ON THE TOTAL QUEUE LENGTH IN MULTIPHASE QUEUES

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Abstract

The paper is designated to the analysis of queueing systems, arising in the network and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). The author investigated multiphase queueing systems and presents heavy traffic limit theorems for the total queue length of customers in multiphase queues. In this work, functional limit theorems are proved for values of important probability characteristics of the queueing system investigated as well as the total queue length of customers.

Keywords: mathematical models of technical systems, queueing systems, multiphase queues, heavy traffic, total queue length of customers.

1 Introduction

The paper is designated to the analysis of queueing systems, arising in the networks theory and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). Multiphase queueing systems are of special interest both in theory and in practical applications (message switching systems, processes of conveyor production, retransmission of video images, etc.).

We investigated an important probability characteristic of the multiphase queueing system (a total queue length of customers). In this paper, functional limit theorems under heavy traffic conditions for values of the total queue length of customers are proved. The main tool for the analysis of a multiphase queue in heavy traffic is the functional limit theorem for a renewal process (the proof can be found in [1]).

We investigate here a k-phase multiphase queue (i.e., after a customer has been served in the j-th phase of a multiphase queue, he goes to the $j+1$-st phase of the multiphase queue, and after the customer has been served at the k-th phase of the multiphase queue, he leaves the queueing system). Let us denote t_n as a time of arrival of the n-th customer; $S_n^{(j)}$ as the service time of the n-th customer at the j-th phase of the multiphase queue; $z_n = t_{n+1} - t_n$. Let us introduce mutually independent renewal processes $x_j(t) = \{\max\limits_k \sum\limits_{i=1}^{k} S_i^{(j)} \leq t\}$ (such a total number of customers can be served at the j-th phase of the multiphase queue until time t, if devices are working without time wasted), $e(t) = \{\max\limits_k \sum\limits_{i=1}^{k} z_i \leq t\}$ (the total number of customers which arrive at a multiphase queue until the time moment t).

Next, denote by $\tau_j(t)$ the total number of customers after service departure from the j-th phase of multiphase queueing systems until time t; denote by $Q_j(t)$ the queue length of customers at the j-th phase of the multiphase queue until the time moment t; $V_j(t) = \sum_{i=1}^{j} Q_i(t)$ stands for the total queue length of customers in the j-th phase of the multiphase queue at the time moment t, $j = 1, 2, \ldots, k$ and $t > 0$. Also, let $\hat{Q}_j(t)$ be the queue length of customers at the j-th phase of a modified multiphase queue at time t, $\hat{V}_j(t) = \sum_{i=1}^{j} \hat{Q}_i(t)$ stands for the total queue length of customers at the j-th phase of a modified multiphase queue until t, $j = 1, 2, \ldots, k$ and $t > 0$.

Suppose that the queue length of customers at each phase of the multiphase queue is unlimited, the service principle of customers is “first come, first served" (FCFS). All random variables are defined on one common probability space (Ω, F, P).

Let interarrival times (z_n) at the the multiphase queue and service times $(S_n^{(j)})$ at every phase of the multiphase queue for $j = 1, 2, \ldots, k$ be independent identically distributed random variables.

Let us define $\beta_j = (MS_1^{(j)})^{-1}$, $\beta_0 = (Mz_1)^{-1}$, $\alpha_j = \beta_0 - \beta_j$, $\alpha_0 \equiv 0$,

$$\hat{\sigma}_j^2 = DS_1^{(j)} \cdot (MS_1^{(j)})^{-3} > 0, \;\; \hat{\sigma}_0^2 = Dz_1 \cdot (Ez_1)^{-3} > 0, \; \tilde{\sigma}_j^2 = \hat{\sigma}_0^2 + \hat{\sigma}_j^2,$$

$$\sigma_j^2 = \hat{\sigma}_j^2 + \hat{\sigma}_0^2, \; \hat{x}_j(t) = e(t) - x_j(t), \; \tilde{x}_j(t) = \hat{x}_{j-1}(t) - \hat{x}_j(t), \; j = 1, 2, \ldots, k.$$

Let us consider, as in [3], a sequence of multiphase queues: $S_{m,n}^{(j)}$ are independent identically distributed random variables in the n-th multiphase queue, $j = 0, 1, 2, \ldots, k$, $S_{m,n}^{(0)} = z_{m,n}$, $m \geq 1$, $n \geq 1$. Define $G_{j,n}(x) = \mathbf{P}(S_{1,n}^{(j)} < x)$, $j = 0, 1, 2, \ldots, k$.

