CONSTRUCTION OF QUEUEING NETWORKS WITH STATIONARY PRODUCT DISTRIBUTIONS

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In this article some new product theorems for opened and closed queuing networks with finite number of states have been proved. Each network is characterized by a graph with states in its nodes and positive transition intensities in its edges. For different graphs and systems of motion equations stationary product distributions of Markov processes, which describe queuing networks with different prohibited transitions, have been obtained. Calculation algorithm of route matrices corresponding to modified systems of motion equations has been constructed.

Keywords: product theorems, queueing networks, prohibitions.

1. FORMULATION OF PROBLEM AND MAIN RESULTS

In classical product theorems [1], [2] it is possible to recognize an analogy between opened and closed queuing networks. Closed queuing network may be represented as an opened network in which both arrivals to the network and departures from the network are prohibited. These prohibitions do not prevent to represent stationary distributions of customer numbers at different nodes in product form. Is it possible to widen a set of such prohibitions with the product form of stationary distributions? This problem occurs from modern applications for example retrial queues which are very important in modern applications.

In this article some new product theorems for opened and closed queuing networks with finite number of states have been proved. Each network is characterized by a graph with states in its nodes and positive transition intensities in its edges. For different graphs and systems of motion equations stationary product distributions of Markov processes, which describe queuing networks with different prohibited transitions, have been obtained. Calculation algorithm of route matrices corresponding to modified systems of motion equations has been constructed.

Consider an opened queuing network G with m nodes (one-server queuing systems), input intensity $\lambda > 0$, serving intensities $\mu_1 > 0, \dots, \mu_m > 0$ and route matrix $\Theta = (\theta_{ij})_{k,i=0}^{m}$:

$$\theta_{00} = 0, \quad 0 \le \theta_{ki} \le 1, \quad (k,i) \ne (0,0), \quad \sum_{i=0}^{m} \theta_{ki} = 1, \quad 0 \le k, i \le m.$$

Here θ_{ki} is a probability of customer transition from i-th node to j-th node after its serving in i-th node (the 0-th node is an external source or external space).

Denote $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_m = (0, \ldots, 0, 1)$ basic vectors in the linear space E^m and let $Z^m = \{\mathbf{N} = (n_1, \ldots, n_m),\ n_1 \geq 0, ..., n_m \geq 0\}$. Suppose that $L(\mathbf{N}, \mathbf{J})$, $\mathbf{J} = (j_1, \ldots, j_m)$, $j_1, j_2, ..., j_m \geq 0$ are transition intensities of discrete Markov process $\mathbf{N}(t)$ with states in Z^m which describe numbers of customers in the nodes of opened Jackson network with the route matrix Θ and input intensity λ. Positive transition intensities of Markov process $\mathbf{N}(t)$ are the following

$$L(\mathbf{N}, \mathbf{N} + \mathbf{e}_k) = \lambda\theta_{0k},\ \ \mathbf{N} \in Z^m,\ \ L(\mathbf{N}, \mathbf{N} - \mathbf{e}_k) = \mu_k\theta_{k0},\ \ \mathbf{N} \in Z^m,\ \ n_k > 0, \tag{1}$$

$$L(\mathbf{N}, \mathbf{N} - \mathbf{e}_k + \mathbf{e}_i) = \mu_k\theta_{ki},\ \ \mathbf{N} \in Z^m,\ \ n_k > 0,\ \ 1 \leq k \neq i \leq m. \tag{2}$$

To describe the network G with prohibitions consider non-oriented graph Γ with nodes from states set $\mathcal{L} \subseteq Z^m$ with edges represented in the form

$$[\mathbf{N}, \mathbf{N} + \mathbf{e}_k],\ \ \mathbf{N} \in Z^m,\ \ 1 \leq k \leq m, \tag{3}$$

$$[\mathbf{N}, \mathbf{N} - \mathbf{e}_k + \mathbf{e}_i],\ \ \mathbf{N} \in Z^m,\ \ n_k > 0;\ \ 1 \leq k \neq i \leq m. \tag{4}$$

If (3) edge belongs to the graph Γ then there is an allowance for a customer transition from k-th node to 0-th node and from 0-th node to k-th node, $1 \leq k \leq m$. If (4) edge belongs to the graph Γ then there is an allowance for customer served in i-th node to transit to k-th node and vica versa. If some of (3), (4) edges does not belong to the graph Γ then appropriate transitions are prohibited in both sides.

If (3) (or (4)) edge is prohibited then a customer arrived in k-th node from i-th node returns to i-th node and vica versa, $0 \leq k \neq i \leq m$ and so transition intensity (1) (or transition intensity (2)) becomes equal to 0. Such allowances and prohibitions define a protocol in the network G for example retrial queues protocol.

