WAVELET-BASED JUMP DETECTION IN TIME SERIES WITH HEAVY-TAILED NOISE

M.S. Abramovich, M.M. Mitskevich
Belarusian State University
Minsk, BELARUS
e-mail: `Abramovichms@bsu.by`

Abstract

Wavelet-based robust tests for jump detection in time series with heavy-tailed noise are proposed. An efficiency of these tests is analyzed by statistical modelling.

1 Introduction

Wavelet analysis is widely used for jump detection in time series observed with noise [1, 4, 5]. Usage of an wavelet transformation for jump detection is based on the maximality of absolute values of wavelet coefficients in jump times. By checking the absolute values of wavelet coefficients we can detect jumps even in the presence of additive noise. In parametric tests for jump detection the Gaussian distribution of the noise is usually assumed [1]. But in practice the noise distibution is often different from the Gaussian, for example the Student t_3 - noise heavy-tailed distibution with finite variance is often used.

The main goal of this paper is to build robust tests for jump detection in time series.

2 Mathematical model

Let $T = 2^M$ and $X(t)$ be a time series of the following form:

$$X(t) = f(t) + z_t, \qquad f(t) = \begin{cases} \mu, \ 0 \le t \le t_0 - 1, \\ \mu + \tau, \ \ t \ge t_0, \end{cases}, \qquad t = 0, ..., T-1. \tag{1}$$

Here z_t are independent identically distributed (iid) random variables (rv's) with zero mean and a finite variance σ^2, τ is the jump of the mean μ at unknown time moment t_0. We suppose that tailes of z_t distribution are more heavy than ones of the Gaussian distribution.

Let a hypothesis H_0 consist in $\tau = 0$ and an alternative hypothesis $H_1 = \overline{H_0}$ consist in $\tau \neq 0$.

Discrete wavelet transformation (DWT) of the time series $X(t)$ is defined as follows:

$$d_{j,k}^{(\psi)} = \sum_{t=0}^{T-1} X(t)\psi_{j,k}(t), \ j = 1, \ldots, M, \ k = 0, \ldots, 2^{M-j} - 1, \tag{2}$$

where $\psi_{j,k}(t) = 2^{-\frac{j}{2}}\psi(2^{-j}t - k)$, $\psi(t)$ is a basic wavelet, j is a scale parameter (so-called resolution level), k is a shift parameter. We use the Haar wavelet basis:

$$\psi_{j,k}(x) = \begin{cases} 2^{-j/2},\ 2^j k \le x \le 2^j (k + 1/2), \\ -2^{-j/2},\ 2^j (k+1/2) \le x \le 2^j (k+1), \\ 0, \quad otherwise. \end{cases}$$

It is easy see that for this basis $E\left\{d_{j,k}^{(\psi)}\right\} = 0$ and $D\left\{d_{j,k}^{(\psi)}\right\} = \sigma^2$.

In [4] were shown that the Haar wavelet coefficients (2) are independent on each resolution level. Note that coefficients (2) are identically distributed on each resolution level as borelic functions of identically distributed random values.

3 Jump detection tests

A. The first proposed test for jump detection is based on the assumption that the wavelet coefficients of the time series (1) exceed some threshold in the jump time t_0.

In [5] were shown that if Y_i, $i = 0, \dots, n-1$, are iid rv's, then

$$P((Y_i - u)_+ \le x |\, Y_i > u) = 1 - P((Y_i - u)_+ \ge x |\, Y_i > u) \approx H(x; \sigma, \gamma)$$

for all $i = 0, \dots, n-1$ and sufficiently large u. Here $(x-u)_+ = \max(x-u,\, 0)$ and

$$H(x; \sigma, \gamma) = \begin{cases} 1 - (1 - \gamma x/\sigma)^{1/\gamma},\ if\ \gamma \ne 0, \\ 1 - e^{-x/\sigma},\ if\ \gamma = 0, \end{cases}$$

is the cumulative Generalized Pareto Distribution (GPD) function with parameters γ and σ [3].

Let $q_{(j,1)} \ge q_{(j,2)} \ge \dots \ge q_{(j,2^{M-j}-1)}$ be ordered absolute values of the wavelet coefficients $q_{j,k} = \left|d_{j,k}^{(\psi)}\right|$, $k = 0, \dots, 2^{M-j} - 1$, and let $u_j = (q_{(j,2^{M-j-1}-1)} + q_{(j,2^{M-j-1})})/2$ be the corresponding median for each resolution level $j = 1, \dots, M$. Define the statistics $T_{j,l} = q_{(j,l)} - u_j$ and choose some thresholds $C_{j,l}$, $l = 0, \dots, 2^{M-j-1} - 1$. For each resolution level j individual hypotheses H_{0j} and H_{1j} are tested in the following way: H_{0j} is accepted if $T_{j,l} \le C_{j,l}$ for all $l = 0, \dots, 2^{M-j-1} - 1$ and rejected otherwise. The union hypothesis H_0 is accepted, if all H_{0j}, $j = 1, \dots, M$, are accepted, and rejected otherwise.

Let α be a significance level of the test. The significance level α^* for the resolution level must be chosen such that $\alpha^* \le \alpha/M$ (see [2]). It should be noted that the condition $\sum_{l=0}^{2^{M-j-1}-1} P(T_{j,l} > C_{j,l}) = \alpha^*$ holds due to independence of the wavelet coefficients on each resolution level. We calculate the significance level for individual hypothesis H_{0j}.

