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ON THE STABILITY OF INVARIANT SETS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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Introduction. Let $t \in \mathbb{R}_+ = [0, \infty)$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n, |x| = \sqrt{\sum_{i=1}^n (x^i)^2}, y = (y^1, \dots, y^m)$, $|y| = \sqrt{\sum_{s=1}^m (y^s)^2}$, $z = (x, y) = (z^1, \dots, z^{n+m}) \in \mathbb{R}^{n+m}$, $|z| = \sqrt{|x|^2 + |y|^2}$. For a given $h > 0$, C^n and C^m denote the spaces of continuous functions mapping $[-h, 0]$ into $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^{n+m}) = (\psi, \lambda)$, where $\psi = (\psi^1, \dots, \psi^n) \in C^n$, $\lambda = (\lambda^1, \dots, \lambda^m) \in C^m$, $C = C^n \times C^m$. Denote

$$\|\psi\| = \sup(|\psi^i(\theta)|, \text{ under } -h \le \theta \le 0, \ 1 \le i \le n),$$

$$\|\lambda\| = \sup(|\lambda^j(\theta)|, \text{ under } -h \le \theta \le 0, \ 1 \le j \le m),$$

$$\|\varphi\| = \max(\|\psi\|, \|\lambda\|),$$

$$C_H = \{\varphi \in C : \|\psi\| \le H, \ \|\lambda\| < +\infty\}.$$

If z is a continuous function of u defined on $-h \le u < A, A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then z_t denotes the restriction of z to the segment $[t-h, t]$ so that $z_t = (z_t^1, \dots, z_t^{n+m}) = (x_t, y_t)$ is an element of C defined by $z_t(\theta) = z(t+\theta)$ for $-h \le \theta \le 0$.

Consider a system of functional differential equations

$$\frac{dz(t)}{dt} = Z(t, z_t). \tag{1}$$

In this system dz/dt denotes the right-hand derivative of z at t, t is time, and $Z(t, \varphi) = (X(t, \varphi), Y(t, \varphi)) \in \mathbb{R}^{n+m}$ is defined on $\mathbb{R}_+ \times C_{H_1}$; $X \in \mathbb{R}^n, Y \in \mathbb{R}^m$.

According to T.Burton [1], we denote by $z(t_0, \varphi) = (x(t_0, \varphi), y(t_0, \varphi))$ a solution of (1) with initial condition $\varphi \in C_{H_1}$, where $z_{t_0}(t_0, \varphi) = \varphi$ and we denote by $z(t, t_0, \varphi)$ the value of $z(t_0, \varphi)$ at t and $z_t(t_0, \varphi) = z(t+\theta, t_0, \varphi)$, $-h \le \theta \le 0$.

It is assumed that the vector-valued functional $Z(t, \varphi)$ is continuous on $\mathbb{R}_+ \times C_{H_1}$ so that a solution will exist for each continuous initial condition. We suppose that each solution $z(t_0, \varphi)$ is defined for those $t \ge t_0$, such that $\|x_t(t_0, \varphi)\| < H_1$.

Let $V(t, \varphi)$ be a continuous functional defined for $t \ge 0, \varphi \in C_{H_1}$.

Consider the set

$$M := \{\varphi \in C : \|\psi\| = 0, \|\lambda\| < \infty\}. \tag{2}$$

The necessary and sufficient conditions of the uniform asymptotic stability of the invariant set M of system (1) were obtained in [2]. In that paper, the method of Lyapunov functionals, founded by [3], was used. It was proved there that for uniform asymptotic stability of M it is necessary and sufficient the existence of continuous functional $V : \mathbb{R}_+ \times C_H \to \mathbb{R} \quad (H < H_1)$ such that

$$a(\|x_t\|) \leq V(t, z_t) \leq b(\|x_t\|), \quad a, b \in \mathcal{K}, \tag{3}$$

$$\frac{dV}{dt} \leq -c(\|x_t\|), \quad c \in \mathcal{K} \tag{4}$$

along solutions of system (1). Here $\mathcal{K}$ denotes the class of Hahn's functions,that is $r \in \mathcal{K}$ if $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonically increasing function such that $r(0) = 0$.

The purpose of this paper is twofold. First we consider the system

$$\frac{dz(t)}{dt} = Z(t, z_t) + R(t, z_t) \tag{5}$$

for which M is also the invariant set. In section 1 the restrictions on R are stated under which the uniform asymptotic stability of M of system (1) implies the uniform asymptotic stability of invariant set (2) of system (5). In section 2 we also consider the particular case of system (1) when Z is an almost periodic function of t. It is shown that for asymptotic stability of M of system (1) it is sufficiently the existence of a functional V which has more weak properties than (3) and (4).

1. On the Uniform Asymptotic Stability of M to Perturbed Systems.

Definition 1. *We shall say that a functional $Q : \mathbb{R}_+ \times C_H \to \mathbb{R}^{n+m}$ satisfies condition (B_1) if there is a $\beta > 0$ $(\beta < H)$ such that for any $\xi \in (0, \beta)$ there exist a $\tau_\xi \geq 0$ and a function $g_\xi(t)$, continuous on $[\tau_\xi, \infty)$ such that $|Q_i(t, \varphi)| \leq g_\xi(t) \quad (i = 1, \ldots, n)$ for $\varphi \in C_\beta \setminus C_\xi, t \in [\tau_\xi, \infty)$, and*

$$\lim_{t\to\infty} G_\xi(t) = 0$$

where $G_\xi(t) = \int_t^{t+1} g_\xi(s)ds$.

