Construction of Copula Models

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Abstract: *In this paper the device of models copula and opportunity of application is considered at the analysis of time series. For the purpose the copula definition and its properties, are discussed then different copulas families. The illustrative examples of modeling of distribution by various kinds copulas are described, using a package R.*

Keywords**:** Copula, elliptical, archimedean, extreme.

1. INTRODUCTION

The device copulas has received wide application to such financial tasks, as an estimation of risk, choice of an optimum investment portfolio, estimation of cost of difficult financial tools, management of currency risk of banks. Copulas already have found successful application at modeling the natural phenomena for example, on oceanology (Michele C., 2007), at research of ways of hashing given as means of protection of a database (Sarathy R., 2002), at the analysis of the microeconomic data (Smith M., 2003; Genius M., Strazzera E., 2004; Sun J., 2008; Kim G., 2007).

By results of events of the crisis period 2008 - 2009 Basel Committee on bank supervision has published the document “Bank regulation and supervision (BCBS II)”, the describing order of an estimation is brave for the purpose of account of sufficiency of the capital of banks. In the document the preference is given up to models “copula” in comparison with the approaches, most widespread in bank practice, (summation, simple diversification and variance- covariance the approach). It is explained to that the models “copula” allow to simulate joint multidimensional of distribution (including asymmetric), which are not normal. In October, 2010 in the document Basel Committee “Development of the approaches to modeling aggregation is brave” the basic supervision on questions aggregation are generalized is brave, allowed to allocate advantages received from application of models “copulas”. Thus study of models “copula” represents the special urgency in view of the legislative initiatives on regulation is brave of commercial banks and control of a level of sufficiency of their capital.

2. DEFINITION OF TWO-DIMENSIONAL COPULA AND BASIC PROPERTIES

Definition. Function $C(x, y)$ is called subcopula two variable x and y, defined on $A\times B$, where $A\in[0,1], B\in[0,1]$, with range space $[0,1]$ and satisfy the conditions :

1. $C(x,0)=0, C(0,y)=0$;
2. $C(x,1)=x, C(1,y)=y$;
3. $C(x_2,y_2)+C(x_1,y_1)-C(x_2,y_1)-C(x_1,y_2)\ge 0$,

where $(x_1,y_1)\in[0,1]^2$, $(x_2,y_2)\in[0,1]^2$ such as $x_1\le y_1, x_2\le y_2$

Definition. **Copula** – is a subcopula in cases when $A=[0,1], B=[0,1]$.

Properties of the copula:

1. Boundedness: $0\le C(x,y)\le 1$.
2. Any copula lays in borders Frechet-Hoeffding: $\max(x+y-1,0)\le C(x,y)\le \min(x,y)$.
3. Domination. Copula C_2 dominate above copula C_1, or $C_1\prec C_2$, in case, when for $\forall x,y$ it it true $C_1(x,y)\le C_2(x,y)$.
4. For $(x_1,x_2)\in[0,1]^2, (y_1,y_2)\in[0,1]^2$ it is true $|C(x_2,y_2)-C(x_1,y_1)|\le|x_2-x_1|+|y_2-y_1|$.

Use copulas for modeling joint probability of distributions is based on conclusions of the theorem Sklar, entered in 1959.

Sklar’s theorem. Let $x, y\in R$ random variables, marginal distributions functions $F_Y(y)=P(Y\le y)$ $F_X(x)=P(X\le x)$, two-dimesional the function of distribution looks $F_{XY}(x,y)=P(X\le x, Y\le y)$. Then exists $C(x,y)$, such as:

$$F_{XY}(x,y)=C(F_X(x), F_Y(y)). \qquad (1)$$

If functions $F_X(x)$ и $F_Y(y)$ are continuous, then copula (1) is unique, otherwise, copula $C(x,y)$ can be always determined on area of meanings of functions $F_X(x)$ and $F_Y(y)$. Correctly and return statement, if $C(x,y)$ - copula, $F_X(x)$ and $F_Y(y)$ - marginal distributions functions, then function $F_{XY}(x,y)$, determined above, is two-dimensional function of distribution of a random vector (X, Y).

Copula is probability of approach of joint event for random variables $x, y\in R$, that is

$$C(x,y)=F_{XY}(F_X^{-1}(x), F_Y^{-1}(y)).$$

Thus, copula – is function, allowing to proceed from one-dimensional distributions of random variables to joint distribution.

