INVESTIGATION OF IDLE TIME MODEL IN COMPUTER NETWORKS

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Abstract

An open queueing network model in light traffic has been developed. The probability limit theorem for the idle time process of customers in heavy traffic in open queueing networks has been presented. Finally, we present an application of the theorem - an idle time model from computer network practice.
Keywords: mathematical models of technical systems, performance evaluation, queueing theory, open queueing network, light traffic, probability limit theorem, idle time process of customers.

1 Introduction

One can apply the theory of queueing networks to obtain probability characteristics of technical systems (for example, the idle function of computer networks). The idle function of computer networks shows which part of time computer network is not busy (idle). So in this paper, we present a probability limit theorem for the idle time process of customers in light traffic in the queueing network. The service discipline is "first come, first served" (FCFS).

We consider open queueing networks with the FCFS service discipline at each station and general distributions of interarrival and service time.

The queueing network we studied has k single server stations, each of which has an associated infinite capacity waiting room.

Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables. The basic components of the queueing network are arrival processes, service processes, and routing processes.

We begin with a probability space (Ω, B, P) on which these processes are defined. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, ..., k$; defined on the probability space. Random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have support in $\{0, 1, 2, ..., k\}$. We define $\mu_j = \left(E\left[S_n^{(j)}\right]\right)^{-1}$, $\sigma_j = D\left(S_n^{(j)}\right)$ and $\lambda_j = \left(E\left[z_n^{(j)}\right]\right)^{-1}$, $a_j = D\left(z_n^{(j)}\right)$, $j = 1, 2, ..., k$; with all of these terms assumed finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right)$, $i, j = 1, 2, ..., k$. The $k \times k$ matrix $P = (p_{ij})$ is assumed to have a spectral radius strictly smaller than a unit. The matrix P is called a routing matrix.

In the context of the queueing network, the random variables $z_n^{(j)}$ function as inter-arrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network.

At first, let us define $I_j(t)$ as the idle time process of customers at the jth station of the queueing network in time t (time t, which an open queueing network is not busy (idle) serving customers at the jth station of the queueing network), $\hat{\beta}_j = 1 - \frac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij}}{\mu_j}$, $\hat{\sigma}_j^2 = \sum_{i=1}^{k} p_{ij}^2 \cdot \mu_i \cdot \left(\sigma_j + \left(\frac{\mu_i}{\mu_j}\right)^2 \cdot \sigma_i\right) + \lambda_j \cdot \left(\sigma_j + \left(\frac{\lambda_j}{\mu_j}\right)^2 \cdot a_j\right)$, $j = 1, 2, \cdots, k$ and $t > 0$.

We suppose that the following conditions are fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} < \mu_j, \;\; j = 1, 2, \ldots, k. \tag{1}$$

In addition, we assume throughout that

$$\max_{1 \leq j \leq k} \sup_{n \geq 1} E\left\{\left(z_n^{(j)}\right)^{2+\gamma}\right\} < \infty \;\; \text{for some} \;\; \gamma > 0, \tag{2}$$

$$\max_{1 \leq j \leq k} \sup_{n \geq 1} E\left\{\left(S_n^{(j)}\right)^{2+\gamma}\right\} < \infty \;\; \text{for some} \;\; \gamma > 0. \tag{3}$$

Conditions (2) and (3) imply the Lindeberg condition for the respective sequences (usually $\gamma = 1$ works).

One of the results of the paper is a following probability limit theorem for the idle time process of customers in an open queueing network (proof can be found in [1]).

Theorem 1. *If conditions (1) - (3) are fulfilled, then*

$$\lim_{n \to \infty} P\left(\frac{I_j(nt) - \beta_j \cdot n \cdot t}{\hat{\sigma}_j \cdot \sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2/2) dy,$$

$0 \leq t \leq 1$ *and* $j = 1, 2, \ldots, k$.

