The Market Price of Risk For Affine Interest Rate Term Structures

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Abstract

This paper examines the market price of risk for discount bond prices under an affine term structure of interest rates. The usual relation plays two roles. First, it is the definition of market price of risk and, second, it provides a no arbitrage condition for the discount bond market. Here the relation defines the market price of risk for more general situations, but includes the processes which give rise to markets with no arbitrage. This allows a separate study of the no arbitrage condition. We solve for the parameters in the general case of affine term structure with constant parameters. The parameters depend explicitly on the market price of risk. Moreover, observations of the yield rate process do not, in general, uniquely determine the market price of risk.

Résumé

Cet article étudie le prix de marché du risque pour des prix d'obligations zéro-coupon, sous une structure par terme affine des taux d'intérêts. La relation habituelle joue un double rôle. Premièrement, c'est la définition du prix du risque, et deuxièmement, elle fournit une condition de non-arbitrage pour le marché des zéro-coupons. Ici, la relation définit le prix de marché du risque pour des situations plus générales, mais inclut les processus qui conduisent des marchés sans opportunités d'arbitrage. Ceci permet une étude séparée de la condition de non-arbitrage. Nous déterminons les paramètres dans le cas d'une structure par terme affine des taux d'intérêts paramètres constants. Les paramètres dépendent de facon claire du prix de marché du risque. De plus, l'observation du processus du taux de rendement ne détermine en général pas uniquement le prix de marché du risque.

Keywords

Stochastic process, interest rate, bond price, yield to maturity, market price of risk, no arbitrage condition, term structure of interest rates, maximum likelihood estimation.

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1 Introduction

Consider a market based on a *short rate* process $r(t)$ that follows the stochastic differential equation

$$dr(t) = \mu(t,r)dt + \sigma(t,r)dW(t) \tag{1}$$

where $W(t)$ is a standard Wiener process and the functions $\mu(t,r)$ and $\sigma(t,r)$ are the instantaneous drift and variance of the process, respectively. The underlying short rate $r(t)$ determines the stochastic properties of the bond price $P(t)$ and the yield to maturity $R(t)$. Let S denote the maturity date and let $T = S - t$ denote the number of years to the maturity date. If the bond price $P(t,r;S)$ can be represented in the form

$$P(t,r;S) = \exp\{A(t,S) - rB(t,S)\} \tag{2}$$

and

$$P(S,r;S) = 1$$

then one says that the model admits an affine term structure [6]. A sufficient condition for an affine term structure is that the coefficients in the equation 1 have the following forms:

$$\mu(t,r) = \alpha(t)r + \beta(t) \tag{3}$$

$$\sigma(t,r)^2 = \gamma(t)r + \delta(t)$$

By the Itô formula the bond price process $P(t,r(t);S)$ follows the stochastic differential equation

$$dP = Pf(t,r;S)dt + Pg(t,r;S)dW(t) \tag{4}$$

where $W(t)$ is the same Wiener process as in equation 1 and the coefficients f and g have the form:

$$\begin{aligned} f(t,r;S) &= A_t(t,S) \\ &\quad - rB_t(t,S) - \mu(t,r)B(t,S) + 0.5\sigma^2(t,r)B^2(t,S) \qquad (5) \\ g(t,r;S) &= \sigma(t,r)B(t,S) \qquad (6) \end{aligned}$$

If the quantity

$$q(t,r;S) = \frac{f(t,r;S) - r}{g(t,r;S)} \tag{7}$$

is independent of the maturity date S one says that the (local) no arbitrage condition is fulfilled and $q(t,r;S)$ is called the *market price of risk*.

The yield to maturity $R(t,T)$ is the internal rate of return at time t on a bond with maturity date $S = t + T$,

$$R(t,T) = -\frac{1}{T}\log P(t,r;t+T) \text{ for } T > 0 \tag{8}$$

The yield to maturity is also a stochastic process. It satisfies the stochastic differential equation

$$dR = u(t, R; T)dt + v(t, R; T)dW(t) \tag{9}$$

where $W(t)$ is again the same Wiener process as in the equation 1 and the coefficients u and v are determined by Itô's formula.