Let

$$\mathbf{D}S_{1,n}^{(j)} \cdot (\mathbf{E}S_{1,n}^{(j)})^{-3} \to \hat{\sigma}_j^2 > 0, \; j = 0, 1, 2, \ldots, k. \tag{1}$$

For simplicity, we omit the index n in the sequel.

First, we investigate a modified multiphase queueing system (in which devices are working without time out) and a usual multiphase queueing system. Limiting distributions for these queueing systems in heavy traffic conditions are the same (see, for example, [2]).

Thus,

$$\frac{Q_j(nt) - \hat{Q}_j(nt)}{\sqrt{n}} \Rightarrow 0, \;\; j = 1, 2, \ldots, k. \tag{2}$$

In [3], the relations

$$Q_j(t) = \tau_{j-1} - \tau_j(t), \tag{3}$$
$$Q_j(t) = f_t(\tau_{j-1}(\cdot) - x_j(\cdot)), \tag{4}$$

are obtained for $(j = 1, 2, \ldots, k)$ and $f_t(x(\cdot)) = x(t) - \inf_{0 \leq s \leq t} x(s)$.

Next, using (3) - (4), we obtain that

$$\hat{V}_j(t) = e(t) - \tau_j(t), \tag{5}$$

$$\hat{V}_j(t) = \hat{x}_j(t) - \inf_{0\le s\le t}(\hat{x}_j(s) - \hat{V}_{j-1}(s)), \tag{6}$$

$j = 1, 2, \ldots, k,\ \ V_0(\cdot) \equiv 0.$

2 Main Results

Let us investigate the case, where

$$(\beta_{j-1} - \beta_j)\cdot\sqrt{n} \to A_j < \infty,\ \ j = 1, 2, \ldots, k. \tag{7}$$

Finally we will prove such a functional limit theorem.

Theorem 1. *If conditions (1) and (7) are fulfilled, then*

$$\left(\frac{V_1(nt)}{\sqrt{n}}; \frac{V_2(nt)}{\sqrt{n}}; \ldots; \frac{V_k(nt)}{\sqrt{n}}\right) \Rightarrow (V_1(t); V_2(t); \ldots; V_k(t)),$$

where $V_j(t)$ are fulfilled equation

$$V_j(t) = \sigma_j\cdot z_j(t) - \inf_{0\le s\le t}(\sigma_j\cdot z_j(s) - V_{j-1}(s) + A_j\cdot s),$$

$z_j(t)$ are independent standard Wiener processes, $j = 1, 2, \ldots, k,\ V_0(\cdot) \equiv 0,\ 0 \le t \le 1.$

Proof. Denote families of random functions in D as follows:

$$V_j^n(t) = \frac{V_j(nt) - \alpha_j\cdot n\cdot t}{\sqrt{n}},\ \ \hat{V}_j^n(t) = \frac{\hat{V}_j(nt) - \alpha_j\cdot n\cdot t}{\sqrt{n}},$$

$$\hat{X}_j^n(t) = \frac{\hat{x}_j(nt) - \alpha_j\cdot n\cdot t}{\sqrt{n}},\ j = 1, 2, \ldots, k,\ \ 0 \le t \le 1.$$

So from (6) we get that

$$\begin{aligned}\hat{V}^n(t) &= \hat{X}_j^n(s) - \frac{\inf\limits_{0\le s\le nt}(\hat{x}_j(s) - \hat{V}_{j-1}(s))}{\sqrt{n}} = \hat{X}_j^n(s) - \frac{\inf\limits_{0\le s\le t}(\hat{x}_j(ns) - \hat{V}_{j-1}(ns))}{\sqrt{n}} = \\ &= \hat{X}_j^n(s) - \inf_{0\le s\le t}(\hat{X}_j^n(s) - \hat{V}_{j-1}^n(s) - (\beta_{j-1} - \beta_j)\cdot\sqrt{n}\cdot s), j = 1, 2, \ldots, k\end{aligned} \tag{8}$$

and $0 \le t \le 1$.

But

$$\hat{X}_j^n(t) \Rightarrow \sigma_j\cdot z_j(t), \tag{9}$$

where $z_j(t)$ are standard Wiener processes, $j = 1, 2, \ldots, k,\ 0 \le t \le 1$ (see, for example, [2]).

Applying (6), (8), and (9), we get the proof of the theorem.

□

References

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