EXACT D-OPTIMAL DESIGNS OF EXPERIMENTS FOR LINEAR MULTIPLE MODEL WITH HETEROSCEDASTICAL OBSERVATIONS

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Abstract

The problem of construction exact D-optimal designs of experiments for linear multiple model with heteroscedastical observations is investigated. Is indicated too the structure of those designs for two factors.

Consider the linear multiple model of heteroscedastical observations:

$$y_i = \theta_1 x_{1i} + \ldots + \theta_m x_{mi} + \varepsilon(x^{(i)}), \quad i = 1, \ldots, n; \quad n \geq m, \tag{1}$$

where y_i are observed variables; $\theta_1, \ldots, \theta_m$ are unknown parameters; $x^i = (x_{1i}, \ldots, x_{mi})$ are m-vectors of controllable variables and components of these vectors are taken from interval $[-1; 1]$; $\varepsilon(x^{(i)})$ are uncontrollable and uncorrelated random errors of observations with mean zero and limited variance $D\varepsilon(x^{(i)}) = d(x_{1i}, \ldots, x_{mi}) > 0$ for each realization $x^{(i)}$ from m-dimensional unit cube, $-1 \leq x_i \leq 1, i = 1, \ldots, m$.

For classical model of observations (1) ($d(x_1, \ldots, x_m) = \text{const}$) the problem of construction D-optimal designs of experiments has been well investigated in [1, 2]. For model of observations (1) with linear variance of observations

$$d(x_1, \ldots, x_m) = a_0 + a_1 x_1 + \ldots + a_m, a_0 > 0, |a_1| + \ldots + |a_m| < a_0$$

problem of construction D-optimal designs has been investigated in [3].

In this paper we will investigate the question about construction exact D-optimal designs of experiments (1) for wider class of heteroscedastical observations then in [3]:

$$d(x_1, \ldots, x_i, \ldots, x_m) \geq \frac{1}{4}\left\{[d_+^{(i)} - d_-^{(i)}]x + d_+^{(i)} + d_-^{(i)}\right\}, x \in [-1, 1], i = 1, \ldots, m, \tag{2}$$

where $d_-^{(i)} = d(x_1, \ldots, -1, \ldots, x_m), d_+^{(i)} = d(x_1, \ldots, 1, \ldots, x_m)$ and $-1, 1$ stays in the place of variable x_i.Variabes $x_1, \ldots, x_m$ in (2) belongs to unit m-dimensional cube. It is easy note that inequality (2) convert to equality when $x_i = \pm 1$. Linear variance of observations $a_0 + a_1 x_1 + \ldots + a_m x_m$ also belong to class functions (2), i.e. inequality (2) determine wider class heteroscedastical observations then class observations with linear variance.

Theorem 1. *Exist D-optimal design of experiments* (1),(2) *all spectrum points of it lies in the vertices of m-dimensional unit cube.*

Proof. Let ε is design of experiments (1), (2) and $|x_{ij}| < 1$ for some i, j. Without loss of generality we can take $i = j = 1$. Make this point x_{11} "floating", i.e. $x_{11} = x \in [-1, 1]$. Other variables x_{ij} we take without changing. Denote the design obtained from ε by changing x_{11} by x as ε_x. Demonstrate than maximizing $|M(\varepsilon)|$ by $x \in [-1, 1]$ we can go to design ε_1 for which $|M(\varepsilon)| \leq |M(\varepsilon_1)|$ and design ε_1 is design ε_x with $x = \pm 1$. Information matrix of ε_x can be written as $M(\varepsilon_x) = A(x) + B$, where

$$A(x) = \frac{1}{d_1(x)} \begin{pmatrix} x \\ t \end{pmatrix} (x, t'), B = \sum_{i=2}^{n} \frac{x^{(i)}(x^{(i)})'}{d_i},$$

