SOME PROPERTIES OF ABSOLUTE ORDER p OF α-STABLE RANDOM VARIABLES

N.N. Troush, L.H. Son
Belarusian State University, Minsk, Belarus
e-mail: TroushNN@bsu.by, lhsondhv@gmail.com

Abstract

In this paper, we introduce the representation of density functions in the form of convergent series and point out some properties of absolute order p of α-stable random variables.

1 Introduction

Stable distributions are a rich class of probability distributions, the lack of closed formulas for densities and distribution functions is sometimes a nuisance when working with stable distributions. In practice, all numerical calcutations of densities are based on other representations of density functions. There are three basic ones that are used: integrals of non-oscillating functions [5, 6], convergent series and asymptotic series [1, 3, 4]. In this paper, we introduce the representation of density functions in the form of convergent series and point out some properties of absolute order p of α-stable random variables.

2 Main result

Let us remind first the definition of α - stable random variables. A random variable X is α - stable, $0 < \alpha < 2$ if its characteristic function has the form

$$\varphi_X(t) = \begin{cases} exp\{-\delta^\alpha |t|^\alpha exp[-i\beta sign(t) K(\alpha)] + i\mu t\} & \text{if } \alpha \neq 1 \\ exp\{-\delta |t| [\frac{\pi}{2} + i\beta sign(t) \ln |t|] + i\mu t\} & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where $K(\alpha) = \alpha - 1 + sign(1-\alpha)$, $\delta > 0$, $\beta \in [-1; 1]$, $\mu \in R$. In this case, we denote $X \sim S_\alpha(\delta, \beta, \mu)$. If $q_X(x, \alpha, \beta)$ is denoted for density function of $X \sim S_\alpha(\delta, \beta, 0)$, then by [1], we have

$$q_X(x, \alpha, \beta) = q_X(-x, \alpha, -\beta). \tag{2}$$

Therefore, it is sufficient to represent $q_X(x, \alpha, \beta)$ while $x > 0$. In fact, the probability density functions of stable random variables exist and are continuous, but with a few exceptions, they are not known in a closed form. However, power series expansions can be derived for $q_X(x, \alpha, \beta)$. The following represention of density functions of α - stable random variables X is given by [1, 4]

$$q_X(x, \alpha, \beta) = \begin{cases} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(\alpha n + 1) sin(n\rho\pi) x^{-n\alpha - 1} & \text{if } 0 < \alpha < 1 \\ \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(\frac{n}{\alpha} + 1) sin(n\rho \frac{\pi}{\alpha}) x^{n-1} & \text{if } 1 < \alpha < 2, \end{cases} \tag{3}$$

where point out that $\rho = \frac{\gamma+\alpha}{2}$, $\gamma = \beta K(\alpha)$, $x > 0$.

The case of $\alpha = 1$, $\beta \neq 0$, [3] shows that, density functions of 1 - stable random variables are represented in the form

$$q_X(x, 1, \beta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1} n b_n(\beta) x^{n-1},$$

where

$$b_n(\beta) = \frac{1}{\Gamma(n+1)}\int_0^{\infty} exp\{-\beta t \ln t\} t^{n-1} sin[(1+\beta)t\frac{\pi}{2}]dt.$$

From [2], we know that, if X is α - stable then its absolute moment order p exists if and only if $0 < p < \alpha < 2$. The cloused formula of absolute moment order p of X, $0 < p < \alpha < 2$, is given by the following theorem.

Theorem 1. *If $X \sim S_\alpha(1, \beta, 0)$, then absolute moment order p of X, $0 < p < \alpha < 2$, is*

$$E|X|^p = \frac{cos(\frac{\gamma p \pi}{2\alpha})\Gamma(1 - \frac{p}{\alpha})}{cos(\frac{p\pi}{2})\Gamma(1-p)}. \tag{4}$$

