ON CONTROLLED BRANCHING PROCESSES WITH IMMIGRATION

Ya.M. Khusanbaev
Institute of Mathematics, Uzbek Academy of Sciences, Tashkent, UZBEKISTAN
e-mail: yakubjank@mail.ru

Abstract

In the work is proved convergence of a branching process with immigration to the solution of a differential equation.

Let for each $n \in \mathbb{N}$, $\left\{\xi_{k,j}^{(n)}(l), l, k, j \in \mathbb{N}\right\}$ and $\left\{\varepsilon_k^{(n)}(l), l, k \in \mathbb{N}\right\}$ be independent totalities of independent nonnegative integer random variables such that distributions of $\xi_{k,j}^{(n)}(l)$ and $\varepsilon_k^{(n)}(l)$ don't depend on k, j. Let $\eta_0^{(n)}$ be a given nonnegative integer random number. For each $n \in \mathbb{N}$ we define the process $X_k^{(n)}$, $k \geq 0$ by the following recurrent relations

$$X_0^{(n)} = \eta_0^{(n)}, \quad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)}\left(X_{k-1}^{(n)}\right) + \varepsilon_k^{(n)}\left(X_{k-1}^{(n)}\right). \tag{1}$$

Such defined process is called the Galton-Watson branching process with state-dependent immigration or controlled branching process with immigration. Suppose that variables $\eta_0^{(n)}$, $\xi_{k,j}^{(n)}(l)$ and $\varepsilon_k^{(n)}(l)$ have finite second moments and denote

$$m_n(x) = \mathbf{E}\xi_{k,j}^{(n)}(x), \quad \lambda_n(x) = \mathbf{E}\varepsilon_k^{(n)}(x), \quad \sigma_n^2(x) = \mathbf{var}\xi_{k,j}^{(n)}(x), \quad b_n^2(x) = \mathbf{var}\varepsilon_k^{(n)}(x).$$

Process (1) is said to be nearly critical if $m_n(x) \to 1$ as $n \to \infty$. Later, let a_n, $n \in \mathbb{N}$ be a sequence of positive non-random numbers. Let $\mathcal{F}_k^{(n)} = \sigma\left\{X_0^{(n)}, X_1^{(n)}, \ldots, X_k^{(n)}\right\}$ be the σ-algebra generated by $X_0^{(n)}, X_1^{(n)}, \ldots, X_k^{(n)}$. Define the step process $X_n(t)$, $t \geq 0$ with trajectories in the Skorokhod space $\mathcal{D}(\mathbb{R}_+)$ by the following rule

$$X_n(t) = a_n X_{[nt]}^{(n)}, \quad t \in \mathbb{R}_+$$

where $[\cdot]$ means the integer part. Below, if we don't specify the other, the limit passage is realized by $n \to \infty$.

Theorem 1. *Let the following conditions hold:*

A) $m_n(x) = 1 + \frac{\alpha_n(x)}{n}$ *where a function* $\alpha_n(x)$ *is such that* $\alpha_0 = \sup_{x,n} |\alpha_n(x)| < \infty$;

B) 1) functions $\widetilde{\alpha}_n(x) = \alpha_n\left(a_n^{-1}x\right)$ *and* $\widetilde{\lambda}_n(x) = n a_n \lambda_n\left(a_n^{-1}x\right)$ *are such that for any* $L \geq 0$ *and* $0 \leq x, y \leq L$ *there exists* C_L *such that*

$$\left|\widetilde{\lambda}_n(x) - \widetilde{\lambda}_n(y)\right| + |\widetilde{\alpha}_n(x) - \widetilde{\alpha}_n(y)| \leq C_L |x - y|,$$

2) $\widetilde{\lambda}_n(x) \le C(1+x),\ x \ge 0$;

C) there exist sequences of numbers σ_n^2, b_n^2 *and* λ_n *such that* $\sigma_n^2(x) \le \sigma_n^2$, $b_n^2(x) \le b_n^2$, $\lambda_n(x) \le \lambda_n$ *for all* $x \ge 0$, *and also* $na_n\sigma_n^2 \to 0$, $na_n^2 b_n^2 \to 0$, $\overline{\lim\limits_{n\to\infty}}\, na_n\lambda_n < \infty$;

D) $\widetilde{\alpha}_n(x) \to \alpha(x)$, $\widetilde{\lambda}_n(x) \to \lambda(x)$;

F) $a_n\eta_0^{(n)} \xrightarrow{P} \eta_0$ *where* η_0 *is a finite random variable, moreover* $\overline{\lim\limits_{n\to\infty}}\, \mathbf{E} a_n\eta_0^{(n)} < \infty$.