$d_1(x) = d(x, x_{21}, \ldots, x_{m1}), d_i = d(x_{1i}, \ldots, x_{mi}), t' = (x_{21}, \ldots, x_{m1})'$. Matrix B does not depend on x and $rank A(x) = 1$. Matrix B and $A(x)$ can be written as $B = (b_1, \ldots, b_m), A(x) = (a_1(x), \ldots, a_m(x))$, where $b_i, a_i(x)$ are m-dimensional columns. Determinant $|A(x) + B|$ may be presented as a sum determinants of matrixes containing different combinations of columns matrixes $A(x), B$. Determinants of these combinations containing more than one column of $A(x)$ are equal to zero. Hence,

$$|A(x) + B| = |a_1(x), b_2, b_3, \ldots, b_m| + |b_1, a_2(x), b_3, \ldots, b_m| + \ldots + |b_1, b_2, b_3, \ldots, a_m(x)| +$$

$$|b_1, b_2, b_3, \ldots, b_m|. \tag{3}$$

Calculating determinants in right part of (3) respect elements of columns $a_1(x), a_2(x), \ldots, a_m(x)$ obtain

$$f(x) = |A(x) + B| = \frac{\alpha x^2 + \beta x + \gamma}{d_1(x)} + \kappa,$$

where

$$\kappa = |B|, \alpha = \left| \sum_{i=2}^{n} \frac{z^{(i)}(z^{(i)})'}{d_i} \right| \geq 0, z^{(i)} = (x_{2i}, \ldots, x_{mi})',$$

and β, γ are constants witch does not depend on x. We may use (2) for obtain upper estimate of $f(x)$:

$$f(x) \leq f_1(x) = \frac{\alpha x^2 + \beta x + \gamma}{d_1(x)} + \kappa, \tag{4}$$

where $a_1 = \frac{1}{4}[d(1, x_{21}, \ldots, x_{m1}) - d(-1, x_{21}, \ldots, x_{m1})]$, $c_1 = \frac{1}{4}\ [d(1, x_{21}, \ldots, x_{m1}) + d(-1, x_{21}, \ldots, x_{m1})]$. In the points 1, -1 inequality (4) convert to equality. Demonstrate that maximum function $f_1(x)$ respect of $x \in [-1, 1]$ we can obtain in the point -1 or 1. Indeed, the derivative of $f_1(x)$ is

$$\frac{df_1(x)}{dx} = \frac{\alpha_1 x^2 + 2\alpha c_1 x + c_1\beta - a_1\gamma}{(a_1 x + c_1)^2}. \tag{5}$$

In the case when $\alpha = 0$ we have

$$\frac{df_1(x)}{dx} = \frac{c_1\beta - a_1\gamma}{(a_1 + c_1)^2},$$

i.e. derivative of $f_1(x)$ is positive or negative in $[-1,1]$ (it depend on sign of $c_1\beta - a_1\gamma$). In this case function $f_1(x)$ increase or decrease in $[-1,1]$. Let $D = 4(\alpha c_1 - a_1(c_1\beta - a_1\gamma))$ the discriminate of numerator in (5) and $\alpha > 0$. If $D > 0$, than function $f_1(x)$ is convex downwards. Indeed,

$$\frac{d^2 f_1(x)}{dx^2} = \frac{D}{2\alpha(a_1 x + c_1)^3} > 0.$$

Hence, in any case $f_1(x)$ (also $f(x)$) takes maximum in $x = \pm 1$. Theorem 1 is proved. □

Theorem 1 help us construct numerically D-optimal designs for model (1), (2). D-optimal design must have following structure $\varepsilon^0 = (x^{(1)}, n_1; x^{(2)}, n_2; \ldots, x^{(l)}, n_l)$, where $x^{(1)}, \ldots, x^{(l)}, l = 2^m$ are vertices of m-dimensional unit cube; $n_1, \ldots, n_l$ are numbers of observations in the points $x^{(1)}, \ldots, x^{(l)}; n_1 + \ldots + n_l = n$. Calculating $|D(\varepsilon^0)|$ and maximizing it respect of $n_1, \ldots, n_l$ we can construct D-optimal design of (1), (2). For example, in the case when $m = 3, n = 40, d(x_1, x_2, x_3) = 1 + 0.2x_1 - 0.3x_2 - 0.1x_3 + (1-x_1^2)(1-x_2^2)(1-x_3^2)$ D-optimal design is $\varepsilon^0 = (x^{(1)}, 10; x^{(2)}, 12; x^{(3)}, 7; x^{(4)}, 11)$, where $x^{(1)} = (1,1,1)$, $x^{(2)} = (-1,1,1)$, $x^{(3)} = (1,1,-1)$, $x^{(4)} = (-1,1,1)$.