2 Market Price of Risk

The quantity $q(t, r; S)$ in 7 carries a double burden. On the one hand the relation 7 defines the market price of risk. On the other hand, one refers to the relation

$$q(t, r; S_1) = q(t, r; S_2)$$

for all S_1 and S_2 as the no arbitrage condition. It seems reasonable to partition these burdens.

Moreover, there is a question as to why the market price of risk is defined only for the no arbitrage case. In other words, why not let the market price of risk depend on the maturity date S. It is natural to allow for this and, thus, to broaden the concept of arbitrage processes as well. Then the market price of risk will be defined by $q(t, r; S)$ in 7 without the restriction that it be independent of S. Keep in mind that the no arbitrage condition is fulfilled when the $q(t, r; S)$ in 7 does not depend on the maturity date S.

Note that the sign of coefficient σ in the equation 1 does not affect the properties of the short rate process $r(t)$. This is because $V = -W$ is also a standard Weiner process. Replacing W by $-V$ gives an equivalent formulation:

$$dr(t) = \mu(t, r)dt - \sigma(t, r)dV(t)$$

but the sign of the volatility term changed. In the same way, the bond price dynamics can be equivalently formulated as

$$dP = Pf(t, r; S)dt - Pg(t, r; S)dV(t)$$

with the sign of the volatility term changed. In the original formulation "the" market price of risk is $\frac{f-r}{g}$, but in the second it is $\frac{f-r}{-g}$. They are not equal. However, one is independent of S if and only if the other is also. The ambiguity should be eliminated from the definition.

The basic equation for pricing of the bonds in an efficient (no arbitrage) market has the following form:

$$P_t + [\mu(t, r) + q(t, r)\sigma(t, r)]P_r + \frac{1}{2}\sigma^2(t, r)P_{rr} - rP = 0 \tag{10}$$

Here and elsewhere, the subscripts denote partial derivatives. In order maintain an affine term structure framework, one should add to 3 the relation:

$$q(t,r)\sigma(t,r) = \xi(t)r + \eta(t) \tag{11}$$

It follows the affine term the structure of the market price of risk has the following functional form:

$$q(t,r) = \frac{\xi(t)r + \eta(t)}{\sqrt{\gamma(t)r + \delta(t)}} \tag{12}$$

Note that when $\xi(t) = 0$, $\gamma(t) = 0$ and the quantity

$$\frac{\eta(t)}{\sqrt{\delta(t)}}$$

does not depend on r. The market price of risk is a constant as in the Vasicek [8] model; when $\eta(t) = 0$, $\delta(t) = 0$, and the quantity

$$\frac{\xi(t)}{\sqrt{\gamma(t)}}$$

is a constant q the market price of risk is

$$q(t,r) = q\sqrt{r}$$

as in the Cox-Ingersoll-Ross model [4].

Substituting 2, 3, and 11 in the equation 10 gives the equations for the functions $A(t,S)$ and $B(t,S)$ in the following forms:

$$A_t(t,S) = \{\beta(t) + \eta(t)\}\, B(t,S) - 0.5\delta(t)B(t,S)^2 \tag{13}$$

subject to $A(S,S) = 0$ and

$$B_t(t,S) + [\alpha(t) + \xi(t)]B(t,S) - 0.5\gamma(t)B(t,S)^2 + 1 = 0 \tag{14}$$

subject to $B(S,S) = 0$. The explicit form of the functions $A(t,S)$ and $B(t,S)$ can be found by the following steps:

Solve 14.

Substitute the solution of 14 into 13.

Integrate 13.

Using 3, 13, and 14 in 5 and 6 gives the explicit form of the coefficients of equation 4 that, by the formula 7, determine $q(t,r)$.

$$\begin{aligned} f(t,r;S) &= r + [\xi(t)r + \eta(t)]B(t,S) \\ &= r + q(t,r)\sigma(t,r)B(t,S) \qquad (15) \\ g(t,r;S) &= \sigma(t,r)B(t,S) \qquad (16) \end{aligned}$$

