ALMOST SURE VERSIONS OF LIMIT THEOREMS FOR RANDOM SUMS OF MULTIINDEX RANDOM VARIABLES

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Abstract

In the case of the domain of attraction of a p-stable law almost sure versions of limit theorems for random vectors are presented.

1 Introduction

We will suppose that $0 < p \leq 2$. Let's denote by $\xrightarrow{d}$ the convergence in distribution, by $\xrightarrow{w}$ the weak convergence of measures, by μ_ζ the distribution of the random variable ζ and by $\mathbf{R}$ the set of real numbers.

Let ζ_n, $n \in \mathbf{N}$, be a sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Consider the measures

$$Q_n(\omega) = Q_n((\zeta_n))(\omega) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\, \delta_{\zeta_k(\omega)},$$

where $\omega \in \Omega$, $n \in \mathbf{N}$ and δ_x is the point mass at x.

Classical limit theorems deal with the following convergence: $\zeta_n \xrightarrow{d} \zeta$, as $n \to \infty$. In many cases the convergence $\zeta_n \xrightarrow{d} \zeta$, as $n \to \infty$, implies the convergence of measures $Q_n(\omega) \xrightarrow{w} \mu_\zeta$, as $n \to \infty$, for almost all $\omega \in \Omega$. Such limit theorems are called almost sure versions of ordinary limit theorems. Investigations in this field started with Brosambler [3] and Schatte [8], who obtained an almost sure version of the central limit theorem. Then Berkes I. [1] and Ibragimov I.A. [5] generalized their results on the normalized sums of identically distributed random variables that belong to the domain of attraction of a p–stable law. Berkes I. and Csáki E. [2] showed that every weak limit theorem for random variables, subject to minor technical conditions, has an analogous almost sure version. Also a paper of Fazekas I. and Rychlik Z. [4] should be noted. There an almost sure version of the central limit theorem for the sums of multiindex random variables is presented.

Let ξ, ξ_n, $n \in \mathbf{N}$, be independent identically distributed random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, which belong to the domain of attraction of a p–stable law. It means that for some numerical sequence B_n, such that $B_n \longrightarrow \infty$, as $n \to \infty$, the following convergence takes place:

$$S_n \xrightarrow{d} \gamma_p,\ n \to \infty, \tag{1}$$

where $S_n = \frac{1}{B_n}\sum_{i=1}^n(\xi_i - \alpha_n)$, $\alpha_n = E\xi_1 \cdot I_{|\frac{\xi_1}{B_n}|<1}$ and γ_p is a p–stable random variable.

An almost sure version of the limit theorem (1) was obtained by Ibragimov I.A. in [5] and Berkes I. in [1]. Let's consider the sums of random variables with a random index of summation

$$S_n^\nu = \frac{1}{B_n}\sum_{i=1}^{\nu_n}(\xi_i - \alpha_n) = \frac{1}{B_n}\sum_{i=1}^{\infty}\left(\sum_{k=1}^{i}(\xi_k - \alpha_n)\right)\cdot I_{\{\nu_n = i\}}, \tag{2}$$

where $\nu_n, n \in \mathbf{N}$, is a sequence of integer-valued random variables, defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

Renyi in [7] investigated the convergence in distribution of the sequence in (2).

Consider the sequence of random vectors $V_n = (S_n, W_n)$, $n \in \mathbf{N}$, where $W_n = \frac{1}{B_n^2}\sum_{i=1}^n(\xi_i - \alpha_n)^2$. The following limit theorem is an isolated case of the theorem, obtained by Kruglov V. M. and Petrovskaya G.N. [6].

Theorem A *Let $S_n \xrightarrow{d} \gamma_p$, as $n \to \infty$, where γ_p is a p–stable random variable with the characteristic function*

$$f(t) = \exp\left\{\int_{-\infty}^{0}\left(e^{itx} - 1 - it\frac{x}{1+x^2}\right)d\left(\frac{c_1}{|x|^p}\right) + \int_{0}^{\infty}\left(e^{itx} - 1 - it\frac{x}{1+x^2}\right)d\left(-\frac{c_2}{x^p}\right)\right\},$$

where $t \in \mathbf{R}$, $c_1, c_2 \geq 0,\ c_1 + c_2 > 0$.

Then the sequence of distribution functions of random vectors (S_n, W_n), $n \in \mathbf{N}$, weakly converges to the distribution function with the characteristic function

$$f(s,t) = \exp\left\{\int_{-\infty}^{0}\left(e^{isx+itx^2} - 1 - is\frac{x}{1+x^2}\right)d\left(\frac{c_1}{|x|^p}\right) + \right.$$
$$\left. + \int_{0}^{\infty}\left(e^{isx+itx^2} - 1 - is\frac{x}{1+x^2}\right)d\left(-\frac{c_2}{x^p}\right)\right\}, \tag{3}$$

where $s, t \in \mathbf{R}$, $c_1, c_2 \geq 0,\ c_1 + c_2 > 0$.