Consider now model (1), (2) with two variables ($m = 2$). Points of spectrum D-optimal design of these model must lyes in vertices of unit square: $x^{(1)} = (1,1), x^{(2)} = (-1,1), x^{(3)} = (-1,-1), x^{(4)} = (1,-1)$. Let n_i is the number of observations in thy point $x^{(i)}$; $d_i = d(x^{(i)}), i = 1, \ldots, 4$. Determinate of information matrix D-optimal design must have expression

$$|M| = 4\left(\frac{n_1}{d_1} + \frac{n_3}{d_3}\right)\left(\frac{n_2}{d_2} + \frac{n_4}{d_4}\right). \tag{6}$$

From (6) follow that in D-optimal design not all observations may be taken in the points $x^{(1)}, x^{(3)}$ or only in $x^{(2)}, x^{(4)}$. In this case $|M| = 0$. Hence, one parte of observations must be taken in $x^{(1)}$ or $x^{(3)}$ (let the number of these observations is p) , but the other part must be taken in $x^{(2)}$ or $x^{(4)}$. Hence, $|M| = 4F_1(n_3)F_2(n_4)$, where

$$F_1(n_3) = \frac{p - n_3}{d_1} + \frac{n_3}{d_3}, F_2(n_4) = \frac{n - p - n_4}{d_2} + \frac{n_4}{d_4}$$

and n_3, n_4 doesn't depend on each other. Derivative of $F_1(n_3)$ is

$$\frac{dF_1(n_3)}{dn_3} = \frac{1}{d_3} - \frac{1}{d_1}. \tag{7}$$

If $d_1 < d_3$, then derivative (7) is negative and $F_1(n_3)$ take maximum when $n_3 = 0$. If $d_3 < d_1$, then $F_1(n_3)$ take maximum when $n_3 = p$. If $d_1 = d_3$, then

$$F_1(n_3) = \frac{p}{d_1}$$

and function take maximum for all $n_3 \in [1, p]$. The same proof is to demonstrate that $F_2(n_4)$ reaches its maximum in $x^{(2)}$ or $x^{(4)}$ in which variance of observation is minimal

(if $d_2 \neq d_4$). If $d_2 = d_4$, then the number $n - p$ of observations may be distributed arbitrarily among points $x^{(2)}, x^{(4)}$. Suppose now, for example, that $\min\{d_1, d_3\} = d_1, \min\{d_2, d_4\} = d_4$. In this case $|M|$ is equal to

$$|M| = 4\frac{p(n-p)}{d_1 d_4}. \tag{8}$$

From (8) follow that result of maximizing (8) respect of p depends on parity of p. If p is even ($p = 2s$), then necessary make s observations in $x^{(1)}$ and also s observations in $x^{(4)}$. If p is odd ($p = 2s + 1$), then necessary make s or $s + 1$ observations in point $x^{(1)}$ and rest observations in $x^{(4)}$. So we have the following theorem.

Theorem 2. *For model (1),(2) with two variables ($m = 2$) the structure of exact D-optimal designs is following. Make s observation for even $n = 2s$ or s, may be $s + 1$ observations for odd $n = 2s + 1$ in the point $(1, 1)$ or in the point $(-1, 1)$ and the rest observations in other vertices of unit square. Between the points $x^{(1)}, x^{(3)}(x^{(2)}, x^{(4)})$ we choose one of them in which variance is minimal and make observations in it, if there are two such vertices then observations may be distributed arbitrarily among these ones.*

Theorem 2 help us construct D-optimal designs. For example, if $d(x_1, x_2) = 5 + \sqrt{(1 + x_1)(1 + x_2)}$ and $n = 400$ then in D-optimal design 200 observations are in the point $(-1, -1)$ and the rest of its may be distributed arbitrarily among points $(-1, 1), (1, -1)$.

