]. Therefore we consider here the l_n test only. Put

$$\mathbb{P}_\delta^+ = \mathbb{P}_{\theta_\delta}\{W_n^+ > z_\alpha^+\}, \ \mathbb{P}_\delta^- = \mathbb{P}_{\theta_\delta}\{W_n^- < z_\alpha^-\}. \tag{4.1}$$

Clearly, the power of the modified l_n test equals $\mathbb{P}_\delta = \mathbb{P}_\delta^+ + \mathbb{P}_\delta^-$. Note also that for $\delta = 0$ the power $\mathbb{P}_\delta$ turns into the equality $\mathbb{P}_0 = \mathbb{P}_0^+ + \mathbb{P}_0^-$, where $\mathbb{P}_0$ is the first kind error of the modified test.

Theorem 2. *Let be* $\lambda := \delta\sigma^{-1}\Pi_n^{1/2}$. *Suppose that condition of Grigoriev and Ivanov* (1995) *are satisfied. Then* $\mathbb{P}_\delta = \sum_{\nu=0}^{2} \pi_\nu n^{-1/2} + o(n^{-1})$, *where the coefficients* π_ν *are defined as* $\pi_0 = \int_z^\infty f_{\chi^2}(x; 1, \lambda^2)dx$ *and*

$$\pi_\nu = \varphi(z-\lambda)\sum_{j=1}^{r_\nu} \lambda^j S_{\nu j}(z) + \varphi(z+\lambda)\sum_{j=1}^{r_\nu}(-\lambda^j)S_{\nu j}(z), \quad \nu = 1, 2. \tag{4.2}$$

Here $r_1 = 2$, $r_2 = 5$, $f_{\chi^2}(x; 1, \lambda^2)$ *is the density of the noncentral* χ^2*-distribution with one degree of freedom and noncentrality parameter* λ^2, $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ *and* $S_{\nu j}(z)$ *are polynomials in* z *with coefficients depending on the cumulants* $\kappa_{j\nu}(\lambda^r)$, $j = 1, \dots, 4$, *of the random variables* $W_n^\pm$. *These polynomials are defined in the paper.*

5. Comparison of the powers of modified tests

Consider the problem of comparing the powers of the l_n and $\psi_n^{(m)}$ tests based on the statistics $W_n^\pm$ and $W_n^{(m)}$ accordingly. Here $W_n^{(m)}$ is a statistic of the modified test $\psi_n^{(m)}$ that is defined as $T_n^{(m)} = W_n^{(m)2} + \xi n^{-3/2}$, where ξ is a random variable with uniformly bounded according to $\theta \in Q$ quickly decreasing tail of distribution.

We call the function $\Delta_s = \pi_2 - \pi_2^{(s)}$ as comparison function or *defect* of l_nth and $\psi_n^{(s)}$th tests. Then $\Delta_{rs} := \Delta_s - \Delta_r = \pi_2^{(r)} - \pi_2^{(s)}$ is defect of $\psi_n^{(r)}$th test with regard to $\psi_n^{(s)}$th test. The defect Δ_{rs} for $r, s = 0, 1, 2$ has been researched in [1]. Denote $\alpha_1^{(s)}$, $s = 0, 1, 2$, the adjustment coefficients of $\psi_n^{(s)}$th test. They are obtain in [1] and equal $\alpha_1^{(s)} = s$.

Let be γ_s is the sth cumulant of the random variable ε_j. Put

$$I_{ijk} = \frac{\gamma_{i+j+k}}{\sigma^{2(i+j+k)}}(\sigma^2 n^{-1}\Lambda_n)^{(i+2j+3k)/2}\sum_{a=1}^{n} g_1^i(a,\theta)g_2^j(a,\theta)g_3^k(a,\theta), \tag{5.1}$$

$$B_{1n} = I_{02} - I_{11}^2,\ B_{2n} = I_{21} - I_3 I_{11},\ B_{3n}(\alpha,\beta) = \alpha I_4 + \beta I_3^2,\ \alpha,\beta \in \mathbb{R}^1. \tag{5.2}$$

The quantities $B_{\nu n}$, $\nu = 1, 2, 3$, are statistical invariants of the observation model (1.1). The value $\sqrt{B_{1n}}$ is called *Efron's curvature* [3].

Theorem 3. *Suppose that condition of Grigoriev and Ivanov* (1995) *are satisfied. Then the defects* Δ_s *and* Δ_{rs} *are the linear combination of the invariants* B_{1n} *and* B_{2n} *depending on the coefficients* $\alpha_1^{(s)}$ *only.*

We can find according to the Theorem 4 the *preference regions* of the tests l_n and $\psi_n^{(s)}$, $s = 0, 1, 2$ in the half-plane $OB_{1n}B_{2n}$ of invariants (see Tab. 1 and Fig. 1). Let be $z^2 > 2$. Then the inequality $D_s \geq 0$ and the inequality $D_{rs} \geq 0$ are equivalent to

$$\frac{B_{2n}}{B_{1n}} \geq -\frac{\alpha_1^{(s)}z^2 - 1}{8(2z^2-2)}, \quad \frac{B_{2n}}{B_{1n}} \geq -\frac{(\alpha_1^{(s)} + \alpha_1^{(r)})z^2 - 2}{8(2z^2-2)}, \quad r, s = 0, 1, 2. \tag{5.3}$$