Substituting 15 and 16 in the formula 7 to determine the market price of risk gives the identity. Find the explicit solution form for the equations 13 and 14 in the case when the functions α, β, γ, δ, ξ, and η in equalities 3 and 12 are the constants. The solution of the equation 14 has the following form:

$$B(t,S) = \frac{2\left(\exp\{\varepsilon(S-t)\}-1\right)}{\varepsilon+\alpha+\xi+(\varepsilon-\alpha-\xi)\exp\{\varepsilon(S-t)\}} \tag{17}$$

where $\varepsilon = \sqrt{(\alpha+\xi)^2+2\gamma}$. Using this representation in 14 we obtain

$$A(t,S) = \left(\frac{\delta}{\gamma} - \frac{\lambda\vartheta}{\gamma^2}\right)(S-t) - \frac{\delta}{\gamma}B(t,S) - \frac{2\lambda}{\gamma^2}\log\left(1-\frac{\vartheta}{2}B(t,S)\right) \tag{18}$$

where for compactness of notation we use

$$\lambda = \delta(\alpha+\xi) - \gamma(\beta+\eta), \text{ and } \vartheta = \varepsilon-(\alpha+\xi). \tag{19}$$

Note that formulas 17 and 18 are of independent interest because they give the general solution of the problem for an affine term structure with constant parameters. When $\delta=0$, $\eta=0$ the expressions 17 and 18 give the corresponding functions for the Cox-Ingersoll-Ross model (the function $B(t,S)$ has the same form). When $\gamma=0$, $\xi=0$, the expressions 17 and 18 give the corresponding functions of the Vasicek model (in order to obtain the explicit form of the function $A(t,S)$, it is necessary to first decompose the function $B(t,S)$ in a series in γ up to second order terms).

For the determination of the market price of risk, Vasicek [8, 1977, page 184] suggests using the following formula:

$$\lim_{T\to 0^+} R_T = \frac{1}{2}\left[\mu(t,r)+q(t,r)\sigma(t,r)\right] \tag{20}$$

Using the relations 2, 3, 8 and 12, the equality 20 may be rewritten in the following form:

$$\lim_{T\to 0^+} \frac{A(t,t+T)-TA_S(t,t+T)}{T^2} + r\frac{TB_S(t,t+T)-B(t,t+T)}{T^2} =$$

$$\frac{1}{2}\left(\left[\alpha(t)+\xi(t)\right]r+\beta(t)+\eta(t)\right) \tag{21}$$

where the subscript S denotes the partial derivative with respect to the second argument of the functions $A(t,S)$ and $B(t,S)$. However, equation 21 does not allow us to determine the market price of risk (at least for the general affine term structure with constant parameters). Indeed, expand the functions $A(t,S)$ and $B(t,S)$ for the general affine term structure with the constant parameters as in equations 17 and 18 as power series in $T = S - t$:

$$\begin{aligned} B(T) &= T + 0.5(\alpha + \xi)T^2 + o(T^2) \\ A(T) &= -0.5(\beta + \eta)T^2 + o(T^2) \end{aligned} \tag{22}$$

Because $A_S(t, t+T) = A_t(T)$ and $B_S(t, t+T) = B_t(T)$, the equality 21 takes the form

$$\lim_{T \to 0^+} (0.5(\beta + \eta) + 0.5r(\alpha + \xi) + o(T)) = 0.5[(\alpha + \xi)r + \beta + \eta] \tag{23}$$

Thus we obtain again an identity that does not specify the market price of risk. Thus we see that an explicit form of the market price of risk is not provided by an affine term structure framework. Some authors (Duffie [6], Hull [3], Björk [1]) specify no arbitrage processes in the affine term structure framework by writing $q(t, r) = 0$. Regardless of whether or not there is an affine term structure, if there is no arbitrage, $q(t, r; S)$ is independent of S. Then the change of measure obtained by setting $dV = dW + q(t, r)dt$, and replacing dW by $dV - q(t, r)dt$ gives a reformultion in which

$$dP = rPdt + gPdV.$$

In the new formulation, the market price of risk is zero.

Finally, the market price of risk is defined by expression 7 only for bond prices that follow the stochastic differential equation 4. At the same time, bond price changes can be described by other stochastic mathematical models (for example by Markov chains or autoregresssion processes). The definition of the market price of risk should be independent of the way of bond price changes are described mathematically. In this sense, the definition of market price of risk needs to be revised.