3. RESEARCH OF VARIOUS FAMILIES COPULAS

All copulas can be related to three families: elliptical (Gaussian, Student’s), archimedean copula (Frank, Clayton, Gumbel, Ali-Mikhail-Haq), extreme-value (Galambos, Husler-Reiss).

Before to proceed to consideration of separate families copulas, is necessary to enter such concepts, as product, maximal and minimal copulas.

Copula $C(x,y)$ for two variables X, Y is called *product copula*, if $C(x,y)=xy$.

construction copulas and contour diagram.

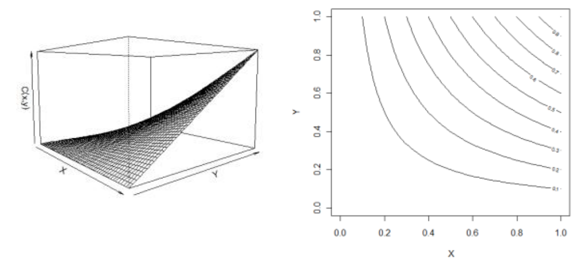

Fig. 1- The product copula and its level curves.

The minimum copula- it is the bottom border for all copul, has the following definition:

$$C(x,y)=\max(x+y-1,0) \qquad (2)$$

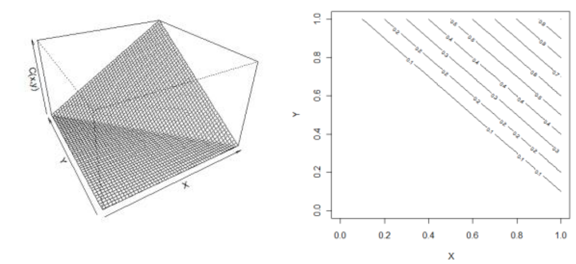

Fig. 2- The minimum copula and its level curves.

The maximum copula- is the top border for all copul, has the following definition:

$$C(x,y)=\min(x,y) \qquad (3)$$

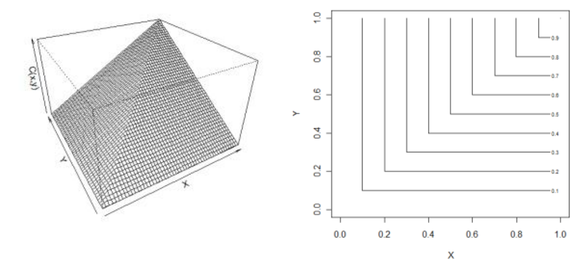

Fig. 3- The maximum copula and its level curves.

For construction of the basic families copulas environment R, intended for automation statistical of calculations was used. The examples of generation of two joint distributions with the different forms of interrelation are resulted.

Definition. **Elliptical copula** satisfies to the following formula:

$$C(x,y)=F_{XY}(F_X^{-1}(x),F_Y^{-1}(y)), \qquad (4)$$

где $F_{XY}(x,y), F_X(x), F_Y(y)$ functions of distribution of the normal laws.

Exist Gaussian and Student's elliptical copulas. They occur from the analytical forms of record two-dimensional of normal distribution and distribution student's.

Example 1. The Gaussian copula is defined as follows:

$$C(x,y)=\int_{-\infty}^{F_X^{-1}(x)}\int_{-\infty}^{F_Y^{-1}(y)}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left(\frac{2\theta z_1z_2-z_1^2z_2^2}{2(1-\theta^2)}\right)dz_1dz_2 \qquad (5)$$

where $\theta\in[-1,1]$-parameter of copula.

Is investigated Gaussian copula at various meanings of parameter θ. With the help of a statistical package R the construction copulas and contour diagram.

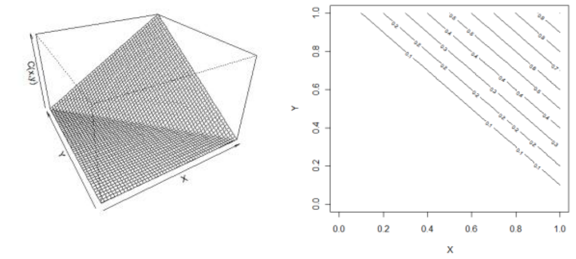

Fig. 4- The Gaussian copula and its level curves, with θ=-1.