Theorem 3. *D-optimal estimates $\hat{\theta}_1, \hat{\theta}_2$ for model (1), (2) with two variables ($m = 2$) doesn't depend on d_1, d_2, d_3, d_4.*

Proof. Without loss of generality assume that $\min\{d_1, d_3\} = d_1$, $\min\{d_2, d_4\} = d_2$. Exact D-optimal design (see theorem 2) is $\varepsilon^0 = ((1, 1), n_1; (-1, 1), n_2), n_1 + n_2 = n$. Using design ε^0 and omitting long calculations we have

$$\hat{\theta_1} = \frac{1}{2n_1 n_2}\left(n_2 \sum_{i=1}^{n_1} y_i - n_1 \sum_{j=1}^{n_2} y_j\right), \hat{\theta_2} = \frac{1}{2n_1 n_2}\left(n_2 \sum_{i=1}^{n_1} y_i + n_1 \sum_{j=1}^{n_2} y_j\right), \tag{9}$$

where y_i are observations in the point $(1, 1)$ and y_j are observations in the point $(-1, 1)$. D-optimal design for homoscedastical observations ($d(x_1, x_2) = const$) has such estimates as (9). □

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ABOUT BEST POLYNOMIAL MULTIDIMENSIONAL-MATRIX REGRESSION

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Abstract

The problem of finding the best multidimensional-matrix polynomial regression is formulated. System of the equations for parameters calculation of the polynomial regression with any degree is received. Also, expressions for the parameters of quadratic regression are received.

1 Introduction

The paper is devoted to problem of searching for best multivariate-matrix polynomial regression. The solution of the problem provides the possibilities for building the nonlinear algorithms of multidimensional data forecasting.

2 Formulation of the problem

We shall consider two multidimension random matrixes: q-dimension matrix $\xi = (\xi_{j_{(q)}})$, $j_{(q)} = (j_1, j_2, \ldots, j_q)$, and p-dimension matrix $\eta = (\eta_{i_{(p)}})$, $i_{(p)} = (i_1, i_2, \ldots, i_p)$. We aproximate conditional average $A_{\eta/\xi}(x) = E(\eta/\xi = x)$ by polynomial of degree m:

$$A_{\eta/\xi}(x) = \phi(x) = \sum_{k=0}^{m} {}^{0,kq}(C_{(p,kq)}x^k) = \sum_{k=0}^{m} {}^{0,kq}(x^k C_{(kq,p)}), \quad m = 0, 1, 2, \ldots, \qquad (1)$$

where $C_{(p,kq)}$ – $(p + kq)$-dimension matrices of the coefficients,

$$C_{(p,kq)} = (c_{i_{(p)},\bar{j}_{(k)}}), \quad i_{(p)} = (i_1, i_2, \ldots, i_p), \quad \bar{j}_{(k)} = (j_{(q),0}, j_{(q),1}, \ldots, j_{(q),k}),$$

symmetrical relatively to q-multiindexes $j_{(q),0}, j_{(q),0}, \ldots, j_{(q),k}$ and subjected to condition

$$C_{(p,kq)} = C_{(kq,p)}^{H_{p+kq,kq}}, \quad C_{(kq,p)} = C_{(p,kq)}^{B_{p+kq,kq}}.$$

Here $H_{p+kq,kq}$ and $B_{p+kq,kq}$ – substitutions of transpose of the type "back" and "onward" respectively [1]. We shall denote

$$z = (z_{i_{(p)}}) = \eta - \sum_{k=0}^{m} {}^{0,kq}(C_{(p,kq)}\xi^k) = \eta - \sum_{k=0}^{m} {}^{0,kq}(\xi^k C_{(kq,p)}), \quad , i_{(p)} = (i_1, i_2, \ldots, i_p), \quad (2)$$

and formulate the necessary extremal problem

$$f = E({}^{0,p}(zz)) \to \min_{C_{(p,0)}, C_{(p,q)}, \ldots, C_{(p,mq)}}. \qquad (3)$$