Let $\mathbf{k} = (k_1, k_2, ..., k_d)$, $\mathbf{n} = (n_1, n_2, ..., n_d), ... \in \mathbf{N}^d$, $|\mathbf{n}| = \prod_{i=1}^d n_i$ and $|\log \mathbf{n}| = \prod_{i=1}^d \log_+ n_i$, $\mathbf{n} \in \mathbf{N}^d$, where $\log_+ x = \log x$, if $x \geq e$, and $\log_+ x = 1$, if $x < e$.

Let $\zeta_{\mathbf{n}}$, $\mathbf{n} \in \mathbf{N}^d$, be a sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Consider the measures $Q_{\mathbf{n}}(\omega) = Q_{\mathbf{n}}((\zeta_n))(\omega) = \frac{1}{|\log \mathbf{n}|}\sum_{\mathbf{k}\leq\mathbf{n}}\frac{1}{|\mathbf{k}|}\delta_{\zeta_{\mathbf{k}}(\omega)}$. The multiindex version of the ordinary almost sure limit theorem is the following:

$$Q_{\mathbf{n}}((\zeta_n))(\omega) \xrightarrow{w} \mu_\zeta,\ n \to \infty,$$

for almost all $\omega \in \Omega$.

Let $\xi_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{N}^d$, be the multiindex sequence of independent identically distributed random variables, which belong to the domain of attraction of a p–stable law. Theorem A remains valid in the case of the multiindex sequences $V_{\mathbf{n}} = (S_{\mathbf{n}},\ W_{\mathbf{n}})$, $\mathbf{n} \in \mathbf{N}^d$, where

$$S_{\mathbf{n}} = \frac{1}{B_{|\mathbf{n}|}}\sum_{\mathbf{i}\leq\mathbf{n}}(\xi_{\mathbf{i}} - \alpha_{|\mathbf{n}|}),\ \ W_{\mathbf{n}} = \frac{1}{B_{|\mathbf{n}|}^2}\sum_{\mathbf{i}\leq\mathbf{n}}(\xi_{\mathbf{i}} - \alpha_{|\mathbf{n}|})^2,$$

$\alpha_{|\mathbf{n}|} = E\xi_1 I_{\left|\frac{\xi_1}{B_{|\mathbf{n}|}}\right|<1}$, $B_{|\mathbf{n}|}$ is a numerical sequence, such that $B_{|\mathbf{n}|} \to \infty$, as $\mathbf{n} \to \infty$, and the convergence $S_{\mathbf{n}} \xrightarrow{d} \gamma_p$, as $\mathbf{n} \to \infty$, takes place.

Let $\nu_{\mathbf{n}} = (\nu_{1\mathbf{n}}, \nu_{2\mathbf{n}}, ..., \nu_{d\mathbf{n}})$, where $\nu_{1\mathbf{n}},\ \nu_{2\mathbf{n}}, ..., \nu_{d\mathbf{n}} : \Omega \longrightarrow \mathbf{N}$, be sequences of integer-valued random vectors.

Our aim is to generalize Theorem A to the case of 2-dimensional random vectors $(S^{\nu}_{\mathbf{n}}, W^{\nu}_{\mathbf{n}})$ with the coordinates $S^{\nu}_{\mathbf{n}} = \frac{1}{B_{|\mathbf{n}|}} \sum_{\mathbf{i} \le \nu_{\mathbf{n}}} (\xi_{\mathbf{i}} - \alpha_{|\mathbf{n}|})$ and $W^{\nu}_{\mathbf{n}} = \frac{1}{B^2_{|\mathbf{n}|}} \sum_{\mathbf{i} \le \nu_{\mathbf{n}}} (\xi_{\mathbf{i}} - \alpha_{|\mathbf{n}|})^2$, and to get an almost sure version of this result.

2 Main results

Below we formulate our first result providing the convergence in distribution of random vectors $V^{\nu}_{\mathbf{n}} = (S^{\nu}_{\mathbf{n}}, W^{\nu}_{\mathbf{n}})$.