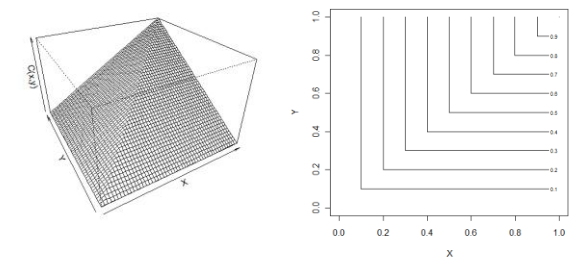

Fig. 5- The Gaussian copula and its level curves, with θ=-1.

Example 2. Student's copula is defined as follows:

$$C(x,y)=\int_{-\infty}^{F_X^{-1}(x)}\int_{-\infty}^{F_Y^{-1}(y)}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left(1+\frac{z_1^2+z_2^2-2\theta z_1z_2}{\nu(1-\theta^2)}\right)^{-\frac{\nu+2}{2}}dz_1dz_2 \qquad (6)$$

where $\theta\in[-1,1]$-parameter of copula , ν- number of degrees of freedom copula.

Is submitted Student's copula at various value of parameter θ.

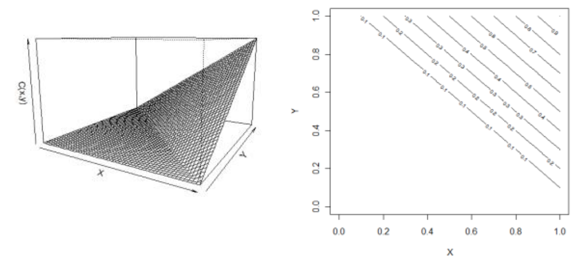

Fig. 6-The Student's copula and its contour diagram, if θ=-1.

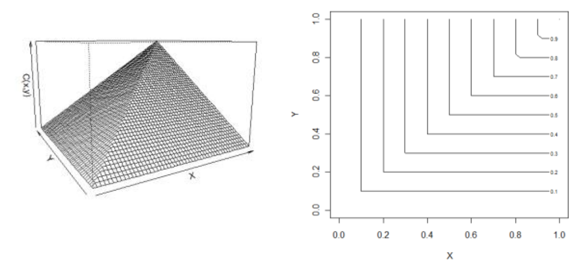

Fig. 7- The Student's copula and its contour diagram, if θ=1.

Archimedean copulas may be constructed using a function $\varphi:[0,1]\to[0,+\infty]$ is called a generator and has the following properties:

1. $\varphi(1)=0,\ \varphi(0)=+\infty$
2. Continuous;
3. Decreasing: $\varphi'(t)<0,\quad t\in(0,1)$
4. Convex: $\varphi''(t)>0,\quad t\in(0,1)$

The pseudo-inverse of *φ(t)* must also be defined, as follows:

$$\varphi^{[-1]} = \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0) \\ 0, & \varphi(t) \le t \le +\infty \end{cases}$$

and $\varphi^{[-1]}(\varphi(t)) = t$, for every $t \in [0,1]$.

Definition. **Archimedean copula** is generated as follows:

$$C(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)],$$

Basic archimedean copula the following are: Frank, Clayton, Gumbel, Ali-Mikhail-Haq.

Example 3. The Ali-Mikhail-Haq copula is defined as follows:

$$C(x, y) = \frac{xy}{1-\theta(1-x)(1-y)} \tag{7}$$

where $\theta \in [-1,1]$ – parameter of copula, describing a degree of dependence of random variables. To be under construction Ali-Mikhail-Haq copula in a statistical package R for various value of parameter θ

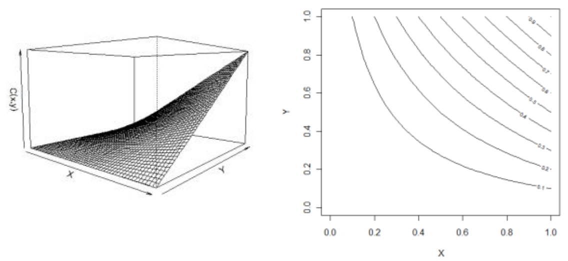

Fig. 8- The Ali-Mikhail-Haq copula and its contour diagram, if θ=-1.

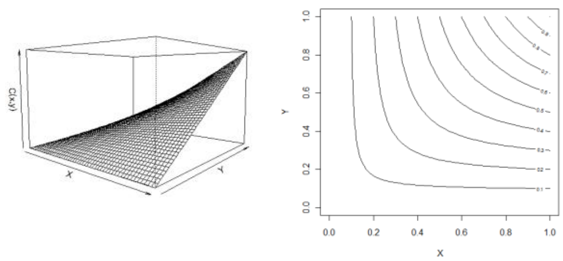

Fig. 9- The Ali-Mikhail-Haq copula and its contour diagram, if θ=1.