Theorem 1. *Let M be a uniformly asymptotic stable set of system (1), and its domain of attraction contains C_H. If $R(t, \varphi)$ satisfies condition (B_1), then M is also a uniformly asymptotic stable set of system (5), and there exists a positive η $(\eta < H)$ such that the domain of attraction of M of system (5) contains C_η.*

2. On the Stability of a Positive Invariant Set in Almost Periodic Systems.

Definition 2. *The solution $z(t_0, \varphi)$ of system (1) is called x-eventually nonzero if for every $t > t_0$ there exists $t_* > t$ such that $|x(t_*, t_0, \varphi)| \neq 0$.*

Theorem 2. *Let functional differential equations (1) satisfy the above conditions; let any solution $z(t_0, \varphi)$ such that $z_t(t_0, \varphi) \in C_H$ be y-bounded, and there exists a continuous functional $V(t, \varphi) : \mathbb{R} \times C_H \to \mathbb{R}$, which is locally Lipschitz in φ, such that the following conditions are fulfilled on the set $\mathbb{R} \times C_H$:*

- *$V(t, 0, \lambda) \equiv 0, \quad a(|\psi(0)|) \leq V(t, \varphi) \leq b(\|\psi\|)$, where $a, b \in \mathcal{K}$;*
- *$V(t, \varphi)$ is almost periodic in t for each fixed $\varphi \in C_{A,B}$ $(0 < A \leq H, B > 0)$;*
- *$dV/dt \leq 0$, $dV/dt \not\equiv 0$ on each x-eventually nonzero solution of system (1).*

Then M is asymptotically stable set of system (1).

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DIRECT NUMERICAL SIMULATION OF MAGNETOHYDRODYNAMIC TURBULENCE BASED ON THE LEAST DISSIPATIVE MODES

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1. Problem formulation. We consider the case of a space periodic flow in a $3D$ cubic box Ω of size L under imposed homogeneous and steady magnetic field $\mathbf{B} = B_0 \cdot \mathbf{e}_z$. In the frame of the low-Rm approximation, the governing equations can be reduced to a single one involving the velocity $\mathbf{u}$ and pressure p only (see [2]). Using a reference length L_{ref} we shall write it in a non dimensional form as

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nabla^2\mathbf{u} - Ha^2\nabla^{-2}\frac{\partial^2\mathbf{u}}{\partial z^2} + G\mathbf{f}(\mathbf{x},t), \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{1}$$

where following notations are used $\text{Ha} = L_{\text{ref}} B_0 \sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number and $G = \frac{L_{\text{ref}}^{3/2}}{\nu^2}||\mathbf{f}||$ is the Grashof number, $\mathbf{u}(\mathbf{x},t)$ is the velocity-vector of the flow, $\mathbf{f}(\mathbf{x},t)$ is the external forcing, $\mathbf{x} = (x,y,x)$ is the spatial variable, t is time, ρ is the density, p is the pressure, ν is the viscosity, σ is the electrical conductivity, B_0 is the imposed magnetic field. Additionally, we will use another non-dimensional parameter Reynolds number $Re = \frac{UL_{\text{int}}}{\nu}$ based on integral length scale L_{int} (see [3]) and reference velocity U. The addition of periodic boundary conditions and zero initial condition $\mathbf{u}(\mathbf{x},0) = 0$ completely determine the problem.

We present numerical study using pseudo-spectral method based on a decomposition of the velocity $\mathbf{u}$ over the orthonormal basis of the eigenfunctions $\mathbf{v}_{\mathbf{k}}$ of the linear operator $D_{Ha} = \nabla^2 - \text{Ha}^2\nabla^{-2}\frac{\partial^2}{\partial z^2}$, which corresponds to the linear part of the problem (1). These eigenfunctions are in a subset of the Fourier space used in the standard DNS schemes (see [3]). The aim is to show that properly chosen subset of least dissipative modes reduces the costs of the numerical simulations without loosing precision. It makes sense to consider eigenvalues $\lambda_{\mathbf{k}}$ which represents the rate of dissipation of mode $\mathbf{k}$

$$\lambda_{\mathbf{k}} = \lambda_{(k_x,k_y,k_z)} = -(k_x^2 + k_y^2 + k_z^2) - Ha^2\frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}. \tag{2}$$

Since $\lambda_{\mathbf{k}} < 0$, $\lambda_{\mathbf{k}}$ can be arranged by growing dissipation so the spectral decomposition of $\mathbf{u}$ can be written as $\mathbf{u} = \sum\limits_{|\lambda_{\mathbf{k}}|<|\lambda^{\max}|} c_{\lambda_{\mathbf{k}}}\mathbf{v}_{\lambda_{\mathbf{k}}}$, where $\lambda^{\max}$ defines the maximum resolution required to resolve the flow completely. This yields a natural spectral parameter $\lambda_{\mathbf{k}}$ that already incorporates anisotropy. In the case of $Ha = 0$, $|\lambda_{\mathbf{k}}|^{1/2}$ reduces to $||\mathbf{k}||$ which is the usual spectral parameter in non-MHD isotropic turbulence. As mentioned by [1], the set of least dissipative eigenmodes of D_{Ha} required to describe the flow exhibits the rate of anisotropy expected for such flow from previous heuristic consideration. In short, one could see $\lambda_{\mathbf{k}}$ as an anisotropic generalization of the usual $\mathbf{k}$-sequence.