Let $C_{j,l} = H^{-1}\left(\left(1 - \frac{\alpha^*}{2^{M-j-1}}\right)^{\frac{1}{2^{M-j-1}-l}}\right)$, where

$$H^{-1}(x) = \begin{cases} \frac{\sigma}{\gamma}\left(1 - (1-x)^{\gamma}\right), \quad \gamma \ne 0, \\ -\sigma \ln(1-x), \quad \gamma = 0. \end{cases}$$

Then

$$\begin{aligned}P(T_{j,l} > C_{j,l}) &= 1 - P(T_{j,l} \le C_{j,l}) = 1 - P(\max(T_{j,l}, \dots, T_{2^{M-j-1}-1}) \le C_{j,l}) \\ &= 1 - P(T_{j,l} \le C_{j,l}, \dots, T_{2^{M-j-1}-1} \le C_{j,l}) = 1 - P(\bigcap_{i=l}^{2^{M-j-1}-1} (T_{j,i} \le C_{j,l})) \\ &= 1 - \prod_{i=l}^{2^{M-j-1}-1} P(T_{j,i} \le C_{j,l}) = 1 - \prod_{i=l}^{2^{M-j-1}-1} H^{-1}(C_{j,l}; \sigma, \gamma) \\ &= 1 - \left(H^{-1}(C_{j,l}; \sigma, \gamma)\right)^{2^{M-j-1}-l} = \alpha^*/2^{M-j-1}.\end{aligned}$$

To complete the test construction we must estimate the parameters σ and γ of GPD. We use a method of moments from [3] and find estimators $\hat{\sigma} = \frac{1}{2}\overline{x}(\overline{x}^2/s^2 + 1)$ and $\hat{\gamma} = \frac{1}{2}(\overline{x}^2/s^2 - 1)$. The sample mean $\overline{x}$ and variance s^2 are calculated using statistics $T_{1,l}$ for the high-frequency level ($j = 1$).

B. The second test for jump detection use the maximum sum of wavelet coefficients on the resolution levels.

Define the following statistic:

$$V_j = \sqrt{2^j} \sum_{k=0}^{2^{M-j}-1} d_{j,k}^{(\psi)}, j = 1, \dots, M.$$

It should be noted that the wavelet coefficients $d_{j,k}^{(\psi)}$ with fixed j and k don't depend on T. Using Central Limit Theorem we can see that if H_0 holds then the statistic $\frac{V_j}{\sqrt{T}}$ has asymptotically the Gaussian distribution $N(0, \sigma^2)$.

Let $V = \max(|V_1|, |V_2|, \dots, |V_M|)$ and $\Delta = -\Phi^{-1}\left(\frac{1}{2}\left(1 - (1-\alpha)^{\frac{1}{M}}\right)\right)$, where $\Phi^{-1}(\cdot)$—quantile of the standard Gaussian distribution. Using the same arguments as in the previous test we can see that

$$P(\frac{V}{\sigma\sqrt{T}} > \Delta) = \alpha.$$

Take the following robust estimator of the unknown standart deviation σ:

$$\hat{\sigma} = \underset{k}{median}\left|d_{1,k}^{(\psi)} - \underset{k}{median}(d_{1,k}^{(\psi)})\right|/0.6745. \quad (3)$$

Now decition rule for the second test is defined as follows:

accept H_0, $if \quad V/\hat{\sigma}\sqrt{T} \le \Delta$, and H_1 otherwise.

C. The third test use the sum of wavelet coefficients.

Define the statistic: $Q = \sum_{j=1}^{M} V_j$. Then statistic $Q\Big/\sqrt{MT}$ has asymptotically the Gaussian distribution $N(0, \sigma^2)$ as a sum of asymptotically Gaussian random values $V_j\Big/\sqrt{T} \sim N(0, \sigma^2)$.

We find threshold value Δ for jump test similarly to the previous tests: $\Delta = \Phi^{-1}(1 - \alpha/2)$ and take a robust estimator (3) as an estimator of σ.

The decision rule for this test is defined as follows:
accept $H_0,\ if\ \ |Q|/\hat{\sigma}\sqrt{MT} \le \Delta$, and H_1 otherwise.

4 Experimental results

In order to estimate the type-1 and type-2 error probabilities we perform simulation experiments with time series with Student t_3-noise. The simulated time series consist of the two homogeneous fragments of length $T_1 = T_2 = T/2$, and the jumps $\tau \in \{0.1, 0.2, \dots, 0.9\}$ are simulated at the moment $t_0 = 2^{11}$. The length of the time series is equal to $T = 2^{12}$, the number of the simulated time series is equal to $K = 10\ 000$. The significance level of the tests is equal to $\alpha = 0.05$.

The calculated estimate of the type-1 error probability for the test **A** is equal to 0.0002; for the test **B** is equal to 0.0141; for the test **C** is equal to 0.092. The calculated estimates of the type-2 error probability with respect to jump for the tests **A**, **B**, **C** are presented in table 1.

Table 1: Estimatets of probabilitity of the type 2 error

Test	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
A	0.9983	0.8928	0.3675	0.0571	0.0057	0.001	0.0003	0	0.0001
B	0.8699	0.2455	0.006	0	0	0	0	0	0
C	0.8484	0.6633	0.4201	0.2067	0.0724	0.02	0.0039	0.0004	0

The results above confirm efficiency of the consructed wavelet-based tests for jump detection in time series with heavy-tailed noise.

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