Example 4. The copula Clayton is defined as follows:

$$C(x, y) = x + y - 1 + \left((1-x)^{-\frac{1}{\theta}} + (1-y)^{-\frac{1}{\theta}} - 1 \right)^{-\theta} \tag{8}$$

where $\theta \in [-1,0) \cup (0,+\infty)$

In a package R is under construction Clayton copula and its contour diagram at various value of parameter θ.

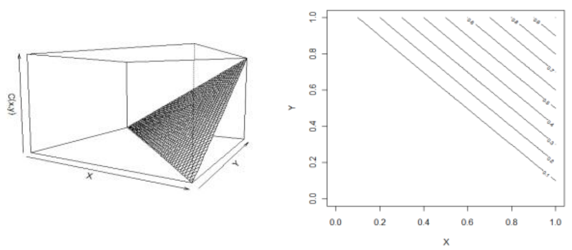

Fig. 10- The Clayton copula and its level curves, with θ=-1.

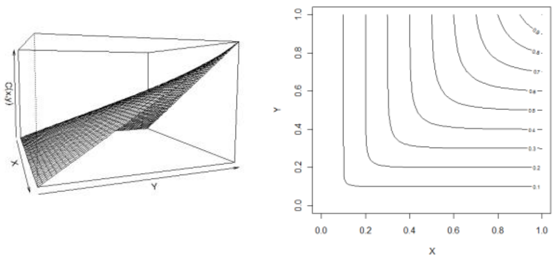

Fig. 11- The Clayton copula and its level curves, if θ=5.

Example 5. The Frank copula is defined as follows

$$C(x, y) = -\frac{1}{\theta} \ln\left[1 + \frac{(e^{-\theta x} - 1)(e^{-\theta y} - 1)}{e^{-\theta} - 1} \right], \tag{9}$$

where $\theta \in (-\infty,0) \cup (0,+\infty)$.

Is submitted Frank copula at various value of parameter θ.

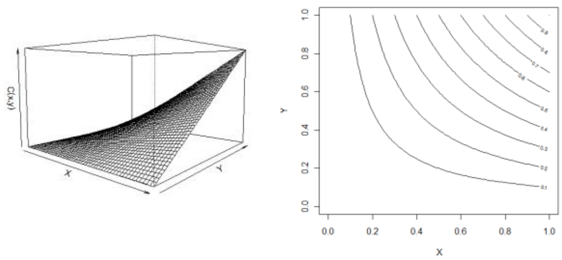

Fig. 12- The Frank copula and its contour diagram, if θ=0,02.

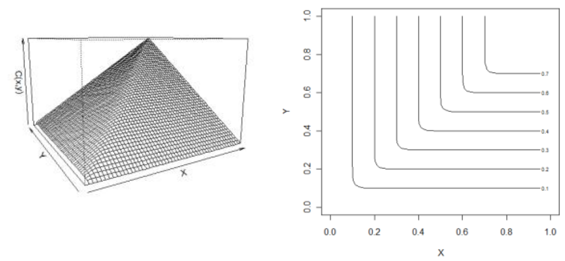

Fig. 13- The Frank copula and its contour diagram, if θ=50.

Example 6. The Gumbel copula is defined as follows:

$$C(x, y) = \exp\left\{ -\left((-\ln x)^{\theta} + (-\ln y)^{\theta} \right)^{1/\theta} \right\} \tag{10}$$

where $\theta \in [-1,+\infty)$.

In a statistical package R is constructed Gumbel copula with the contour diagram for various value of parameter θ.

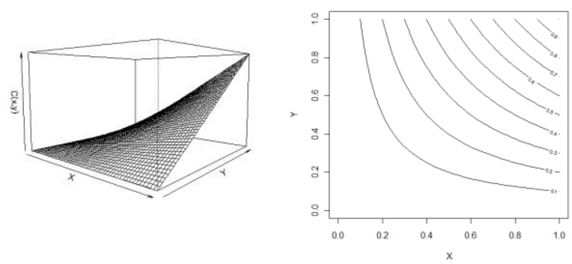

Fig. 14- The Gumbel copula and its contour diagram projection on an axis x,y, if θ=1.