Then for any $T > 0$

$$\sup_{0\le t\le T} |X_n(t) - \eta(t)| \xrightarrow{P} 0$$

where a process $\eta(t)$ *is the solution of the differential equation*

$$d\eta(t) = (\alpha(\eta(t))\,\eta(t) + \lambda(\eta(t))\,dt$$

with the initial condition $\eta(0) = \eta_0$.

Note that if $\xi_{k,j}^{(n)}(x)$ and $\varepsilon_k^{(n)}(x)$ don't depend on x, $a_n = n^{-1}$ and $\eta(0)^{(n)} = 0$, then the obtained result is compatible with the result of theorem 2.1 in [3].

Proof. Rewrite equation (1) in the form

$$X_k^{(n)} = X_{k-1}^{(n)} + \left(m_n\left(X_{k-1}^{(n)}\right) - 1\right) X_{k-1}^{(n)} + \lambda_n\left(X_{k-1}^{(n)}\right) + M_k^{(n)} \qquad (2)$$

where

$$M_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)}\left(X_{k-1}^{(n)}\right) - m_n\left(X_{k-1}^{(n)}\right)\right) + \varepsilon_k^{(n)}\left(X_{k-1}^{(n)}\right) - \lambda_n\left(X_{k-1}^{(n)}\right).$$

Evidently, $M_k^{(n)}$ form the martingale-difference concerning to the stream $\left\{\mathcal{F}_k^{(n)}, k \ge 0\right\}$. Set $\eta_{nk} = a_n X_k^{(n)}$, $k \ge 0$. Write relation (2) in the form

$$\eta_{nk} = \eta_{nk-1} + \left(\widetilde{\alpha}_n(\eta_{nk-1})\,\eta_{nk-1} + \widetilde{\lambda}_n(\eta_{nk-1})\right) \cdot \frac{1}{n} + a_n M_k^{(n)}. \qquad (3)$$

If we now prove for any $T > 0$

$$\sup_{0\le t\le T} \left| a_n \sum_{k=1}^{[nt]} M_k^{(n)} \right| \xrightarrow{P} 0, \qquad (4)$$

then we obtain by conditions of theorem and according to theorem 3.1 [1]

$$\max_{1\le k\le nT} |\eta_{nk} - Z_{nk}| \xrightarrow{P} 0$$

where Z_{nk} satisfy the formula

$$Z_{nk} = Z_{nk-1} + \left(\widetilde{\alpha}_n(Z_{nk-1}) \cdot Z_{nk-1} + \widetilde{\lambda}_n(Z_{nk-1})\right) \cdot \frac{1}{n}\ .$$

Applying further theorem 3.2 [1] we obtain

$$\sup_{0\le t\le T}|Z_n(t)-\eta(t)|=\max_{1\le k\le nT}\left|Z_{nk}-\eta\left(\frac{k}{n}\right)\right|\xrightarrow{P}0$$

where $Z_n(t)=Z_{n[nt]}$. Then

$$\sup_{0\le t\le T}|X_n(t)-\eta(t)|\le\max_{1\le k\le nT}|\eta_{nk}-Z_{nk}|+\max_{1\le k\le nT}\left|Z_{nk}-\eta\left(\frac{k}{n}\right)\right|\xrightarrow{P}0,$$

as was to be proved . Therefore we prove (4). For this it is sufficient to prove

$$a_n\sum_{k=1}^{[nt]}M_k^{(n)}\xrightarrow{P}0 \tag{5}$$

and

$$a_n^2\sum_{k=1}^{[nt]}\mathbf{E}\left(\left(M_k^{(n)}\right)^2\Big/\mathcal{F}_k^{(n)}\right)\to 0. \tag{6}$$