3 System of the equations for coefficients of regression function

Since in (3)

$$^{0,p}(zz) = \sum_{i_{(p)}} z^2_{i_{(p)}}, \quad i_{(p)} = (i_1, i_2, \ldots, i_p),$$

then problem takes the following form

$$f = \sum_{i_{(p)}} E(z^2_{i_{(p)}}) \to \min_{C_{(p,0)}, C_{(p,q)}, \ldots, C_{(p,mq)}}. \tag{4}$$

It is seen from expression (2) that

$$z_{i_{(p)}} = \eta_{i_{(p)}} - \sum_{k=0}^{m} {}^{0,kq}(c_{i_{(p)},\bar{j}_{(k)}}\xi^k) = \eta_{i_{(p)}} - \sum_{k=0}^{m} {}^{0,kq}(\xi^k c_{\bar{j}_{(k)}}, i_{(p)}),$$

where $c_{i_{(p)},\bar{j}_{(k)})}$ under fixed meaning of the multiindex $i_{(p)}$ is kq-dimension matrix, symmetrical with respect to q- multiindexes $j_{(q),0}, j_{(q),1}, \ldots, j_{(q),k}$. If we note that

$$r_{i_{(p)}} = E(z^2_{i_{(p)}}), \tag{5}$$

that problem (4) will take the next form

$$f = \sum_{i_{(p)}} r_{i_{(p)}} \to \min_{C_{(p,0)}, C_{(p,q)}, \ldots, C_{(p,mq)}}. \tag{6}$$

Since each value $r_{i_{(p)}}$ (5) non-negative and depends only from their own matrixes $c_{i_{(p)},\bar{j}_{(0)})}$, $c_{i_{(p)},\bar{j}_{(1)})}$, $\ldots$, $c_{i_{(p)},\bar{j}_{(m)})}$, that minimization of the functions f in (6) is reduced to minimization each summand $r_{i_{(p)}}$:

$$r_{i_{(p)}} \to \min_{c_{i_{(p)},\bar{j}_{(0)})}, c_{i_{(p)},\bar{j}_{(1)})}, \ldots, c_{i_{(p)},\bar{j}_{(m)})}}. \tag{7}$$

For solution of the problem (7) we write the expression for $r_{i_{(p)}}$ (5):

$$r_{i_{(p)}} = E\Big(z^2_{i_{(p)}}) = E((\eta_{i_{(p)}} - \sum_{k=0}^{m} {}^{0,kq}(c_{i_{(p)},\bar{j}_{(k)}}\xi^k))(\eta_{i_{(p)}} - \sum_{l=0}^{m} {}^{0,lq}(\xi^l c_{\bar{j}_{(l)},i_{(p)}}))\Big) =$$

$$= E(\eta_{i_{(p)}}\eta_{i_{(p)}}) - 2\sum_{l=0}^{m} {}^{0,lq}(E(\eta_{i_{(p)}}\xi^l)c_{\bar{j}_{(l)},i_{(p)}}) + \sum_{k=0}^{m}\sum_{l=0}^{m} {}^{0,kq}\Big(c_{i_{(p)},\bar{j}_{(k)}})^{0,lq}(E(\xi^k\xi^l)c_{\bar{j}_{(l)},i_{(p)}})\Big). \tag{8}$$

A solution of the problem (7) is reduced to a solution of the system of equations

$$\frac{\partial r_{i_{(p)}}}{\partial c_{\bar{j}_{(\nu)},i_{(p)}}} = 0, \quad \nu = \overline{0,m}.$$

