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DIRECT NUMERICAL SIMULATION OF MAGNETOHYDRODYNAMIC TURBULENCE BASED ON THE LEAST DISSIPATIVE MODES

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1. Problem formulation. We consider the case of a space periodic flow in a $3D$ cubic box Ω of size L under imposed homogeneous and steady magnetic field $\mathbf{B} = B_0 \cdot \mathbf{e}_z$. In the frame of the low-Rm approximation, the governing equations can be reduced to a single one involving the velocity $\mathbf{u}$ and pressure p only (see [2]). Using a reference length L_{ref} we shall write it in a non dimensional form as

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nabla^2\mathbf{u} - Ha^2\nabla^{-2}\frac{\partial^2\mathbf{u}}{\partial z^2} + G\mathbf{f}(\mathbf{x},t), \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{1}$$

where following notations are used $\mathrm{Ha} = L_{\text{ref}}B_0\sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number and $G = \frac{L_{\text{ref}}^{3/2}}{\nu^2}||\mathbf{f}||$ is the Grashof number, $\mathbf{u}(\mathbf{x},t)$ is the velocity-vector of the flow, $\mathbf{f}(\mathbf{x},t)$ is the external forcing, $\mathbf{x} = (x,y,x)$ is the spatial variable, t is time, ρ is the density, p is the pressure, ν is the viscosity, σ is the electrical conductivity, B_0 is the imposed magnetic field. Additionally, we will use another non-dimensional parameter Reynolds number $Re = \frac{UL_{\text{int}}}{\nu}$ based on integral length scale L_{int} (see [3]) and reference velocity U. The addition of periodic boundary conditions and zero initial condition $\mathbf{u}(\mathbf{x},0) = 0$ completely determine the problem.

We present numerical study using pseudo-spectral method based on a decomposition of the velocity $\mathbf{u}$ over the orthonormal basis of the eigenfunctions $\mathbf{v}_\mathbf{k}$ of the linear operator $D_{Ha} = \nabla^2 - \mathrm{Ha}^2\nabla^{-2}\frac{\partial^2}{\partial z^2}$, which corresponds to the linear part of the problem (1). These eigenfunctions are in a subset of the Fourier space used in the standard DNS schemes (see [3]). The aim is to show that properly chosen subset of least dissipative modes reduces the costs of the numerical simulations without loosing precision. It makes sense to consider eigenvalues $\lambda_\mathbf{k}$ which represents the rate of dissipation of mode $\mathbf{k}$

$$\lambda_\mathbf{k} = \lambda_{(k_x,k_y,k_z)} = -(k_x^2+k_y^2+k_z^2) - Ha^2\frac{k_z^2}{k_x^2+k_y^2+k_z^2}. \tag{2}$$

Since $\lambda_\mathbf{k} < 0$, $\lambda_\mathbf{k}$ can be arranged by growing dissipation so the spectral decomposition of $\mathbf{u}$ can be written as $\mathbf{u} = \sum\limits_{|\lambda_\mathbf{k}|<|\lambda^{\max}|} c_{\lambda_\mathbf{k}}\mathbf{v}_{\lambda_\mathbf{k}}$, where $\lambda^{\max}$ defines the maximum resolution required to resolve the flow completely. This yields a natural spectral parameter $\lambda_\mathbf{k}$ that already incorporates anisotropy. In the case of $Ha = 0$, $|\lambda_\mathbf{k}|^{1/2}$ reduces to $||\mathbf{k}||$ which is the usual spectral parameter in non-MHD isotropic turbulence. As mentioned by [1], the set of least dissipative eigenmodes of D_{Ha} required to describe the flow exhibits the rate of anisotropy expected for such flow from previous heuristic consideration. In short, one could see $\lambda_\mathbf{k}$ as an anisotropic generalization of the usual $\mathbf{k}$-sequence.

