ROBUSTNESS OF AUTOREGRESSIVE FORECASTING UNDER BILINEAR DISTORTIONS

Yu.S. Kharin, O.N. Radzieuskaya
Belarusian State University
Minsk, BELARUS, e-mail: kharin@bsu.by

Abstract

The paper is devoted to the investigation of bilinear stochastic time series model BL(p,0,1,1). The autoregressive forecasting statistic is considered under the mean square risk criterion; its robustness under bilinear distortions is evaluated.

1 Introduction

There is a growing interest in investigation of nonlinear models of time series that is caused by nonlinearity of real processes. Bilinear model of time series was proposed by Granger and Andersen in 1978 [1, 2]. This model gives a possibility to describe "sufficiently large stochastic outbursts" that appear in applications to seismological and financial data analysis.

2 Models BL(p,0,1,1), AR(p) and stationarity

Bilinear model BL(p,0,1,1) is a simple modification of the well known and intensively used autoregressive model AR(p). Therefore it is quite reasonable to consider these models together and apply some earlier obtained results of model AR(p) to model BL(p,0,1,1).

The time series \{x^0_t\}_{t \in \mathbb{Z}} is said to correspond with autoregressive model AR(p) if it satisfies the linear stochastic difference equation

\[ x^0_t = \sum_{j=1}^{p} \alpha_j x^0_{t-j} + u_t, \quad t \in \mathbb{Z}, \]  

(1)

where \( u_t \sim IIDN(0, \sigma^2) \) and \( \{\alpha_j\}_{j=1}^{p} \) are autoregressive coefficients.

Respectively, the time series \{x_t\}_{t \in \mathbb{Z}} is said to correspond with bilinear model BL(p,0,1,1) if it satisfies the bilinear stochastic difference equation

\[ x_t = \sum_{j=1}^{p} \alpha_j x_{t-j} + \beta x_{t-1} u_{t-1} + u_t, \quad t \in \mathbb{Z}, \]  

(2)

where \( \beta \) is the bilinearity coefficient. Note, if \( \beta = 0 \), then model (1) and (2) are the same.
Further only stationary models AR(p) (1) and BL(p,0,1,1) (2) will be considered. The conditions \( \rho(A) < 1 \) and \( \rho(A \otimes A + \sigma^2 B \otimes B) < 1 \) provide their stationarity respectively. Here matrices A and B depend on coefficients \( \{\alpha_j\}_{j=1}^{p} \) and bilinearity coefficient \( \beta \) defined in stochastic difference equations (1), (2).

3 Autoregressive forecasting of bilinear time series

The significant and complicated problem is to find a forecasting statistic \( \hat{x}_{T+\tau} \) as an estimator of the future value \( x_{T+\tau} \) of the time series optimal w.r.t. some criterion by the observed history \( X = \{x_1, \ldots, x_T\} \) of the length T. Here \( \tau, \, \tau = 1, \ldots, p \), is a horizon of prediction. As a criterion of optimality we will use a set of scalar risks \( \tau \in \{\hat{x}_{T+\tau} - x_{T+\tau}\} \) of the forecast (4) are provided.

Corollary 1. Under the conditions of Theorem 1 scalar risks of the forecast (4) are

\[
\begin{align*}
& r(\tau) = \sigma^2 \sum_{j=1}^{\tau} (S_1^{-1}(1,j))^2 + \beta^2 \sigma^2 c(0) \sum_{j=1}^{\tau} (S_1^{-1}(1,j))^2 + \beta^2 \sigma^4 \left( \sum_{j=1}^{\tau} (S_1^{-1}(1,j))^2 \right) + \\
& \quad + \sum_{j=1}^{p} \alpha_j (I_p^{-1,0,1} - I_p) (S_1^{-1})(1,j)^2,
\end{align*}
\]

where \( c(0) = E\{x_1^2\} \) is the second order moment of the bilinear time series, \( I_p^{-1,0,1} \) are some constant \( (p \times p) \)-matrices independent of model parameters.

Theorem 1. For the stationary model BL(p,0,1,1) defined by (2) the autoregressive forecast (4) has the following matrix mean square risk:

\[
\begin{align*}
R &= \sigma^2 (S_1)^{-1}(S_1^{-1})' + \beta^2 \sigma^2 c(0)(S_1)^{-1}(S_1^{-1})' + \beta^2 \sigma^4 (S_1)^{-1}(I_p^0 + I_p)(S_1^{-1})' + \\
&\quad + \frac{2\beta^2 \sigma^4}{1 - \sum_{j=1}^{p} \alpha_j} (S_1)^{-1}(I_p^{-1,0,1} - I_p)(S_1^{-1})',
\end{align*}
\]

where \( c(0) = E\{x_1^2\} \) is the second order moment of the bilinear time series, \( I_p^0, I_p^{-1,0,1} \) are some constant \( (p \times p) \)-matrices independent of model parameters.