Proof. From (1), when $\alpha \neq 1$ and $x > 0$, we have

$$\begin{aligned} q_X(x, \alpha, \beta) = q_X(-x, \alpha, -\beta) &= \frac{1}{\pi} Re \int_0^{\infty} e^{itx}\varphi_X(t)dt \\ &= \frac{1}{\pi} Re \int_0^{\infty} e^{itx} exp\{-t^\alpha exp[i\gamma\frac{\pi}{2}]\}dt \\ &= \frac{1}{\pi} Im \int_0^{\infty} exp\{-tx - t^\alpha exp(-i\rho\pi)\}dt. \end{aligned}$$

Therefore, using Jordan lemma we have

$$\begin{aligned} I(p, \alpha, \gamma) &= \int_0^{\infty} x^p q_X(x, \alpha, \beta)dx = \frac{1}{\pi}\int_0^{\infty} x^p Im \int_0^{\infty} exp\{-tx - t^\alpha exp(-i\rho\pi)\}dtdx \\ &= \frac{1}{\pi}\int_0^{\infty} x^p e^{-tx}dx Im \int_0^{\infty} exp\{-t^\alpha exp(-i\rho\pi)\}dt \\ &= \frac{1}{\pi}\Gamma(p+1) Im \int_0^{\infty} t^{-(p+1)} exp\{-t^\alpha exp(-i\rho\pi)\}dt \\ &= \frac{1}{\pi\alpha}\Gamma(p+1) Im\{\frac{1}{[exp(-i\rho\pi)]^{-\frac{p}{\alpha}}}\int_0^{\infty}[z.exp(-i\rho\pi)]^{-\frac{p}{\alpha}-1} exp[-z.exp(-i\rho\pi)]dz.exp(-i\rho\pi)\} \\ &= \frac{1}{\pi\alpha}\Gamma(p+1) Im[exp(-i\rho\pi)]^{\frac{p}{\alpha}}\Gamma(-\frac{p}{\alpha}) \\ &= -\frac{1}{\pi\alpha}\Gamma(p+1) sin(\frac{\rho p \pi}{\alpha})\Gamma(-\frac{p}{\alpha}) = \frac{sin(\frac{\rho p\pi}{\alpha})\Gamma(1-\frac{p}{\alpha})}{sin(p\pi)\Gamma(1-p}. \end{aligned}$$

Using the above, we see that

$$
\begin{aligned}
E|X|^p &= \int_{-\infty}^{\infty} |x|^p q_X(x,\alpha,\beta)dx = \int_{-\infty}^{0} (-x)^p q_X(x,\alpha,\beta)dx + \int_{0}^{\infty} x^p q(x,\alpha,\beta)dx \\
&= \int_{0}^{\infty} x^p q_X(x,\alpha,\beta)dx - \int_{0}^{\infty} x^p q_X(-x,\alpha,\beta)dx \\
&= \int_{0}^{\infty} x^p q_X(x,\alpha,\beta)dx + \int_{0}^{\infty} x^p q_X(x,\alpha,-\beta)dx \\
&= I(p,\alpha,\gamma) + I(p,\alpha,-\gamma) = \frac{cos(\frac{\gamma p\pi}{2\alpha})\Gamma(1-\frac{p}{\alpha})}{cos(\frac{p\pi}{2})\Gamma(1-p)}.
\end{aligned}
$$

Because of the continuous property, formula (4) is also true in the case of $\alpha = 1$.
This completes the proof.

Stable densities are supported on either the whole real line or a half line. The latter situation can only occur when $\alpha < 1$ and $\beta = 1$ or $\beta = -1$. By the assessment of the support of the stable distribution in [7], we see that, if X is a α-stable random variable then for any p, $0 < p < \alpha < 2$, $|X|^p$ is not a stable random variable. The following theorem will give the density functions of $|X|^p$ in the form of convergent series.

Theorem 2. *Let $X \sim S_\alpha(1,\beta,0)$, then for all p, $p > 0$, the density function of $|X|^p$ has the form*

$$
q_{|X|^p}(x,\alpha,\beta) = \begin{cases} \frac{2}{p\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\Gamma(\alpha n+1)sin(n\frac{\alpha\pi}{2})cos(n\frac{\gamma\pi}{2})x^{-\frac{n\alpha}{p}-1} & if\ 0<\alpha<1 \\ \frac{2}{p\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\Gamma(\frac{2n+1}{\alpha}+1)cos(\frac{(2n+1)\gamma\pi}{2\alpha})x^{\frac{2n+1}{p}-1} & if\ 1<\alpha<2, \end{cases} \tag{5}
$$

where $x > 0$.