Differentiation of the scalar function $r_{i_{(p)}}$ (8) by her symmetrical multivariate-matrix arguments $c_{\bar{j}_{(\nu)},i_{(p)}}$ with use the existing rules of the differentiation [1] brings about following system of the equations

$$-E\Big(\eta_{i_{(p)}}\xi^{\lambda}\Big)_r^{(1)} + \sum_{k=0}^{m} {}^{0,kq}\Big(c_{i_{(p)},\bar{j}_{(k)}}\, E\Big(\xi^{\lambda+k}\Big)_r^{(1)}\Big) = 0, \quad \lambda = \overline{0,m},$$

or

$$\sum_{k=0}^{m} {}^{0,kq}\Big(C_{(p,kq)}E\Big(\xi^{\lambda+k}\Big)\Big) = E\Big(\eta\xi^{\lambda}\Big), \quad \lambda = \overline{0,m}.$$

We shall denote

$$\nu_{\xi^{\lambda+k}} = E\Big(\xi^{\lambda+k}\Big), \quad \nu_{\eta\xi^{\lambda}} = E\Big(\eta\xi^{\lambda}\Big), \tag{9}$$

and write these system as follows

$$\sum_{k=0}^{m} {}^{0,kq}\Big(C_{(p,kq)}\nu_{\xi^{\lambda+k}}\Big) = \nu_{\eta\xi^{\lambda}}, \quad \lambda = \overline{0,m}. \tag{10}$$

Given system of the equations represents itself decision of the delivered problem.

4 The best quadratic regression

We shall consider case of the best multidimensional-matrix quadratic regression

$$y = \phi(x) = C_{(p,0)} + {}^{0,q}(C_{(p,q)}x) + {}^{0,2q}(C_{(p,2q)}x^2). \tag{11}$$

In this case in system of the equations (10) $m = 2$, and we have a system from three equations

$$\left\{\begin{array}{c} C_{(p,0)} + {}^{0,q}(C_{(p,q)}\nu_{\xi}) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^2}) = \nu_{\eta}, \\ {}^{0,0}(C_{(p,0)}\nu_{\xi}) + {}^{0,q}(C_{(p,q)}\nu_{\xi^2}) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^3}) = \nu_{\eta\xi}, \\ {}^{0,0}(C_{(p,0)}\nu_{\xi^2}) + {}^{0,q}(C_{(p,q)}\nu_{\xi^3}) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^4}) = \nu_{\eta\xi^2}. \end{array}\right.$$

We shall solve this system by process of Gauss elimination . We shall multiply first equation on the right by ν_{ξ}, ν_{ξ^2} in the sense of $(0,0)$-convolute multiplying and shall subtract it from the second and the third equations accordingly. Thus, we get system:

$$\left\{\begin{array}{c} C_{(p,0)} + {}^{0,q}(C_{(p,q)}\nu_{\xi}) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^2}) = \nu_{\eta}, \\ {}^{0,q}(C_{(p,q)}(\nu_{\xi^2} - \nu_{\xi}^2)) + {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^3} - \nu_{\xi^2}\nu_{\xi})) = \nu_{\eta\xi} - \nu_{\eta}\nu_{\xi}, \\ {}^{0,q}(C_{(p,q)}(\nu_{\xi^3} - \nu_{\xi}\nu_{\xi^2})) + {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^4} - \nu_{\xi^2}\nu_{\xi^2})) = \nu_{\eta\xi^2} - \nu_{\eta}\nu_{\xi^2}. \end{array}\right.$$

With notations

$$\mu_{\xi^2} = \nu_{\xi^2} - \nu_{\xi}^2, \quad \mu_{\eta\xi} = \nu_{\eta\xi} - \nu_{\eta}\nu_{\xi}$$

we have

$$\left\{\begin{array}{c} C_{(p,0)} + {}^{0,q}(C_{(p,q)}\nu_{\xi}) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^2}) = \nu_{\eta}, \\ {}^{0,q}(C_{(p,q)}\mu_{\xi^2}) + {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^3} - \nu_{\xi^2}\nu_{\xi})) = \mu_{\eta\xi}, \\ {}^{0,q}(C_{(p,q)}(\nu_{\xi^3} - \nu_{\xi}\nu_{\xi^2})) + {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^4} - \nu_{\xi^2}\nu_{\xi^2})) = \nu_{\eta\xi^2} - \nu_{\eta}\nu_{\xi^2}. \end{array}\right.$$