3 The No Arbitrage Condition

This condition is fulfilled when the quantity 7 does not depend on the maturity date S. From the mathematical point of view this means that the derivative of $q(t, r; S)$ with respect to S is equal to zero. Substitute in 7 the explicit form of coefficients f and g from 5 and 6, compute this derivative and set it equal to zero. After some algebraic transformation, we obtain the no arbitrage condition (NAC) in the following form:

$$\begin{aligned} &\frac{1}{2}\sigma(t, r)^2 B(t, S)^2 B_S(t, S) = \\ &\quad A_t(t, S)B_S(t, S) - A_{tS}(t, S)B(t, S) + \\ &\quad r\left[B_{tS}(t, S)B(t, S) - B_t(t, S)B_S(t, S) - B_S(t, S)\right] \text{ for } t \le S \end{aligned} \tag{24}$$

Expression 24 provides some interesting conclusions:

1. The NAC is not linked explicitly with the instantaneous drift $\mu(t,r)$ (the terms that contain the drift cancel). The dependence of the NAC on the properties of the instantaneous drift comes through the functions $A(t,S)$ and $B(t,S)$ (see equations 13 and 14).
2. The NAC is linked explicitly to the properties of the instantaneous variance $\sigma(t,r)^2$. In order for the NAC to hold, the instantaneous variance must be linear function of the short rate r. Note that this property is consistent with the variance form in 3
3. Because $B_S(t,S) > 0$ (when the maturity date increases, the bond price has to decrease for any fixed short rate), the left side of 24 is positive. This means that the NAC must satisfy the inequality:

$$\begin{aligned} A_t(t,S)B_S(t,S) - A_{tS}(t,S)B(t,S) > \\ r\left([B_t(t,S)+1]\,B_S(t,S) - B(t,S)B_{tS}(t,S)\right) \end{aligned} \quad (25)$$

4. If the instantaneous variance $\sigma(t,r)$ is independent of the short rate r (the coefficient $\gamma(t)$ in 3 is equal to zero), then the NAC holds for an affine term structure where the function $B(t,S)$ satisfies the additional equation:

$$B(t,S)B_{tS}(t,S) - B_t(T,S)B_S(t,S) - B_S(t,S) \quad = \quad 0 \quad (26)$$

   Moreover, from 2 the function $B(t,S)$ must have the properties $B(S,S) = 0$, $B_t(t,S) < 0$, and $B_S(t,S) > 0$
5. If the instantaneous variance $\sigma(t,r)^2$ is proportional to the short rate r (the coefficient $\delta(t)$ in 3 is equal to zero), the NAC holds for an affine term structure for which the functions $A(t,S)$ and $B(t,S)$ satisfy the equation:

$$A_t(t,S)B_S(t,S) - A_{tS}(t,S)B(t,S) \quad = \quad 0 \quad (27)$$

   subject to $A(S,S) = 0$.

It is easy to check the models: Vasicek [8], Cox-Ingersoll-Ross [4], Ho-Lee (in the description of Hull [3]), Hull-White [7, 1993] (extended Vasicek and extended CIR) satisfy the NAC. The models of Dothan [5] and Black-Derman-Toy [2] do not satisfy the NAC, but they are not affine term structures either (Duffie [6]).

The NAC 24 is interesting because it is written only in terms of the functions $A(t,S)$ and $B(t,S)$ and can be used to check the NAC when the bond price is given in the form 2 but equation 4 is not given. At the same time, the equality 24 required in order to fulfill the NAC. This appears to take into account equations 13 and 14 for the functions $A(t,S)$ and $B(t,S)$ in equality 24. The equality is transformed in equality 3 for $\sigma(t,r)^2$. Thus the expressions 3 are sufficient conditions in order for the bond price to have the form 2, and the second equality 3 is a necessary and sufficient condition that the bond price process 4 is a no arbitrage process.

4 Estimation of the Market Price of Risk

Although the market price of risk is not determined explicitly in the affine term structure framework (see above), one hope to estimate it from observed data. The market price of risk is a function of the unobserved short rate $r(t)$. The financial press quotes only the asset price or the yield rate. Consider the opportunity to estimate the market price of risk by the observating yield rates. From the equalities 2 and 8 we get the following dependence between the yield rate $R(t,T)$ and the short rate $r(t)$

$$R(t,T) = h(t,r) = \frac{r(t)B(t,t+T) - A(t,t+T)}{T} \tag{28}$$

This dependence allows construction of the stochastic differential equation for the yield rate in 9. The coefficients of this equation are determined by the Itô formula applied to the function $h(t,r)$ in 28:

