ASYMPTOTIC EXPANSION IN THE CENTRAL LIMIT THEOREM FOR AN AR(1) PROCESS

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Abstract

Consider an AR(1) process with i.i.d. innovations. We assume that the innovations have the finite fourth moment and density satisfying the integral Lipschitz condition. For such a process the asymptotic expansion of order 1 in the central limit theorem is obtained.

1 Asymptotic expansion

The subject of interest in the following is the AR(1) process

$$X_t = \rho X_{t-1} + u_t, \qquad t = 0, \pm 1, \pm 2, \dots, \tag{1}$$

where $|\rho| < 1$, $\{u_t\}$ — i.i.d. random variables with $\mathsf{E}\{u_0^4\} < \infty$. Denote the moments of u_0: $\mathsf{E}\{u_0\} = m$, $\mathsf{D}\{u_0\} = \sigma^2$, $\mathsf{E}\{(u_0 - \mathsf{E}\{u_0\})^3\} = \mu_3$.

If only the second moment exists: $\mathsf{E}\{u_0^2\} < \infty$, the central limit theorem holds for the process (1) (see e.g. [1, Theorem 7.7.8]). If higher moments are available and u_0 has the density then the asymptotic expansion of the corresponding order in the central limit theorem has been proved [4]. In particular, if $\mathsf{E}\{u_0^4\} < \infty$ the expansion of order 1 is obtained in [4] with the error $o\left(\frac{1}{\sqrt{n}}\right)$. We develop the latter case and give, under the additional assumption on the density of u_0, the asymptotic expansion of order 1 with the error $\mathcal{O}\left(\frac{\ln^3 n}{n}\right)$.

Introduce the normalized sum S_n:

$$S_n = \frac{1}{B_n}\left(\sum_{t=1}^{n} X_t - \frac{mn}{1-\rho}\right).$$

Here

$$B_n^2 = \mathsf{D}\left\{\sum_{t=1}^{n} X_t\right\} = \frac{\sigma^2}{(1-\rho)^2}\left(n - 2\rho\frac{1-\rho^n}{1-\rho^2}\right) \qquad \text{and} \qquad \mathsf{E}\{X_t\} = \frac{m}{1-\rho}.$$

So $\mathsf{E}\{S_n^2\} = 1$. The explicit value of $\mathsf{E}\{S_n^3\}$ will also be of use:

$$\mathsf{E}\left\{S_n^3\right\} = \frac{\mu_3}{\sigma^3} \cdot \frac{n - 3\rho\frac{1-\rho^n}{1-\rho^3}\frac{1+\rho+\rho^2-\rho^{n+1}}{1+\rho}}{\left(n - 2\rho\frac{1-\rho^n}{1-\rho^2}\right)^{\frac{3}{2}}}.$$

Theorem 1. *Consider the AR(1) process* (1) *with* $\mathsf{E}\{u_0^4\} < \infty$*. Suppose that* u_0 *has the density* $p(x)$ *satisfying the following condition:*

$$\int\limits_{-\infty}^{+\infty} |p(x+h) - p(x)|dx \le C|h| \quad \textit{for some } C > 0 \textit{ and all } h \in \mathbb{R}. \tag{2}$$

Then the following asymptotic expansion holds: as $n \to \infty$,

$$\sup_{x\in\mathbb{R}} \left| \mathsf{P}\{S_n < x\} - \Phi(x) + \frac{\mathsf{E}\{S_n^3\}}{6}\phi(x)(x^2-1) \right| = \mathcal{O}\left(\frac{\ln^3 n}{n}\right), \tag{3}$$

where $\Phi(x)$ *and* $\phi(x)$ *are the cumulative distribution function and density of the standard normal distribution, respectively.*

Remark 1. *The condition* (2) *is satisfied if the density* $p(x)$ *is absolutely continuous and* $\int\limits_{-\infty}^{+\infty} |p'(x)|dx = \mathsf{E}\left\{\left|\frac{\partial \ln p(u_0)}{\partial u_0}\right|\right\} < \infty$*. Another sufficient condition can be stated in terms of the characteristic function of* u_0, $f(t)$*:* (2) *holds if* $\int\limits_{-\infty}^{+\infty} |tf(t)|dt < \infty$*.*