Then we multiply the second equation on the right by ${}^{0,q}(\mu_{\xi^2}^{-1}(\nu_{\xi^3}-\nu_\xi\nu_{\xi^2}))$ in the sense of $(0,q)$-convolute multiplying and subtract it from the third equation. As a result we get following equations

$$C_{(p,0)} + {}^{0,q}(C_{(p,q)}\nu_\xi) + {}^{0,2q}(C_{(p,2q)}\nu_{\xi^2}) = \nu_\eta, \tag{12}$$

$${}^{0,q}(C_{(p,q)}\mu_{\xi^2}) + {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^3}-\nu_{\xi^2}\nu_\xi)) = \mu_{\eta\xi}, \tag{13}$$

$$\begin{aligned}{}^{0,2q}(C_{(p,2q)}(\nu_{\xi^4}-\nu_{\xi^2}\nu_{\xi^2} - {}^{0,q}({}^{0,q}((\nu_{\xi^3}-\nu_{\xi^2}\nu_\xi)\mu_{\xi^2}^{-1})(\nu_{\xi^3}-\nu_\xi\nu_{\xi^2})))) = \\ = \nu_{\eta\xi^2} - \nu_\eta\nu_{\xi^2} - {}^{0,q}({}^{0,q}(\mu_{\eta\xi}\mu_{\xi^2}^{-1})(\nu_{\xi^3}-\nu_\xi\nu_{\xi^2})).\end{aligned} \tag{14}$$

Now from equation (14) get the expression for $C_{(p,2q)}$:

$$C_{(p,2q)} = {}^{0,2q}(B_{(p,2q)}\, A_{(2q,2q)}^{-1}), \tag{15}$$

where

$$B_{(p,2q)} = \nu_{\eta\xi^2} - \nu_\eta\nu_{\xi^2} - {}^{0,q}({}^{0,q}(\mu_{\eta\xi}\mu_{\xi^2}^{-1})(\nu_{\xi^3}-\nu_\xi\nu_{\xi^2})), \tag{16}$$

$$A_{(2q,2q)} = (\nu_{\xi^4}-\nu_{\xi^2}\nu_{\xi^2} - {}^{0,q}({}^{0,q}((\nu_{\xi^3}-\nu_{\xi^2}\nu_\xi)\mu_{\xi^2}^{-1})(\nu_{\xi^3}-\nu_\xi\nu_{\xi^2}))), \tag{17}$$

in which $A_{(2q,2q)}^{-1}$ is a matrix, $(0,2q)$-inverse to matrix $A_{(2q,2q)}$. From equation (13) we get the expression for $C_{(p,q)}$:

$$C_{(p,q)} = {}^{0,q}((\mu_{\eta\xi} - {}^{0,2q}(C_{(p,2q)}(\nu_{\xi^3}-\nu_{\xi^2}\nu_\xi)))\mu_{\xi^2}^{-1}). \tag{18}$$

Finally, from equation (12) we find the expression for $C_{(p,0)}$:

$$C_{(p,0)} = \nu_\eta - {}^{0,q}(C_{(p,q)}\nu_\xi) - {}^{0,2q}(C_{(p,2q)}\nu_{\xi^2}). \tag{19}$$

Substituting (19) in (11), we get the following expression the best multidimensional-matrix quadratic regression:

$$y = \phi(x) = \nu_\eta + {}^{0,q}(C_{(p,q)}(x-\nu_\xi)) - {}^{0,2q}(C_{(p,2q)}(x^2-\nu_{\xi^2})). \tag{20}$$

We bring the sequence of actions required for calculation of the best multidimensional-matrix quadratic regression:

1) calculate $B_{(p,2q)}$ by formula (16);
2) calculate $A_{(2q,2q)}$ by formula (17);
3) calculate $C_{(p,2q)}$ by formula (15);
4) find $C_{(p,q)}$ by formula (18);
5) get y by formula (20).

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