2 Idle Time Function of Computer Network

Now we present a technical example from the computer network practice. Assume that queues of customers requests arrive at the computer v_j at a rate λ_j per hour during business hours, $j = 1, 2, \ldots, k$. These queues are served at the rate μ_j per hour by the computer v_j, $j = 1, 2, \ldots, k$. After service in the computer v_j, with probability p_j (usually $p_j \geq 0.9$), they leave the network and with probability p_{ji}, $i \neq j$, $1 \leq i \leq k$ (usually $0 < p_{ji} \leq 0.1$) arrive at the computer v_i, $i = 1, 2, \ldots, k$. Also, we assume the computer v_j to be idle when the idle time of waiting for service computer is less than k_j, $j = 1, 2, \ldots, k$.

In this section, we will prove the following theorem on the idle time function of computer network (probability of idle in computer network). Computer network is idle when it is not busy.

Theorem 2. *If* $t \geq \max_{1 \leq j \leq k} \frac{k_j}{\hat{\beta}_j}$ *and conditions (1) - (3) are fulfilled, all computer in the network are idle.*

Therefore, using Theorem 1 we get for $0 < \varepsilon < 1$

$$\lim_{n \to \infty} P\left(\frac{I_j(n) - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2/2) dy, \; j = 1, 2, \ldots, k. \quad (4)$$

Let us investigate a computer network which consists of the elements (computers) v_j, $j = 1, 2, \ldots, k$.

Denote

$$X_j = \begin{cases} 1, & \text{if the element } v_j \text{ is idle} \\ 0, & \text{if the element } v_j \text{ is not idle,} \end{cases}$$

$j = 1, 2, \ldots, k$.

Note that $\{X_j = 1\} = \{I_j(t) < k_j\}$, $j = 1, 2, \ldots, k$.

Denote the structural function of the system of elements connected by scheme 1 from k (see, for example, [2]) as follows:

$$\phi(X_1, X_2, \ldots, X_k) = \begin{cases} 1, & \sum_{i=1}^{k} X_i \geq 1 \\ 0, & \sum_{i=1}^{k} X_i < 1. \end{cases}$$

Denote $y = \sum_{i=2}^{k} X_i$. Estimate the idle function of the system (computer network) using the formula of the full conditional probability

$$h(X_1, X_2, \ldots, X_k) = E\phi(X_1, X_2, \ldots, X_k) = P(\phi(X_1, X_2, \ldots, X_k) = 1) =$$

$$P(\sum_{i=1}^{k} X_i \geq 1) = P(X_1 + y \geq 1) = P(X_1 + y \geq 1 | y = 1) \cdot P(y = 1) +$$

$$P(X_1 + y \geq 1 | y = 0) \cdot P(y = 0) = P(X_1 \geq 0) \cdot P(y = 1) + P(X_1 \geq 1) \cdot P(y = 0) \leq$$

$$P(y = 1) + P(X_1 \geq 1) = P(y = 1) + P(X_1 = 1) \leq P(y \geq 1) + P(X_1 = 1)$$

$$= P(\sum_{i=2}^{k} X_i \geq 1) + P(X_1 = 1) \leq \cdots \leq \sum_{i=1}^{k} P(X_i = 1) = \sum_{i=1}^{k} P(I_i(t) \leq k_i).$$

Thus,

$$0 \leq h(X_1, X_2, \ldots, X_k) \leq \sum_{i=1}^{k} P(I_i(t) \leq k_i). \tag{5}$$

Applying Theorem 1 (with $t = 1$), we obtain that

$$0 \leq \lim_{t\to\infty} P(I_j(t) < k_j) = \lim_{n\to\infty} P(I_j(n) < k_j) =$$

$$\lim_{t\to\infty} P\left(\frac{I_j(n) - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}} < \frac{k_j - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}}\right) = \int_{-\infty}^{-\infty} \exp(-y^2/2)dy = 0. \tag{6}$$

Thus (see (6)),

$$\lim_{t\to\infty} P\left(I_j(t) < k_j\right) = 0,\ j = 1, 2, \ldots, k. \tag{7}$$

So, (see (5) and (7)), $h(X_1, X_2, \ldots, X_k) = 0$. The proof of the theorem is completed.

References

[1] Minkevičius S. (2007). (to be published). The Probability Limit Thereom for The Idle Time Process in Open Queueing Networks.

[2] Morder J.J., Elmaghraby S. E. (eds.) (1978). *Handbook of Operational Research Models and Applications*. Van Nostrand Reinhold, New York.