THE INVESTIGATION OF ESTIMATES OF CHARACTERISTICS OF RANDOM PROCESS WITH NON-REGULAR OBSERVATIONS

T.I. Varatnitskaya
Belarusian State University
Minsk, BELARUS
e-mail: iliukevich@bsu.by

Abstract

In this paper the amplitude modulated version of random prosess is investigated. There is considered the case when irregularities in observations are defined as arbitrary sequence of independent random values. In this case the estimations of mathematical expectation, covariance function and spectral density have been constructed. The statistical properties of the esimations have been studied.

1 Introduction

In different practical applications we frequently deal with stationary processes with non-regular observations. The estimations of main characteristics of the processes give common information about studied phenomenon. Asymptotic methods of time series' analysis allow to find limiting distribution of the estimations when the number of observations tends to infinity.

Parzen [6] introduced the sequence

$$Y(t) = d(t)X(t), t \in Z, \tag{1}$$

which is called an amplitude modulated version of $X(t)$. It is supposed that $d(t)$ doesn't depend on the $X(t)$ process. A number of examples of $d(t)$ have been considered in [1-5,7].

Let $X(t)$, $t \in Z$ be stationary independent random process. This process has mathematical expectations $m^X = 0$, covariance functions $R^X(\tau), \tau \in Z$ and spectral density $f^X(\lambda), \lambda \in \Pi = [-\pi, \pi]$. The sequence of independent random values is $d(t), t \in Z$. Mathematical expectations and dispersion of $d(t)$ are $m^d \neq 0$ and D^d respectively.

2 The estimation of mathematical expectation

Let

$$Y(0), Y(1), ..., Y(T-1) \tag{2}$$

be T consecutive in equal time period observations of the process $Y(t), t \in Z$. The relation between processes $X(t)$ and $Y(t)$ is given by (1).

Using observations (2) of the process $Y(t), t \in Z$ the estimate of mathematical expectation of process $X(t)$ can be constructed as

$$\widehat{m}^X = \frac{1}{Tm^d} \sum_{t=0}^{T-1} Y(t) \tag{3}$$

The given below theorems are correct for the estimate (3).

Theorem 1. *The statistics (3) is asymptotically unbiased estimate and*

$$D\widehat{m}^X = \frac{2}{T} \sum_{\tau=0}^{T-1} \left(1 - \frac{\tau}{T}\right) R^X(\tau) + \frac{R^X(0)D^d}{T(m^d)^2} + \frac{(m^X)^2(1-m^d)}{Tm^d},$$

$$D\widehat{m}^X = \frac{2\pi}{T} \int\limits_{\Pi} f^X(y)\Phi_T(y)dy + \frac{D^d}{T(m^d)^2} \int\limits_{\Pi} f^X(y)dy + \frac{(m^X)^2(1-m^d)}{Tm^d},$$

where $\Phi_T(\lambda)$ *is the Fejer kernel,* $\lambda \in \Pi$.

It is possible to prove this theorem using the independence $X(t)$ and $d(t)$, the representation of spectral density through the covariance function, the definition of the Fejer kernel.

Further taking into consideration the results of the theorem 1 and the properties of the Fejer kernel asymptotic behaviour $D\hat{m}^X$ is investigated when T tends to infinity.

Theorem 2. *If covariance function* $R^X(\tau)$ *meet the condition* $\sum\limits_{\tau=0}^{\infty} R^X(\tau) < \infty$ *then*

$$\lim_{T\to\infty} TD\widehat{m}^X = 2\sum_{\tau=0}^{\infty} R^X(\tau) + \frac{R^X(0)D^d}{(m^d)^2} + \frac{(m^X)^2(1-m^d)}{m^d}.$$

If spectral density $f^X(\lambda)$*is continuous at point* $\lambda = 0$ *and bounded in* Π *then*

$$\lim_{T\to\infty} TD\widehat{m}^X = 2\pi f^X(0) + \frac{D^d}{(m^d)^2} \int\limits_{\Pi} f^X(y)dy + \tfrac{(m^X)^2(1-m^d)}{m^d}.$$

Note that in the conditions of the theorem 2 the statistics (3) is mean-square consistent.

3 The estimation of covariance function

The statistics

$$\widehat{R}^X(\tau) = \frac{1}{(T-\tau)C_\tau^d} \sum_{t=0}^{T-\tau-1} Y(t+\tau)Y(t), \tau = \overline{0, T-1}, \tag{4}$$

$$\widehat{R}^X(\tau) = \widehat{R}^X(-\tau), \widehat{R}^X(\tau) = 0, |\tau| > T,$$

where

$$C_\tau^d = \begin{cases} (m^d)^2, \tau \neq 0 \\ D^d + (m^d)^2, \tau = 0, \end{cases} \tag{5}$$

is considered as estimate of covariance function of process $X(t)$.