Remark 2. *If* $\mu_3 = 0$ *(e.g. the distribution of* $u_0 - m$ *is symmetric) then* $\mathsf{E}\{S_n^3\} = 0$ *and* (3) *takes the form of the convergence rate in the central limit theorem.*

Examples of the innovation process $\{u_t\}$ that satisfies the conditions of Theorem 1 include (we name the distributions of u_0) the Student's distribution with $\nu \ge 5$ degrees of freedom, Laplace distribution and Skew-normal distribution. The latter is defined to have the density $p(x) = 2\phi(x)\Phi(cx)$, where $c \in \mathbb{R}$ is the shape parameter.

2 Proof

The proof of Theorem 1 will be carried out for the case $m = 0$, $\sigma^2 = 1$. The general case is then obtained by applying Theorem 1 to the "normalized" process $\bar{X}_t = \rho\bar{X}_{t-1} + \bar{u}_t$, where $\bar{X}_t = X_t - \frac{m}{1-\rho}$ and $\bar{u}_t = \frac{u_t - m}{\sigma}$.

The proof is based on our result [6] giving the asymptotic expansion of order 1 in the central limit theorem for strong mixing processes.

The notion of strong mixing was introduced by M. Rosenblatt (1956) together with the central limit theorem for strong mixing processes. Since then different types of mixing for random processes have been introduced. Mixing describes the weak dependence structure of the process when the "past" and "future" of the process become asymptotically independent in a certain sense. Different common problems including the central limit theorem have been extended to mixing processes. See [2] for the details and definition of strong mixing.

The process (1) will be shown to be strong mixing under the assumptions of Theorem 1. In fact, Theorem 1 is an application of our previous Theorem [6] to AR(1) processes. We cite that Theorem from [6] for the ease of further reference.

Let $X_1, X_2, \ldots$ be a sequence of random variables with $\mathbf{E}\{X_t\} = 0$, $\mathbf{E}\{X_t^2\} < \infty$, and a strong mixing coefficient $\alpha(\cdot)$. Denote, as usual, $B_n^2 = \mathbf{D}\left\{\sum_{t=1}^n X_t\right\}$ and $S_n = (\sum_{t=1}^n X_t)/B_n$.

We introduce the three conditions which are commonly used (in a stronger or weaker form) to prove the central limit theorem and its extensions for mixing processes:

$$\sup_t \mathbf{E}\left\{|X_t|^{3+\delta}\right\} < \infty, \tag{4}$$

$$B_n^2 \geq C_1 n, \tag{5}$$

$$\alpha(k) = \mathcal{O}\left(e^{-\beta k}\right), \quad k \to \infty. \tag{6}$$

To obtain an asymptotic expansion we will need the additional condition on the characteristic function of S_n, $f_n(t)$, (cf. with the condition (III) in [5, p. 213] for independent sequences):

$$\int\limits_{C_2 T \leq |t| \leq C_3 \varepsilon_n^{-1}} \left|\frac{f_n(t)}{t}\right| dt = \mathcal{O}\left(\varepsilon_n\right), \quad n \to \infty, \tag{7}$$

where $T = T(n)$ is a parameter of Theorem 2 taking values in the interval

$$\left(\sqrt{n}\right)^{1-\delta} (\ln n)^{\delta - \frac{1}{2}} \leq T \leq \left(\sqrt{n}\right)^{1-\varepsilon} \tag{8}$$

and $\varepsilon_n = \varepsilon_n(T) = \frac{T(\ln n)^{5/2}}{n}$.

Theorem 2 ([6])**.** *Let $X_1, X_2, \ldots$ be a strong mixing sequence (not necessarily stationary) of random variables with $\mathbf{E}\{X_t\} = 0$ and a strong mixing coefficient $\alpha(\cdot)$. And let there exist constants $C_i > 0$, $i = 1, 2, 3$, $\delta \in (0, 1]$, $\beta > 0$, $\varepsilon \in (0, \delta)$ and a function $T = T(n)$ from the interval* (8) *such that the conditions* (4)–(7) *are satisfied. Then the following asymptotic expansion holds: as $n \to \infty$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\{S_n < x\} - \Phi(x) + \frac{\mathbf{E}\{S_n^3\}}{6} \phi(x)(x^2 - 1) \right| = \mathcal{O}\left(\varepsilon_n\right). \tag{9}$$

Now we show that the AR(1) process (1) satisfies the conditions of Theorem 2 with $\delta = 1$ and $T = \sqrt{\ln n}$ (note that this choice of parameters produces the strongest possible estimate in (9)).

