SETTING THE METRICS AND MEASURE OF REFUTABILITY ON PREDICATE FORMULAS AS STATEMENTS IN SOME THEORY

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Abstract

The paper discusses statements of experts about objects represented as the predicate formulas consistence with theory T. Techniques for introducing metrics on such statements and measure of refutability and offered. The research can be applied to solving the problems of the best reconciliation of expert statements, to constructing the decision functions in pattern recognition and building the expert systems. The offered functions of measure of refutability satisfies all requirements formulated in [1, 2].

1 Introduction

There is an increasing interest in the construction of decision function on the basis of analysis of information provided in the form of logical "knowledge" of several experts which are represented by the predicate formulas [1-3] (these "knowledge" can be partly or completely contradictory). Obviously, the statements ("knowledge") can differ in their measure of refutability . The measure of refutability represents importance of the information provided by the expert. It involves the problem of reconciliation of statements of the experts about hierarchical objects and the problems of introducing a distance on these statements and of defining their measure of refutability .

This paper is a natural continuation of previous publications [1-2] and we assume that the latter are familiar. The logical statements (of experts) about objects formulated as the logical predicate formulas are discussed in the paper. By making use of the methods of mathematical logic and of the model theory [3] we propose the techniques for introducing metric on such statements and measure of their refutability . Properties of entered metrics and corresponding to measures are investigated.

2 Distances on the statements and its properties

Let L be a first order language consisting of finite number variable of predicate symbols which are selected for recording and studing the connections between variables in particular application area. For each variable x_i there is the unary predicate P_{x_i} corresponding only on the range of values of variable x_i on the set A_n. Let A_n be a nonempty set with the power n. A_n is the join of all values of considering variables which are included in the predicates.

Suppose we have finite number s of experts and the area of possible values of variables. Models (in the sense of the model fixed theory) are specified by experts. Each expert j defines a specific interpretation of each predicate symbol of the language L in terms of relevant relation in model M_j theory T. Thus, we have the "knowledge" of the experts expressed by the formulas which are set by subsets of formulas in each model M_j [3] under the interpretation of the expert j.

We denote a set of all models of the language L=L(T) defined the experts on the set A_n by $Mod_n(L)$.

We consider σ - algebra of subset F on the set $A = A_n^{<\omega} =$
$= \bigcup\limits_{k<\omega} A_n^k \quad (A_n^k = A_n \times ... \times A_n\ )$. Only those subset S_j from F will be interested us, for which the formula ψ of language L (appropriated to "knowledge" of the experts) are found and ψ such, that under the interpretation of "knowledge" of expert j ($j = 1, ..., s$) $S_j = \psi(M_j) = \{\bar{a} \mid M_j \models \psi(\bar{a})\}$ (that is subset of formulas corresponding to the formula ψ in model M_j [3]).

Let a probabilistic measure μ be defined on the sets from F.

Instead of "knowledge" of the expert defined by predicate P^{M_i} in model M_i further we consider it approximation - predicate $\widetilde{P}^{M_i}$.

By the approximation $\widetilde{P}^{M_i}$ of predicate P^{M_i} we will mean the closer definition of the domain of truth of this predicate in model M_i by one of the ways:
1) to leave the relation without changes;
2) to eliminate these elements from P^{M_i} in which truth the expert i is not absolutely sure;
3) to add in relation new elements and eliminate some old, for example, with allowance for the "knowledge" of other experts;
4) to execute items 2) and 3) simultaneously.

Now we introduce a distance on the set of "knowledge" of the experts with the help of models defined by them. The models differ in interpretations.

We define a distance between the subsets of formulas (predicates) for each model $M_i \in Mod_n(L)$ as a measure of symmetric difference between them.

Definition 1. We call $\rho_{M_i}(P_k^{M_i}, P_j^{M_i}) = \mu(\widetilde{P}_k^{M_i} \triangle \widetilde{P}_j^{M_i})$ the distance between predicates $P_k^{M_i}$ and $P_j^{M_i}$ defined in the model M_i.

Remark. This definition is correct if the predicates of one arity and with the same set of variables. If the considered predicates have different arity or different set of variables and if the expert consider insignificant the absent in one of the formulas variable x_i we suppose that it receives anyone from possible values. Otherwise (if it is significant) we determine this variable by adding to the necessary formula the predicate P_{x_i} corresponding to the values of this variable. It's clear that the entered concept is easily spreaded on subsets of formulas. Further we shall study a distance between the formulas of the same variables (one arity).

We define the distance between the formulas determined on the set of models T $Mod_n(L)$ as the mean value on the set of distances in the models.

Definition 2. We call

$$\rho_1(P_k(\bar{x}), P_j(\bar{x})) = \frac{\sum\limits_{M_i \in Mod_n(L)} \rho_{M_i}(P_k^{M_i}, P_j^{M_i})}{\mid Mod_n(L) \mid}$$

the distance between the formulas $P_k(\bar{x})$ and $P_j(\bar{x})$ defined on the set $Mod_n(L)$.

Now we shall consider the method for determining the distance between the propositions (a formulas that does not involve free variables).

We denote the set of models from $Mod_n(L)$ for which the proposition φ is true by $Mod(\varphi)$, i.e., $Mod(\varphi) = \{M_i \in Mod_n(L) \mid$
$\mid \; M_i \models \varphi\}$.

Clear, there are such models for which a proposition is true and such models for which it is false (if it is not a tautology). It is natural to measure the difference in information contained in the propositions by the number of the models where these propositions take different truth values.

