ON GENERALIZATIONS OF INEQUALITIES 
OF CHERNOFF-TYPE

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Abstract

In the work are obtained generalizations of the inequality proved by H. Chernoff for bound on the variance of an absolutely continuous function of a standard normal random variable.

Let $X$ be a standard normal random variable (r.v.). H. Chernoff in [1] proved an inequality playing important role in the theory of statistical inferences:

for any real valued absolutely continuous function $g(x)$,

$$Dg(X) \leq E(g'(X))^2. \quad (1)$$

It should be noted that the mentioned Chernoff inequality is exact since one can easy check that this inequality becomes the equality for linear functions $g(x)$.

A.A. Borovkov and S.A. Utev in [2] obtained an inequality essentially generalized inequality (1), namely, they proved an inequality of type (1) for an arbitrary r.v. with the distribution function having an absolutely continuous component.

Let $ξ$ be a r.v. with the distribution function

$$F_ξ(x) = αF_1(x) + (1 - α)F_2(x) \quad (2)$$

where $0 \leq α \leq 1$, $F_1(x)$ have the probability density $f_1(x)$.

Suppose $F_ξ(x)$ satisfies the conditions:

$$\int_0^∞ xdF_ξ(x) \leq cf_1(u) \quad \text{for} \quad u \geq 0,$$

$$-\int_∞^0 xdF_ξ(x) \leq cf_1(u) \quad \text{for} \quad u < 0 \quad (3)$$

at some $c > 0$.

In [2], it is given the simple proof of the following

Theorem 1. Let $F_ξ(x)$ satisfy conditions (2) and (3). Then for any absolutely continuous function $g(\cdot)$,

$$Dg(ξ) \leq \frac{c}{α}E(g'(ξ))^2. \quad (4)$$
Remark 1. In the case of
\[ P(\xi < x) = F_\xi(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \] (5)

conditions (2) and (3) are realized at \( \alpha = 1, c = 1, \) and
\[ f_1(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \]

To make sure of validity of the last assertion, it is sufficient to differentiate the equality
\[ \int_{-\infty}^{x} uf(u)du = f(x). \]

Thus, inequality (4) generalizes the Chernoff inequality (1) sufficiently.

In the following theorem, we give generalization of inequality of Chernoff-type (4).

Theorem 2. Let \( \xi \) and \( \eta \) be independent r.v.'s and \( F_\xi(x) \) satisfy conditions (2) and (3). Then for any absolutely continuous function \( g(x) \) with \( g(0) = 0, \)
\[ Dg(\xi\eta) \leq c \alpha E \left[ \eta^2 (g'(\xi\eta))^2 \right]. \] (6)

In the case of \( P(\eta = 1) = 1, \) inequality (6) implies estimation (4).

Remark 2. B.L.S. Prakasa Rao in [3] proved inequality (6) in the case of \( \xi \) is a standard normal r.v. (i.e. with distribution function (5)). Denote also that in [3], using a characterization of the normal distribution obtained by Ch. Stein in [4], lower bounds for \( E [g(\xi\eta)]^2 \) are determined.

Further suppose that considered r.v.'s are defined in a probability space \((\Omega, \mathcal{F}, P)\).

The following theorem generalizes inequality (6).

Theorem 3. Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent r.v.'s with a common distribution function \( F(x) \). Let \( F(x) \) satisfy conditions (2) and (3). Let \( \mathcal{F}_i \) be \( \sigma \)-algebras generated by r.v.’s \( \xi_1, \xi_2, \ldots, \xi_i \) for \( 1 \leq i \leq n \) (\( \mathcal{F}_0 = \{\Omega, \emptyset\} \)). Suppose \( Y_i \) and \( T_i \) are \( \mathcal{F}_{i-1} \)-measurable and r.v.’s \( Y_j \) and \( T_j, i \leq j \leq n \) are independent of \( \xi_i \) for \( i \geq 1 \). Then for any partially differentiable functions \( g(\cdot, \ldots, \cdot) \) and \( h(\cdot, \ldots, \cdot) \) from \( \mathbb{R}^n \)
\[ |\text{Cov}(g(\xi_1Y_1, \ldots, \xi_nY_n), h(\xi_1T_1, \ldots, \xi_nT_n))| \leq \sum_{i=1}^{n} \left( E \left[ Y_i \frac{\partial g}{\partial x_i} \right]^2 E \left[ Y_i \frac{\partial h}{\partial x_i} \right]^2 \right)^{1/2}. \] (7)

Remark 3. Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent r.v.’s with common standard normal distribution function (5). Suppose that random vectors \( (Y_1, \ldots, Y_i) \) and \( (T_1, \ldots, T_i) \) are independent of \( (\xi_1, \xi_{i+1}, \ldots, \xi_n) \) at \( 1 \leq i \leq n \). Then (7) holds.

The following results can be obtained as corollaries to Theorem 3.
Corollary 1. Let $X$ be a standard normal r.v., $g(\cdot)$ and $h(\cdot)$ be real valued absolutely continuous functions. Then
\[ |\text{Cov}(g(X), h(X))| \leq \left( \mathbb{E} \left[ \frac{dg}{dX} \right]^2 \cdot \mathbb{E} \left[ \frac{dh}{dX} \right]^2 \right)^{1/2}. \]

Corollary 2. Let $X_1, X_2, \ldots, X_n$ be independent r.v.’s with common distribution function (5). Further suppose that functions $g(\cdot, \ldots, \cdot)$ and $h(\cdot, \ldots, \cdot)$ from $\mathbb{R}^n$ have partial derivatives of the order 1. Then
\[ |\text{Cov}[g(X), h(X)]| \leq \sum_{i=1}^{n} \left( \mathbb{E} \left[ \frac{\partial g}{\partial X_i} \right]^2 \cdot \mathbb{E} \left[ \frac{\partial h}{\partial X_i} \right]^2 \right)^{1/2} \]
here $X = (X_1, \ldots, X_n)$.

Remark 4. If one passes to the limit in inequalities (6), (7), then he can obtain an analog of the inequality of Chernoff-type for stochastic integrals
\[ \int_0^T \alpha(t)dw(t) \]
where a nonrandom function $\alpha(t) \in L_2(0,T)$, $w(t)$ is the standard Wiener process determined on $[0,T]$.

For example, Theorem 2.2 of [3] implies that for any absolutely continuous function $g(x)$,
\[ D \left[ g \left( \int_0^T \alpha(t)dw(t) \right) \right] \leq \int_0^T \alpha^2(t)dw(t) \mathbb{E} \left[ g' \left( \int_0^T \alpha(t)dw(t) \right) \right]^2. \]

Note also that the last inequality can be used for characterization of the Wiener process in the class of random processes with independent increments.

References