Really, if (6) takes place, then the Lindeberg condition holds for the martingale-differences $M_k^{(n)}$, $k\ge 1$ as for any $\varepsilon>0$

$$a_n^2\sum_{k=1}^{[nt]}\mathbf{E}\left(\left(M_k^{(n)}\right)^2 I\left(a_n\left|M_k^{(n)}\right|>\varepsilon\right)\Big/\mathcal{F}_k^{(n)}\right)\le a_n^2\sum_{k=1}^{[nt]}\mathbf{E}\left(\left(M_k^{(n)}\right)^2\Big/\mathcal{F}_k^{(n)}\right)\xrightarrow{P}0,$$

here $I(A)$ is the indicator of the event A. We obtain from here, (5), (6), and applying theorem 11.1.7 [2]

$$a_n\sum_{k=1}^{[nt]}M_k^{(n)}\xrightarrow{J}0$$

where $\xrightarrow{J}$ means weakly convergence in the Skorokhod J-topology. Since the limit process is continuous (It eqauls identically zero), then J-convergence implies U-convergence. (4) follows from these reasonings.

Let's prove (5). We have

$$\mathbf{E}\left(a_n\sum_{k=1}^{[nt]}M_k^{(n)}\right)^2\le a_n^2\sigma_n^2\sum_{k=1}^{[nt]}\mathbf{E}X_{k-1}^{(n)}+a_n^2[nt]\cdot b_n^2. \tag{7}$$

Now estimate $\mathbf{E}X_{k-1}^{(n)}$. (1) implies

$$\mathbf{E}X_k^{(n)}=\mathbf{E}X_{k-1}^{(n)}m_n\left(X_{k-1}^{(n)}\right)+\lambda_n\left(X_{k-1}^{(n)}\right).$$

Applying conditions A and C we obtain from here

$$\mathbf{E}X_k^{(n)}\le m_n\mathbf{E}X_{k-1}^{(n)}+\lambda_n$$

where $m_n = 1 + \dfrac{\alpha_0}{n}$. Solving this inequality we come to the relation

$$\mathbf{E}X_k^{(n)} \leq m_n^k \mathbf{E}X_0^{(n)} + \lambda_n \sum_{j=0}^{k-1} m_n^j \leq m_n^k \mathbf{E}X_0^{(n)} + \alpha_0^{-1} n (m_n^k - 1)\lambda_n. \tag{8}$$

If we apply the last relation and take into consideration that $\left(1 + \frac{\alpha_0}{n}\right)^{[nt]} \sim e^{\alpha_0 t}$ and $m_n \sim 1$ for sufficiently large n, then we obtain from (7)

$$a_n^2 \sum_{k=1}^{[nt]} \mathbf{E}\left(M_k^{(n)}\right)^2 \leq \alpha_0^{-1} \cdot n a_n \sigma_n^2 \cdot \mathbf{E}\left(a_n X_0^{(n)}\right)\left(e^{\alpha_0 t} - 1\right) +$$

$$+ \alpha_0^{-1} \cdot n a_n \sigma_n^2 \cdot n a_n \lambda_n \left(\alpha_0^{-1}\left(e^{\alpha_0 t} - 1\right) - t\right) + n a_n^2 b_n^2 t \to 0 \tag{9}$$

by virtue of C and F. Now applying the Chebyshev ineqaulity we obtain (5).

Let's prove (6). Taking into account C we have

$$a_n^2 \sum_{k=1}^{[nt]} \mathbf{E}\left(\left(M_k^{(n)}\right)^2 \Big/ \mathcal{F}_k^{(n)}\right) \leq a_n^2 \sigma_n^2 \sum_{k=1}^{[nt]} X_{k-1}^{(n)} + n a_n^2 b_n^2 \cdot t. \tag{10}$$

Similarly to reasonings as in (9) one can obtain

$$a_n^2 \sigma_n^2 \sum_{k=1}^{[nt]} \mathbf{E}X_{k-1}^{(n)} \to 0$$

what implies by the Chebyshev inequality

$$a_n^2 \sigma_n^2 \sum_{k=1}^{[nt]} X_{k-1}^{(n)} \xrightarrow{P} 0.$$

Then taking into consideration condition C we obtain from (10) relation (6). □

References

[1] Anisimov V.V., Lebedev E.A. Stochastic Service Nets. Markovian Models. Kiev: "Libid". 1992. 207 p. (in Russian)

[2] Liptser R.Sh., Shiryaev A.N. Martingales Theory. Moscow: Nauka, 1986. 512 p. (in Russian)

[3] Ispany M., Pap G., Van Zuijlen M.C.A. Fluctuation limits of branching processes with immigration and estimation of the means. // Adv. Appl.Probab. 2005, v.37, p.523-538.