$$u = h_t + \mu(t,r)h_r + 0.5\sigma(t,r)^2 h_{rr}$$

and $v = \sigma(t,r)h_r$. Using the explicit form of the function $h(t,r)$ from 28 gives

$$\begin{aligned} u(t,R;T) &= \frac{1}{T}\{[B_t(t,t+T) + B_S(t,t+T)]\,r(R) - \\ &\quad A_t(t,t+T) - A_S(t,t+T) + \mu(t,r(R))B(t,t+T)\} \\ \text{and} & \\ v(t,R;T) &= \frac{1}{T}\sigma(t,r(R))B(t,t+T) \end{aligned}$$

Here, as before, the subscript t denotes the partial derivative with respect to the first variable and the subscript S denotes the partial derivative with respect to the second variable; $r(R)$ denotes the inverse of the function 28, *i.e.*:

$$r(R) = \frac{TR + A(t,t+T)}{B(t,t+T)} \tag{29}$$

For models with constant parameters (*i.e.*:the functions α, β, γ, δ, ξ, and η in 3 and 11 are constant), the following properties hold: $A(t,S) =$

$A(S-t)$ and $B(t,S) = B(S-t)$. Therefore, $A_t = -A_S$, $B_t = -B_S$, and the expressions for $u(t,R;S)$ and $v(t,R;S)$ depend only on R and the difference $S - t = T$. They are reduced to the following form:

$$\begin{aligned} u(R,T) &= \alpha R + \alpha\left(\frac{A(T)}{T} + \frac{\beta}{\alpha}\frac{B(T)}{T}\right) \\ v(R,T) &= \frac{B(T)}{T}\sqrt{\gamma\frac{TR + A(T)}{B(T)} + \delta} \end{aligned} \tag{30}$$

Take the simplifying assumption $\gamma = 0$, and introduce the simplifying notation:

$$\begin{gathered} \sigma = \sqrt{\delta} \\ F = \frac{A(T)}{t} + \frac{\beta}{\alpha}\frac{B(T)}{T} \end{gathered} \tag{31}$$

Then the stochastic differential equation 9 for the yield rate $R(t,T)$ can be rewritten in this form:

$$dR = \alpha R dt + \alpha F dt + \sigma\frac{B(T)}{T}dW(t) \tag{32}$$

For the existence of a stationary solution, take $\alpha < 0$. This equation allows an analytical solution in the following form:

$$R(s) = R(t)\exp\{\alpha(s-t)\} - F\left(1 - \exp\{\alpha(s-t)\}\right) + \zeta(t,s) \tag{33}$$

where $\zeta(t,s)$ is a normal random variable with zero expectation and covariance of the form:

$$\begin{aligned} &\mathrm{Cov}(\zeta(t,s_1),\zeta(t,s_2)) = \\ &\frac{\sigma^2 B(T)^2}{2|\alpha|T^2}\left(\exp\{\alpha|s_1 - s_2|\} - \exp\{\alpha(s_1 + s_2 - 2t)\}\right) \end{aligned} \tag{34}$$

The random variables $\zeta(t_1,t_2)$ and $\zeta(t_3,t_4)$ are mutually independent for any $t_1 < t_2 \le t_3 < t_4$. These properties of the solution 33 allow for estimation of the unknown parameters of equation 32 by maximum likelihood methods. Introduce the simplifying notation

$$G = \frac{\sigma^2 B(T)^2}{2|\alpha|T^2} \tag{35}$$

Consider a sample set of the observations of the yield rate values:

$$\{R_i = R(t_i,T) : 1 \le i \le N\}$$

These sample values are functions of the unknown parameters of the yield rate process 32. These parameters are α, β, σ, ξ, and η. Two of these (ξ and η) determine the market price of risk by formula 12. These parameters determine the functions A and B by equations 13 and 14 and the auxiliary values F and G by formulas 31 and 35. Using the normality and the mutual independence of the random variables $\zeta(t_i, t_{i+1})$ allows us to write the logarithm of the likelihood function in the form:

$$-\frac{1}{2}\sum_{i=1}^{N-1}\left(\frac{\left[R_{i+1} - R_i \exp\{\alpha\tau_i\} + F(1-\exp\{\alpha\tau_i\})\right]^2}{G(1-\exp\{2\alpha\tau_i\})} + \log\left[2\pi G(1-\exp\{2\alpha\tau_i\})\right]\right) \tag{36}$$

In this expression the known values are $\{R_i\}$ and $\{\tau_i = t_{i+1} - t_i\}$. The unknown values are α, F, and G. Maximizing the function 36 with respect to the variables α, F, and G, we find their most likely values α^e, F^e, and G^e. Thus, the estimate α^e provides one of the five unknown parameters.