Definition 3. We call

$$\rho_2(\varphi, \psi) = \frac{\left|Mod((\neg\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi))\right|}{\mid Mod_n(L) \mid}$$

the distance between the propositions φ and ψ.

Let us consider one more method for determining the distance between the formulas. We complement the first order language L with constants from the set $M = Mod_n(L)$. For this set M we consider the arbitrary tuples $\bar{a}$ which lengths are equal to arity $\ell(\bar{a})$ of the formulas. Substituting the tuples into formulas and assuming that formulas have the same arity (the way of doing so was demonstrated above) the formulas become propositions.

Definition 4. We call $\rho_3(\varphi, \psi) = \min\limits_{\bar{a} \in M^{\ell(\bar{a})}} \rho_2(\varphi(\bar{a}), \psi(\bar{a}))$ the distance between formulas φ and ψ.

We have proved the theorem demonstrating that the offered distances are the metrics really. Also we have proved the existence of several specific properties of the introduced distances.

Theorem 1. For any formulas ("knowledge" for T) φ, ψ, χ and for any function ρ_i the following properties are valid:

1. $0 \leq \rho_i(\varphi, \psi) \leq 1$. 2. $\rho_i(\varphi, \psi) = \rho_i(\psi, \varphi)$ (symmetry). 3. If $\rho_i(\varphi, \psi) = \rho_i(\varphi_1, \psi_1)$ and $\rho_i(\varphi_1, \psi_1) = \rho_i(\varphi_2, \psi_2)$ then $\rho_i(\varphi, \psi) = \rho_i(\varphi_2, \psi_2)$ (transitivity).
4. $\rho_i(\varphi, \psi) \leq \rho_i(\varphi, \chi) + \rho_i(\chi, \psi)$ (inequality of a triangle).
5. $\varphi \equiv \psi \iff \rho_i(\varphi, \psi) = 0$ (henceforth, $\varphi \equiv \psi$ denotes equivalence of formulas in relation to all models of the experts, that is, for any expert i (defining the model M_i) $\varphi^{M_i} \equiv \psi^{M_i}$is true).
6. $\varphi \equiv \neg\psi \iff \rho_i(\varphi, \psi) = 1$.
7. $\rho_i(\varphi, \psi) = 1 - \rho_i(\varphi, \neg\psi) = \rho_i(\neg\varphi, \neg\psi)$.
8. $\rho_i(\varphi, \psi) = \rho_i(\varphi \wedge \psi, \varphi \vee \psi)$.
9. $\rho_i(\varphi, \neg\varphi) = \rho_i(\varphi, \psi) + \rho_i(\psi, \neg\varphi)$.

3 Measure of refutability and their properties

From the point of view of importance of the information presented by an expert, it is natural to assume that the more measure of refutability of the statement (formula), the less the number of models for which the statement is true (the less the measure of the set for which the formula is true). Therefore, we introduce the measure of refutability as follows.

Definition 5. We call $I_i(P(\bar{x}) = \rho_i(P(\bar{x}), 1)$ the measure of refutability of formula $P(\bar{x})$, where 1 is an identical true predicate, that is, $\bar{x} = \bar{x}$.

For the introduced distances we obtain :

$$I_1(P) = \frac{\sum\limits_{M_i \in Mod_n(L)} \mu(\neg P^{M_i})}{|Mod_n(L)|}, \text{ if } \rho_1,$$

$$I_2(P) = \frac{\left|Mod(\neg P)\right|}{|Mod_n(L)|}, \text{ if } \rho_2,$$

$$I_3(P) = \min_{\bar{a} \in M^{\ell(\bar{a})}} \frac{\left|Mod(\neg P(\bar{a}))\right|}{|Mod_n(L)|}, \text{ if } \rho_3.$$

The following theorem is proved:

Theorem 2. For any formulas ("knowledge" for T), φ, ψ and for any ρ_i the following assertions are valid:

1. $0 \leq I(\varphi) \leq 1$. 2. $I(1) = 0$.
3. $I(0) = 1$. 4. $I(\varphi) = 1 - I(\neg\varphi)$.
5. $I(\varphi) \leq I(\varphi \wedge \psi)$. 6. $I(\varphi) \geq I(\varphi \vee \psi)$.
7. $I(\varphi \wedge \psi) = \rho(\varphi, \psi) + I(\varphi \vee \psi)$. 8. If $\varphi \equiv \psi$, then $I(\varphi) = I(\psi)$.
9. If $\rho(\varphi, \psi) = 0$, then $I(\varphi \vee \psi) = I(\varphi \wedge \psi) = I(\varphi)$.
10. $I(\varphi \wedge \psi) = (I(\varphi) + I(\psi) + \rho(\varphi, \psi))/2$.
11. $I(\varphi \vee \psi) = (I(\varphi) + I(\psi) - \rho(\varphi, \psi))/2$.

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References

[1] Bloshchitsyn V.Ya. and Lbov G.S. (1978). Measures of Informativeness of Logical Statements.//*The Reports of Respubl. school-seminar "The Technology of Developing Expert Systems".* Kishinev, 1978, pp.12-14.

[2] Vikent'ev A.A., Lbov G.S. (1997). Setting the Metric and Informativeness on Statements of Experts.// *Pattern Recognition And Image Analysis.* 1997, v. 7 (2), pp. 175-189.

[3] Ershov Yu.L, Palyutin E.A. (1991.)*Mathematical Logic.* Moscow: N., 1991, 336 p.

[4] Gaifman H. (1964). Concerning measures in the first order calculi. // *Israel Journal of Mathematics*, v. 2 (1), 1964, pp. 1-18.