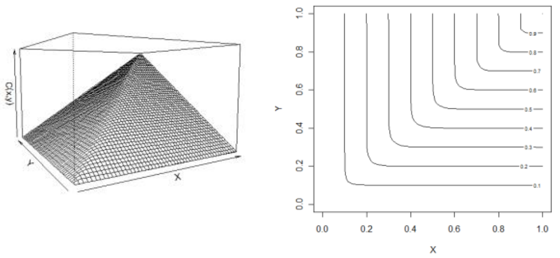

Fig. 15- The Gumbel copula and its contour diagram projection on an axis x,y, if θ=10.

Definition. **Extreme-value copula** is generated as follows:

$$C(x^t, y^t) = C^t(x, y),\ \forall t > 0 .(11)$$

Basic extreme-value copula the following are Galambos, Husler-Reiss.

Example 7. The Galambos copula is defined as follows:

$$C(x, y) = xy \exp\left\{\left((-\ln x)^{-\theta} + (-\ln y)^{-\theta}\right)^{1/\theta}\right\} \quad (12)$$

where $\theta \in [0, +\infty)$.

Is submitted Galambos copula at various value of parameter θ.

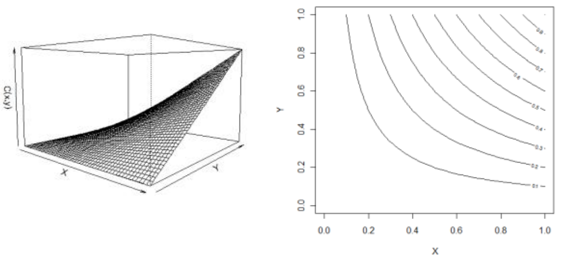

Fig. 16- The Galambos copula and its contour diagram projection on an axis x,y, if θ=0,01.

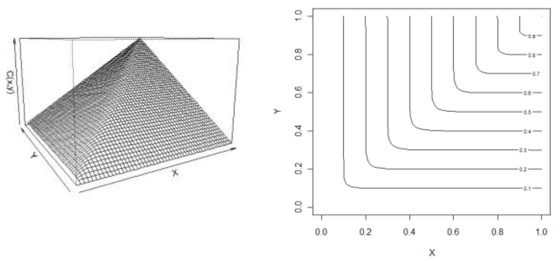

Fig. 17- The Galambos copula and its contour diagram projection on an axis x,y, if θ=10.

Example 8. The Husler-Reiss copula is defined as follows:

$$C(x, y) = \exp\left\{-x\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\frac{\ln x}{\ln y}\right) - y\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\frac{\ln y}{\ln x}\right)\right\} \quad (13)$$

where $\theta \in [0, +\infty)$.

In a statistical package R is constructed Husler-Reiss copula with the contour diagram projection on an axis x,y for various value of parameter θ.

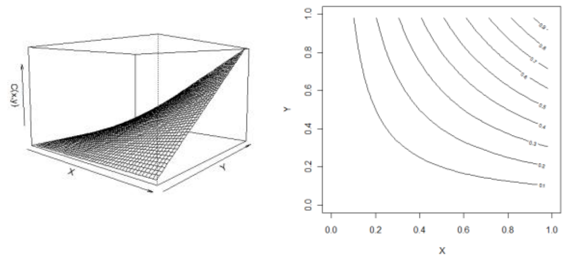

Fig. 18- The Husler-Reiss copula and its level curves, with θ=0,01.

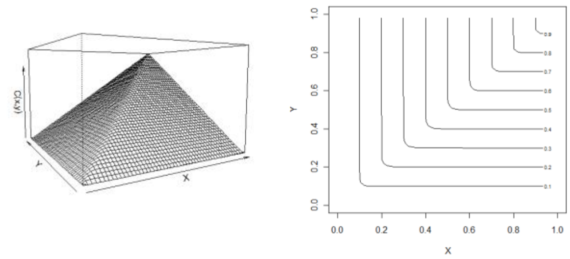

Fig. 19- The Husler-Reiss copula and its level curves, with θ=20.

4. CONCLUSION

In job is given the definition of copula and its properties. The families copulas are investigated, such as elliptical, archimedean, extreme-value. In a statistical package R the following kinds copulas are constructed: Gaussian, Student's, Frank, Clayton, Gumbel, Ali-Mikhail-Haq, Galambos, Husler-Reiss for various value of parameter θ. The device of copulas allows, knowing marginal of distribution to proceed to joint distribution of two random variables. The copula contains the information on a nature of dependence between two random variables, which no in marginals distributions, but does not contain the information about marginals distributions.

5. REFERENCES

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