Theorem 1. *Assume that* $\left(\frac{\nu_{1\mathbf{n}}}{n_1}, \frac{\nu_{2\mathbf{n}}}{n_2}, ..., \frac{\nu_{d\mathbf{n}}}{n_d}\right) \xrightarrow{d} (\nu_1, \nu_2, ..., \nu_d)$, *as* $\mathbf{n} \to \infty$, $\mathbf{n} \in \mathbf{N}^d$, $\{\nu_{\mathbf{n}}\}$ *and* $\{\xi_{\mathbf{n}}\}$ *are independent. Let* $S_{\mathbf{n}} \xrightarrow{d} \gamma_p$, *as* $\mathbf{n} \to \infty$, *where* γ_p *is a p–stable random variable.*

Then $V^{\nu}_{\mathbf{n}} \xrightarrow{d} V^{\nu}$, *as* $\mathbf{n} \to \infty$, *where* V^{ν} *is a random vector with the characteristic function*

$$f^{\nu}(s,t) = \int_0^{\infty} \int_0^{\infty} ... \int_0^{\infty} f^{|\mathbf{u}|}(s,t) d\mu_{\nu_1}(u_1)\, d\mu_{\nu_2}(u_2) ... d\mu_{\nu_d}(u_d), \tag{4}$$

and $f(s,t)$ *is defined by (3).*

The following theorem is an almost sure version of Theorem 1.

Theorem 2. *Assume that* $\nu_{\mathbf{n}}$ *is a sequence of independent random variables,* $\{\nu_{\mathbf{n}}\}$ *and* $\{\xi_{\mathbf{n}}\}$ *are independent,* $\left(\frac{\nu_{1\mathbf{n}}}{n_1}, \frac{\nu_{2\mathbf{n}}}{n_2}, ..., \frac{\nu_{d\mathbf{n}}}{n_d}\right) \xrightarrow{d} (\nu_1, \nu_2, ..., \nu_d)$, *as* $\mathbf{n} \to \infty$. *Let* $S_{\mathbf{n}} \xrightarrow{d} \gamma_p$, *as* $\mathbf{n} \to \infty$, *where* γ_p *is a p–stable random variable.*

Then for almost all $\omega \in \Omega$ *it holds that*

$$Q_{\mathbf{n}}((V^{\nu}_{\mathbf{n}}))(\omega) \xrightarrow{w} \mu_{V^{\nu}},\ \mathbf{n} \to \infty.$$

Corollary 1. *Assume that the conditions of Theorem 2 are valid.*

(a) Let $Q_{\mathbf{n}}(\omega) = Q_{\mathbf{n}}((S^{\nu}_{\mathbf{n}}))(\omega)$. *Then for almost all* $\omega \in \Omega$ *we have*

$$Q_{\mathbf{n}}((S^{\nu}_{\mathbf{n}}))(\omega) \xrightarrow{w} \mu_{\gamma^{\nu,1}_p},\ \mathbf{n} \to \infty.$$

(b) Let $Q_{\mathbf{n}}(\omega) = Q_{\mathbf{n}}((S^{\nu}_{\mathbf{n}}))(\omega)$. *Then for almost all* $\omega \in \Omega$ *we have*

$$Q_{\mathbf{n}}((W^{\nu}_{\mathbf{n}}))(\omega) \xrightarrow{w} \mu_{\gamma^{\nu,2}_p},\ \mathbf{n} \to \infty,$$

where $\gamma^{\nu,1}_p$ *and* $\gamma^{\nu,2}_p$ *are coordinates of the random vector* V^{ν} *from Theorem 2, which are defined by projections of the characteristic function (4) on the first and the second coordinate, respectively, i.e. by characteristic functions* $f(s,0)$ *and* $f(0,t)$.

now we will give an almost sure version of multiindex limit theorem for Student's statistics.

A function $h : \mathbf{R} \times \mathbf{R}_+ \longrightarrow \mathbf{R}$ is denoted in the following way. Let $h(x, y) = x/\sqrt{y}$ for $y \neq 0$, and $h(x, y) = 0$ for $y = 0$. Then we consider the sequence of self-normalized sums $T^\nu_{\mathbf{n}} = h(S^\nu_{\mathbf{n}}, W^\nu_{\mathbf{n}})$.

Theorem 3. *Let $\nu(\{0\}) = 0$, $Q_{\mathbf{n}}(\omega) = Q_{\mathbf{n}}((T^\nu_{\mathbf{n}}))(\omega)$. Then under the conditions of Theorem 2 for almost all $\omega \in \Omega$ we have*

$$Q_{\mathbf{n}}((T^\nu_{\mathbf{n}}))(\omega) \xrightarrow{w} \mu_{T^\nu}, \ \mathbf{n} \to \infty,$$

where μ_{T^ν} is an image of the measure μ_{V^ν} according to the mapping h.

We note that instead of the condition of independency of $\nu_{\mathbf{n}}$ in Theorem 2 it is possible to use the following condition: $\nu_{\mathbf{l}}$ and $\nu_{\mathbf{ln}}$, $\mathbf{l} < \mathbf{n}$, are independent, and $E\nu_{\mathbf{n}} \leq C\mathbf{n}$, where $C > 0$ is some constant, and $\nu_{\mathbf{n}} = \nu_{\mathbf{l}} + \nu_{\mathbf{ln}}$.

Also random sums almost sure limit theorems for the domain of geometric partial attraction of a semistable law and almost sure versions of limit theorems for geometric random sums have been obtained.

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