Proof. For $x > 0$, we have

$$
P(|X|^p < x) = \int_{-x^{1/p}}^{x^{1/p}} q(t,\alpha,\beta)dt.
$$

Thus the density function of $|X|^p$ will be

$$
q_{|X|^p}(x) = \frac{1}{p}q(x^p,\alpha,\beta)x^{1/p-1} + \frac{1}{p}q(x^p,\alpha,\beta)x^{1/p-1}.
$$

The case of $0 < \alpha < 1$, with $\rho' = \frac{-\gamma+\alpha}{2}, by(3)$ we have

$$
\begin{aligned}
q_{|X|^p}(x) &= \frac{1}{\pi p}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\Gamma(\alpha n+1)[sin(n\rho\pi)+sin(n\rho'\pi)]x^{-\frac{n\alpha}{p}-1} \\
&= \frac{2}{\pi p}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\Gamma(\alpha n+1)sin(n\frac{\alpha\pi}{2})cos(n\frac{\gamma\pi}{2})x^{-\frac{n\alpha}{p}-1}.
\end{aligned}
$$

The same procedure is for the case of $1 < \alpha < 2$.
This completes the proof.

Corollary. *If $X \sim S_\alpha(\delta, \beta, 0)$, then*

$$|X|^\alpha \xrightarrow{d} Y^{-1}, \qquad \alpha \longrightarrow 0^+,$$

where Y has the exponential distribution.

Proof. We consider the case of $0 < \alpha < 1$, from (5) with $p = \alpha$, we have

$$\begin{aligned} q_{|X|^\alpha}(x) &= \frac{2}{\pi\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\Gamma(\alpha n+1)sin(n\frac{\alpha\pi}{2})cos(n\frac{\gamma\pi}{2})x^{-n-1} \\ &= \frac{1}{x^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}\Gamma(\alpha n+1)\frac{sin(n\frac{\alpha\pi}{2})}{(n\frac{\alpha\pi}{2})}cos(n\frac{\gamma\pi}{2})x^{-n+1} \\ &\sim \frac{1}{x^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^{-n+1} = \frac{1}{x^2}e^{-\frac{1}{x}}, \end{aligned} \tag{6}$$

when $\alpha \longrightarrow 0^+$.

Moreover, we know that the distribution function of Y^{-1} is

$$P(Y^{-1} < x) = P(Y > \frac{1}{x}) = e^{-\frac{1}{x}}, \quad x > 0.$$

Hence

$$q_{|Y|^{-1}} = \frac{1}{x^2}e^{-\frac{1}{x}}, \quad x > 0. \tag{7}$$

Thus from (6) and (7) infer the proof of corollary.

References

[1] Uchaikin V.V., Zolotarev V.M. (1999). *Change and stability: Stable distributions and their applications, Series modern probab and statist*. Utrecht, CSP, p103-109.

[2] Feller J. (1966). *An introduction to probability theory and its applications*. Vol 2, Wiley, New York, p 312-320.

[3] Zolotarev V.M. (1986). *One dimensional Stable Distributions*. Amer. Math. Soc., Providence, RI.

[4] Tsakalides P. (1995). *Array signal processing with alpha-stable distributions, Dr. Thesis, university of Southern California*, USA, p 26-29.

[5] Weron R. (1996). On the Chambers-Mallows-Stuck method for simulating skewed stable random variables. *Statist. Probab.* Lett. 28, p 165-171.

[6] Borak S., Hgrdle W., Weron. R. (2005). Stable distribution, *SFB 649, Discussion paper, 2005 - 008.* "Economic Risk", Berlin, p 1-25.

[7] Nolan J. P. (2005). Stable distribution models for heavy tailted data, *American University processed January 11, 2005*, p 5-8.