The conditions

\[
\rho(A) < 1 \quad \text{and} \quad \rho(A \otimes A + \sigma^2 B \otimes B) < 1
\]

provide minimum of defined earlier risks can be found from the following system of equations:

\[
S_1 X_{T-1}^{oT+p} = S_2 X_{T-p+1}^{oT}.
\]
\[ + \left( \sum_{j=1}^{\tau} S_{1}^{-1}(1, j) \right)^2 + \frac{4}{\sum_{j=1}^{p} \alpha_j} \sum_{j=1}^{\tau-1} S_{1}^{-1}(1, j)S_{1}^{-1}(1, j+1) \right), \quad \tau = 1, \ldots, p. \]  

Corollary 2. Under the conditions of Theorem 1 and some constraints on autoregressive coefficients, the matrix and the scalar mean square risks satisfy the following asymptotic expansions at \(\beta \to 0\):

\[ R = \sigma^2 S_{1}^{-1}(S_{1}^{-1})' + \beta^2 \sigma^4 \left( W^{-1}(1, 1)S_{1}^{-1}(S_{1}^{-1})' + S_{1}^{-1}(I_{p} + I_{p}) (S_{1}^{-1})' \right) + \frac{2}{1 - \sum_{j=1}^{\tau} \alpha_j} S_{1}^{-1} \left( I_{p,1}^{-1} - I_{p} \right) (S_{1}^{-1})' \right) + o(\beta^2) \Xi_{p,1}. \]

\[ r(\tau) = \sigma^2 \sum_{j=1}^{\tau} (S_{1}^{-1}(1, j))^2 + \beta^2 \sigma^4 \left( (W^{-1}(1, 1) + 1) \sum_{j=1}^{\tau} (S_{1}^{-1}(1, j))^2 + \left( \sum_{j=1}^{\tau} S_{1}^{-1}(1, j) \right)^2 \right) + \frac{4}{1 - \sum_{j=1}^{p} \alpha_j} \left( \sum_{j=1}^{\tau-1} S_{1}^{-1}(1, j)S_{1}^{-1}(1, j+1) \right) \right) + o(\beta^2), \quad \tau = 1, \ldots, p, \]

where \( W \in R^{p+1 \times p+1} \) is a known matrix depended on autoregressive coefficients.

Thus the obtained results represent exact and asymptotic values of matrix and scalar mean square risks (5)–(8) of the autoregressive forecast (4) for the bilinear time series (2).

### 4 Robustness of the autoregressive forecasting statistic under bilinear distortions

The autoregressive forecasting statistic (4) applied to bilinear time series prediction ignores special structural features caused by bilinearity. Therefore we must be cautious while using this statistic and believe in the low level of bilinear distortions. Here the problem of robustness and adequate use of the autoregressive forecasting statistic will be discussed.

Let’s define some functionals of forecast robustness. Assume that the bilinearity level \(\beta\) can vary in the bounds from \(-\beta_+\) to \(\beta_+\) where \(\beta_+ > 0\) is a maximum absolute level of distortions. The instability coefficient \([3]\) is said to be the relative increment of the guaranteed upper risk to the risk of the nondistorted times series: \(k(\tau) = (r_+(\tau) - r(\tau)|_{\beta=0})/r(\tau)|_{\beta=0}\) where \(r_+(\tau) = \sup_{\beta \in [-\beta_+,\beta_+]} r(\tau)\). The \(\delta\)-critical distortion
level [3] is said to be the maximal admissible bilinearity level with the instability coefficient not greater than the given \( \delta > 0 \): 

\[
\beta^+(\delta, \tau) = \sup_{k(\tau) \leq \delta} \beta^+. 
\]

The less instability coefficient \( k(\tau) \) and the greater \( \delta \)-critical distortion level \( \beta^+(\delta, \tau) \) the autoregressive statistic is more robust w.r.t. the bilinear distortions.

**Theorem 2.** For the stationary bilinear model BL\((p,0,1,1)\) (2) with some conditions on autoregressive coefficients the instability coefficient and the \( \delta \)-critical distortion level have the following asymptotic representations at \( \beta \to 0 \):

\[
\kappa(\tau) = \frac{\beta^2_+ \sigma^2}{\sum_{j=1}^{\tau} (S^{-1}_1(1,j))^2} \left( (W^{-1}(1,1) + 1) \sum_{j=1}^{\tau} (S^{-1}_1(1,j))^2 + \left( \sum_{j=1}^{\tau} S^{-1}_1(1,j) \right)^2 \right)
\]

\[
+ \frac{4}{1 - \sum_{j=1}^{\tau} \alpha_j} \sum_{j=1}^{\tau-1} S^{-1}_1(1,j) S^{-1}_1(1,j+1) + o(\beta^2_+),
\]

\[
\beta^+(\delta, \tau) \approx \frac{\delta^{1/2} \sigma^{-1}}{\left( \sum_{j=1}^{\tau} (S^{-1}_1(1,j))^2 \right)^{-1/2}} \left( (W^{-1}(1,1) + 1) \sum_{j=1}^{\tau} (S^{-1}_1(1,j))^2 \right)
\]

\[
+ \left( \sum_{j=1}^{\tau} S^{-1}_1(1,j) \right)^2 \left( \sum_{j=1}^{\tau} \alpha_j \sum_{j=1}^{\tau-1} S^{-1}_1(1,j) S^{-1}_1(1,j+1) \right)^{-1/2}.
\]

Thus for the given \( \delta > 0 \) we can evaluate the bilinearity level for which the autoregressive forecasting statistic (4) can be used without significant increase in the mean square risk.

**References**