Taking into account (5.3) we obtain six independent inequalities. Therefore we see on the Fig. 1 the six straight lines with angular coefficients k_i, where

$$k_1 = -\frac{3z^2-2}{8(2z^2-1)}, \quad k_2 = -\frac{2z^2-1}{8(2z^2-1)}, \quad k_3 = -\frac{2z^2-2}{8(2z^2-1)},$$

$$k_4 = -\frac{z^2-1}{8(2z^2-1)}, \quad k_5 = -\frac{z^2-2}{8(2z^2-1)}, \quad k_6 = \frac{1}{8(2z^2-1)}.$$

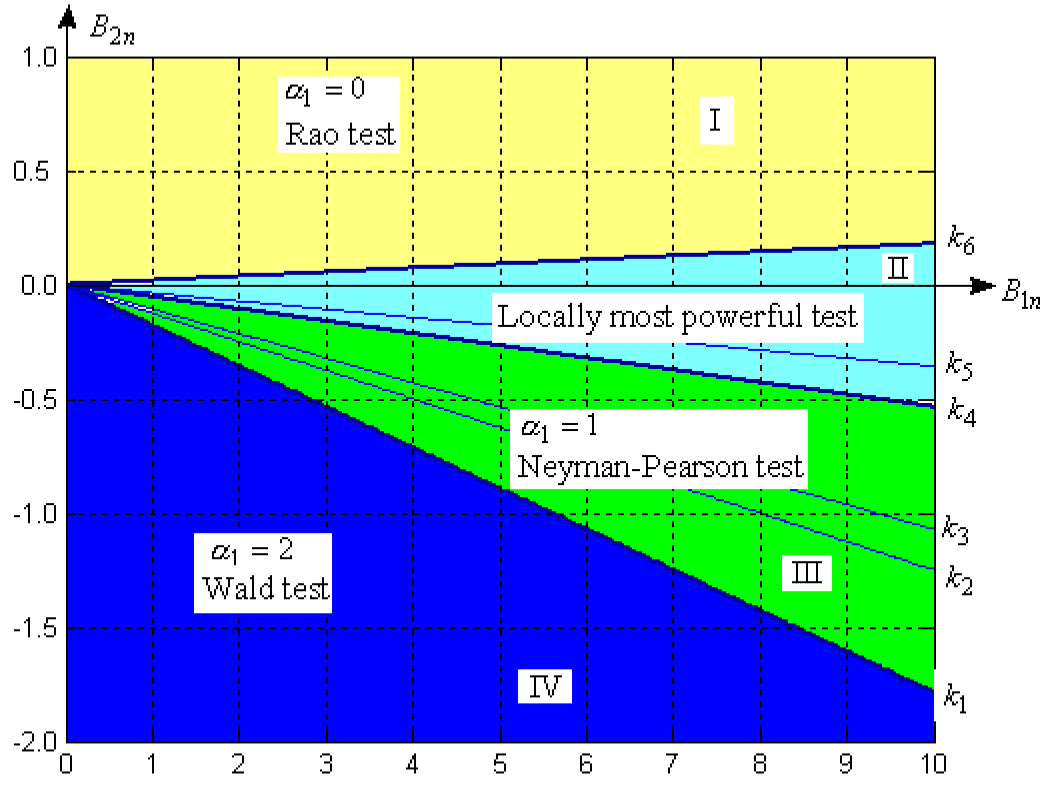

Fig. 1. Preference regions of the tests, $z^2 > 2$.

A more sophistical interpretation of the results obtained is presented in the Tab.1. The six lines in the Fig.1 form seven regions but four preference regions of the tests only. The preference chains of tests for seven regions in the Tab. 1 are shown.

Table 1. Preference regions of the tests

Regions	Test	Limit of B_{2n}/B_{1n}	Ordering of the tests			
I	Rao	(k_6, ∞)	Rao	$\succ$ LMP	$\succ$ NP	$\succ$ Wald
II	Locally most powerful	(k_5, k_6)	LMP	$\succ$ Rao	$\succ$ NP	$\succ$ Wald
	(LMP)	(k_4, k_5)	LMP	$\succ$ NP	$\succ$ Rao	$\succ$ Wald
III	Neyman-Pirson	(k_3, k_4)	NP	$\succ$ LMP	$\succ$ Rao	$\succ$ Wald
	(NP)	(k_2, k_3)	NP	$\succ$ LMP	$\succ$ Wald	$\succ$ Rao
		(k_1, k_2)	NP	$\succ$ Wald	$\succ$ LMP	$\succ$ Rao
IV	Wald	$(-\infty, k_1)$	Wald	$\succ$ NP	$\succ$ LMP	$\succ$ Rao

References

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