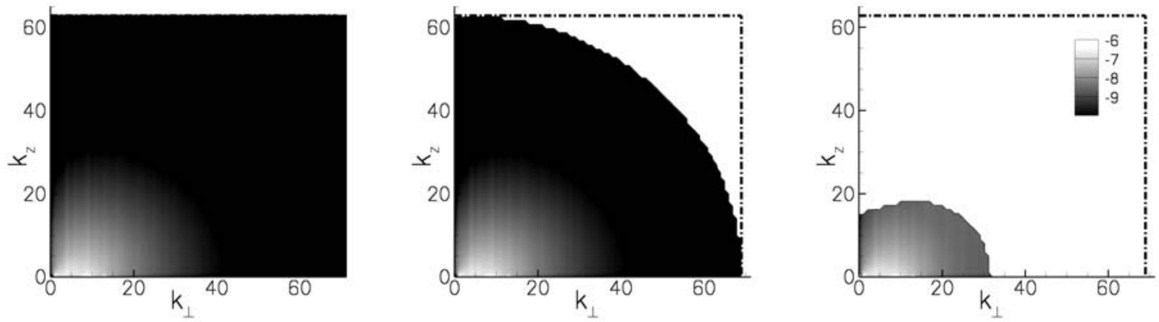

Fig. 1: Logarithmic energy distribution in the $(k_{\perp}, k_z)$-plane.

2. Numerical results. For the simulations we set the following constant values $L = 0.1$ m, $\nu = 3.4 \cdot 10^{-7}$ m^2/s, $\rho = 6.4 \cdot 10^3$ kg/m^3, $\sigma = 3.46 \cdot 10^6$ Ω^{-1}m^{-1}, $B_0 = 0.02$ T. Several simulations were done with different numerical resolutions $n_x \times n_y \times n_z$ and Grashof numbers G. Figure1 shows the time-averaged energy distributions of the 3D anisotropic MHD flow in the $(k_{\perp}, k_z)$-plane (here $k_{\perp} = \sqrt{k_x^2 + k_y^2}$) for fixed $G = 27193$ ($Re = 92$). The first picture corresponds to the $Re = 92$ and highest resolution $128 \times 128 \times 128$ which is necessary to resolve the flow according to the Kolmogorov length scale $k_{\max} = 1.5 Re^{3/4} \approx 45$ (we will use solid line to indicate this case). The second picture is done with the same 128^3 resolution but all the modes $\mathbf{v_k}$ which correspond to the $|\lambda_{\mathbf{k}}|^{1/2} > 68$ are set to be zero (dotted line). And the last one shows the energy of the flow resolved with 64^3 and cutoff for $|\lambda_{\mathbf{k}}|^{1/2} \leq 32$ (dashed line). Thus, the last run uses as much as 8 times less modes as the first classical run. One can see that λ-spectra $E(\lambda)$ obtained in these three runs are very close (see Figure2) and only the case with lowest resolution 32^3 and strongest cut off $|\lambda^{\max}|^{1/2} = 16$ yields the wrong solution (dashed-doted line). Moreover, from the second picture on Figure2 no clear "smallest scale" appears in the traditional $\mathbf{k}-$spectra, so we can not say either this flow resolved or not. It can be already concluded now, that Kolmogorov scaling

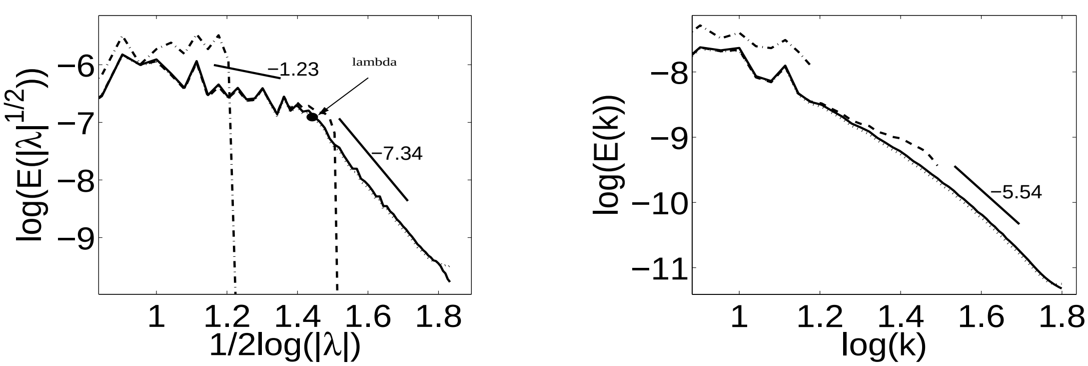

Fig. 2: *Left:* energy spectra in λ-shells, *Right* energy spectra in $\mathbf{k}$-shells.

laws are very often pessimistic and much higher then necessary. The above presented λ-approach can be considered as more realistic for the number of modes required to resolve MHD turbulence completely. The presented cut off of the type $|\lambda_{\mathbf{k}}| < |\lambda^{\max}|$ achieves a good enough precision, even though the Kolmogorov scale is not resolved or uncertain.

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