Solving equations 13 and 14, we obtain

$$A(T) = \left(\frac{\beta+\eta}{\alpha+\xi} + \frac{\sigma^2}{2(\alpha+\xi)^2}\right)[T - B(T)] + \frac{\sigma^2 B(T)^2}{4(\alpha+\xi)} \tag{37}$$

and

$$B(T) = \frac{\exp\{(\alpha+\xi)T\} - 1}{(\alpha+\xi}. \tag{38}$$

Thus for the determination of the four remaining parameters β, σ, ξ, and η, we have only two equations:

$$\left(\frac{\beta+\eta}{\alpha^e+\xi} - \frac{\sigma^2}{2(\alpha+\xi)^2}\right)\left(1 - \frac{B(T)}{T}\right) + \frac{\sigma^2 B(T)^2}{4T(\alpha^e+\xi)} + \frac{\beta}{\alpha^e}\frac{B(T)}{T} = F^e \tag{39}$$

and

$$\frac{\sigma^2 B(T)^2}{2|\alpha^e|T^2} = G^e \tag{40}$$

solving equation (40) for B(T) and substituting in the equation (39), we obtain the equality that links the four unknown parameters:

$$\left(\frac{\beta+\eta}{\alpha^e+\xi} - \frac{\sigma^2}{2(\alpha+\xi)^2}\right)\left(1 - \tfrac{1}{\sigma}\sqrt{2|\alpha^e|G^e}\right) + \frac{2T|\alpha^e|G^e}{4(\alpha^e+\xi)} + \frac{\beta}{\alpha^e}\,\tfrac{1}{\sigma}\sqrt{2|\alpha^e|G^e} = F^e \tag{41}$$

This means that the four parameters β, σ, ξ, and η cannot be estimated uniquely solely by the maximum likelihood method. For their unique estimation, it is necessary to have additional information. We have a more simple case if the parameter ξ is equal to zero (Vasicek model). In this case, the market price of risk is a constant

$$q = \frac{\eta}{\sigma}$$

. Then the value $B(T)$ is determined uniquely by formula 38, $B(T) = B^e$, and the parameter σ can be determined uniquely by formula 40 also: $\sigma = \sigma^e$. However, the two parameters β and η cannot be determined uniquely. They define a two-dimensional surface that is given by the following relation:

$$\beta + \eta \left(1 - \frac{B^e}{T}\right) = \alpha^e F^e + 0.5(\sigma^e)^2 \left(\frac{1}{\alpha^e}\left(1 - \frac{B^e}{T}\right) + \frac{(B^e)^2}{2T}\right) \quad (42)$$

Thus the market price of risk cannot be found by the observing the yield rate process. Note that the parameter β that determines the expectation of the yield rate process in the stationary case cannot be found uniquely either. There is no problem for unique estimation of this parameter when the market price of risk is equal to zero ($\xi = 0$, $\eta = 0$).

5 Conclusions

The market price of risk $q(t, r; S)$ should be redefined so that it applies not only in the case of no arbitrage, but also markets that admit arbitrage. The definition of the market price of risk has to be given in a form that is independent of the mathematical description of the bond price process.

The equalities 17 and 19 give the solution to the general problem of affine term structure with the constant parameters. They show that the market price of risk cannot be found in the way Vasicek [8] suggests.

The no arbitrage condition (NAC) is defined by the equality 24. It can be used to check the NAC when the bond price is given in an affine term structure, but not in the form of a stochastic differential equation.

The NAC 24 coincides with the second equality 3. This means that the equalities 3 are sufficient for an affine term structure of bond prices and the second equality 3 is necessary and sufficient that this structure admit the NAC.

Affine term structure and the bond price process do not determine the value of the market price of risk.

Observations of the yield rate process alone are not sufficient to uniquely estimate the market price of risk.

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