It was proved that the dispersion of estimate (4) meets the following expression

$$\lim_{T\to\infty} (T-\tau) D\widehat{R}^X(\tau) = \left(\frac{(D^d + (m^d)^2)^2}{(m^d)^4} - 1 \right) \left(c_4^X(\tau, 0, \tau) + 2\left(R^X(\tau)\right)^2 + \left(R^X(0)\right)^2 \right) +$$

$$+ \sum_{u=-\infty}^{\infty} J(u, \tau),$$

where

$$J(u,\tau) = c_4^X(u+\tau, u, \tau) + 2\left(R^X(u)\right)^2 + R^X(u-\tau)R^X(u+\tau).$$

The $c_4^X(u+\tau, u, \tau)$ is semi-invariant of process $X(t)$.

Theorem 3. *The estimate of covariance function of process $X(t)$ (4) is asymptotically unbiased estimator and on conditions that*

$$\sum_{u=-\infty}^{\infty} \left(R^X(u)\right)^2 < \infty$$

and

$$\sum_{u=-\infty}^{\infty} c_4^X(u+\tau, u, \tau) < \infty$$

is mean-square consistent, that is $\lim_{T\to\infty} D\widehat{R}^X(\tau) = 0$.

The proof of this theorem is based on the definition of matematical expectation and dispersion, the relations between moments and semi-invariants and the properties of the Fejer kernel.

4 The estimation of spectral density

Here the problem of construction of spectral density's estimate is investigated. It is proposed to consider the following statistics.

$$\widehat{I}^T(\lambda) = \frac{1}{2\pi T} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} \frac{Y(t)Y(s)}{C_{t-s}^d} e^{-i\lambda(t-s)} \lambda \in \Pi \quad , \tag{6}$$

where C_{t-s}^d are defined as (5).

Theorem 4. *Let semi-invariant spectral density of fourth order $f_4^X(\lambda_1, \lambda_2, \lambda_3)$ to be continuous on Π^3 and spectral density $f^X(\lambda)$ to be continuous on Π, then statistics defined as (6) is asimptoticaly unbiased estimate for $f^X(\lambda)$ and*

$$cov\left\{\widehat{I}^T(\lambda_1), \widehat{I}^T(\lambda_2)\right\} \underset{T\to\infty}{\longrightarrow} \begin{cases} 0, \lambda_1 \pm \lambda_2 \neq 0(mod 2\pi), \\ f^X(\lambda_1) f^X(\lambda_2), \lambda_1 \pm \lambda_2 = 0(mod 2\pi). \end{cases}$$

It is easy to receive the proof of this theorem using the definition of covariation, the representation of spectral density and semi-invariant spectral density of fourth order through the covariance function and semi-invariant of fourth order, the properties of the Fejer kernel.

To get the consistent estimate of spectral density $\widehat{f}^T$ it is necessary to smooth this estimate using spectral windows $\varphi^T(k)$.

$$\widehat{f}^T(\lambda_s) = \sum_{k=-\left[\frac{T}{2}\right]+1}^{\left[\frac{T}{2}\right]} \varphi^T(k)\widehat{I}^T(\lambda_{s+k}), \tag{7}$$

$\lambda_s = \frac{2\pi s}{T}$, $-\left[\frac{T}{2}\right]+1 \leq s \leq \left[\frac{T}{2}\right]$, $\left[\frac{T}{2}\right]$ is integer part of number $\frac{T}{2}$.

Theorem 5. *If semi-invariant spectral density of fourth order $f_4^X(\lambda_1, \lambda_2, \lambda_3)$ is continuous on Π^3, spectral density $f^X(\lambda)$ is continuous on Π and* $\sum_{k=-\left[\frac{T}{2}\right]+1}^{\left[\frac{T}{2}\right]} \left[\varphi^T(k)\right]^2 \underset{T\to\infty}{\longrightarrow} 0$ *then statistics defined as (7) is mean-square consistent estimate.*

The proof is based on the previous theorem.

References

[1] Bloomfield P. (1970). Spectral analysis with randomly missed observation. *J. R. Stat. Soc.* Ser. B Stat. Methodol. Vol. **32**, pp. 369-380.

[2] Dunsmuir W., Robinson P.M. (1981). Asymptotic theory for time series containing missing and amplitude modulated observations. *Sankhya*. Ser. A 43, pp. 260-281.

[3] Iliukevich (Varatnitskaya) T.I. (2005). The investigation of the estimation of covariance function of stationary random processes with non-regular observations. *Proceedings of the international conference probability theory, random processes, mathematical statistics and aplications, Minsk, February 21-25.* pp. 103-108(in Russian).

[4] Jiancheng Jiang, Hui Y.V. (2004). Spectral Density Estimation with Amplitude Modulation and Outlier Detection. *Ann. Inst. Statist. Math.* Vol. **56**, No. 4, pp. 611-630.

[5] Myung Sook Lee (2004). Strong Consistency for AR Model with Missing Data. *J. Korean Math. Soc.* Vol. **41**, No. 6, pp. 1071-1086.

[6] Parzen E. (1962). On spectral analysis with missing observations and amplitude modulation. *Tech. Rep.* Vol. **46**.

[7] Troush N.N., Iliukevich (Varatnitskaya) T.I. (2003). The estimation of mathematical expectation of stationary random processes with non-regular observations. *Vestnik BSU.* Vol. **3**, pp. 49-52(in Russian).