It is known that the process (1) is strictly stationary and X_t takes the form $X_t = \sum\limits_{k=0}^{\infty} \rho^k u_{t-k}$, where the series converges in square mean (as well as almost surely). It is seen that $\mathbf{E}\{u_0^4\} < \infty$ implies $\mathbf{E}\{X_t^4\} < \infty$ so (4) holds.

The condition (5) is verified immediately since the variance B_n^2 is calculated directly and is specified at the beginning of the paper.

The strong mixing property of the process (1) with the mixing rate (6) is proven by V. V. Gorodetskii [3] under the assumption (2).

Finally, to verify (7) we first notice that S_n is the infinite sum of independent random variables:

$$S_n = \frac{1}{B_n} \sum_{t=1}^{n} \sum_{k=0}^{\infty} \rho^k u_{t-k} = \frac{1}{B_n} \sum_{k=0}^{\infty} r_k u_{n-k},$$

where $r_k = \sum\limits_{i=0}^{k} \rho^i = \frac{1-\rho^{k+1}}{1-\rho}$ for $k < n$ and $r_k = \sum\limits_{i=k-n+1}^{k} \rho^i = \rho^{k-n+1} \cdot \frac{1-\rho^n}{1-\rho}$ for $k \geq n$. Then for the characteristic function $f_n(t)$ we have the following estimate:

$$|f_n(t)| = \left| \mathbf{E}\left\{ e^{it\frac{1}{B_n}\sum\limits_{k=0}^{\infty} r_k u_{n-k}} \right\} \right| = \left| \prod_{k=0}^{\infty} \mathbf{E}\left\{ e^{it\frac{1}{B_n} r_k u_{n-k}} \right\} \right| \leq \prod_{k=0}^{n-1} \left| f\left(\frac{t}{B_n} r_k \right) \right|,$$

where $f(t)$ is the characteristic function of u_0.

Since $0 < r_- \leq r_k \leq r_+ < \infty$ uniformly over all $k < n$ and all n the argument of $f(\cdot)$ in the last expression behaves like $\frac{t}{\sqrt{n}}$. Then, for small values of t, we use the bound [5, Theorem 2, p. 21]: $|f(t)| \leq 1 - ct^2 \leq e^{-ct^2}$ for $|t| \leq \tau$, where c and τ are some positive constants. For large values of t, since u_0 has the density, we can bound $f(t)$ as follows: $\sup\limits_{|t| \geq a} |f(t)| \leq q < 1$ for arbitrary fixed $a > 0$.

Applying these considerations to our case we obtain

$$\int\limits_{C_2 T \leq |t| \leq C_3 \varepsilon_n^{-1}} \left| \frac{f_n(t)}{t} \right| dt \leq \int\limits_{C_2 T \leq |t| \leq \frac{\tau B_n}{r_+}} \frac{1}{|t|} \prod_{k=0}^{n-1} e^{-c\left(\frac{t}{B_n} r_k\right)^2} dt + \int\limits_{\frac{\tau B_n}{r_+} \leq |t| \leq C_3 \varepsilon_n^{-1}} \frac{q^n}{|t|} dt \leq$$

$$\leq \int\limits_{C_2 T \leq |t| \leq \frac{\tau B_n}{r_+}} \frac{e^{-c_1 t^2}}{|t|} dt + 2q^n \ln(C_3 \varepsilon_n^{-1}) \leq C \left(\frac{e^{-c_1 (C_2 T)^2}}{T^2} + q^n \ln(C_3 \varepsilon_n^{-1}) \right) = o\left(\varepsilon_n\right)$$

if we choose $T = \sqrt{\ln n}$ and large enough C_2.

Thus for the process (1) the asymptotic expansion (9) holds with $\varepsilon_n = \frac{\ln^3 n}{n}$. This coincides with the desired statement (3).

References

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