The network G with states set $\mathcal{L} = Z^m$ and all possible (3), (4) edges (infinite queues in network nodes) is an opened Jackson network. If the route matrix is indivisible:

$$\forall i, j \in \{0, 1, ..., m\}\ \exists\ i_1,\ i_2, \ldots i_r \in \{1, ..., m\}\ : \theta_{ii_1} > 0,\ \theta_{i_1i_2} > 0, \ldots \theta_{i_rj} > 0,$$

then the system of motion equations

$$(\lambda_1, \ldots, \lambda_m) = (\lambda, \lambda_1, \ldots, \lambda_m)\Theta_0 \tag{5}$$

has a unique solution and $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$. Here the matrix Θ_0 is obtained from the matrix Θ by removal of the 0-th column. If $\lambda_1 < \mu_1, \ldots, \lambda_m < \mu_m$ then the discrete Markov process $\mathbf{N}(t)$, $t \geq 0$ is ergodic [3] and its limit distribution [1] is calculated by

$$P(\mathbf{N}) = C\Phi(\mathbf{N}),\ \ \Phi(\mathbf{N}) = \prod_{i=1}^{m} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i},\ \ C = \left[\sum_{\mathbf{N} \in L} \Phi(\mathbf{N})\right]^{-1},\ \ \mathbf{N} \in \mathcal{L}. \tag{6}$$

Theorem 1. *Suppose that the route matrix Θ of the opened network satisfies a condition*

$$0 < \theta_{0k}, \theta_{k0} < 1,\ \ 1 \leq k \leq m \tag{7}$$

and a vector $(\lambda_1, \dots, \lambda_m)$ *is a solution of the system (5). Let* $\mathcal{L}_0$ *is finite subset of* Z^m *so that the graph* Γ *consisting of edges* $\{[\mathbf{N}, \mathbf{N}+\mathbf{e}_i],\ [\mathbf{N}+\mathbf{e}_i, \mathbf{N}+\mathbf{e}_k] :\ \mathbf{N} \in \mathcal{L}_0,\ 1 \le i \ne k \le m\}$ *is connected. Then the discrete Markov process* $\mathbf{N}(t)$ *is ergodic and its limit distribution* $P(\mathbf{N})$, $\mathbf{N} \in \mathcal{L} = \mathcal{L}_0 \bigcup \{\mathbf{N}+\mathbf{e}_i :\ \mathbf{N} \in \mathcal{L}_0,\ 1 \le i \le m\}$ *is calculated by (6).*

Following sets $\{\mathbf{N} \in Z^m :\ \sum_{k=1}^m n_k < M\}$, $\{\mathbf{N} \in Z^m :\ M' < \sum_{k=1}^m n_k < M\}$, $\{\mathbf{N} \in Z^m : M'_k \le n_k < M_k,\ 1 \le k \le m\}$, $0 \le M', M, M'_1, M_1, \dots, M'_m, M_m < \infty$ satisfy conditions of this theorem.

Theorem 2. *Suppose that the conditions (7) are true and* $\mathcal{L}$ *is finite subset of* Z^m *and* Γ *is connected graph with nodes which create a set* $\mathcal{L}$ *and with (3), (4) edges. If elements of the matrix* Θ *and numbers* $\lambda_1 > 0, \dots, \lambda_m > 0$ *satisfy equalities*

$$\lambda_k\theta_{k0} = \lambda\theta_{0k},\quad 1 \le k \le m, \tag{8}$$

$$\lambda_i\theta_{ik} = \lambda_k\theta_{ki},\quad 1 \le k \ne i \le m \tag{9}$$

then the discrete Markov process $\mathbf{N}(t)$ *is ergodic and its limit distribution* $P(\mathbf{N})$, $\mathbf{N} \in \mathcal{L}$ *is calculated by (6).*

Consider now closed Jackson network G' with states set $\mathcal{L} = \{\mathbf{N} \in Z^m,\ \sum_{i=1}^m n_i = M\}$ and all (4) edges. Here M is the total number of customers in the set G'. The set G' is described by discrete Markov process $\mathbf{N}(t)$, $t \ge 0$ with states set $\mathcal{L}$ and transition intensities (2). If route matrix $\Theta' = \left(\theta'_{ki}\right)_{k,i=1}^m$ of the set G' satisfies the condition

$$0 < \theta'_{ki} < 1,\quad 1 \le k, i \le m \tag{10}$$

then appropriate system of motion equations [2]

$$(\lambda_1, \dots, \lambda_m) = (\lambda_1, \dots, \lambda_m)\Theta' \tag{11}$$

has infinite number of solutions $\lambda_1, \dots, \lambda_m$ with nonnegative components. For any natural M and for any solution of the system (11) with nonnegative components the formula (6) is true.

Theorem 3. *Suppose that route matrix* Θ' *of closed network satisfies the condition (10) and a vector* $(\lambda_1, \dots, \lambda_m)$ *is a solution of the system (11) with nonnegative components. Suppose that* $\mathcal{L}_0$ *is finite subset of* Z^m *so that the graph* Γ *consisting of edges* $\{[\mathbf{N}+\mathbf{e}_i, \mathbf{N}+\mathbf{e}_k] :$ $\mathbf{N} \in \mathcal{L}_0,\ 1 \le i \ne k \le m\}$ *is connected. Then discrete Markov process* $\mathbf{N}(t)$ *is ergodic and its limit distribution* $P(\mathbf{N})$, $\mathbf{N} \in \mathcal{L} = \{\mathbf{N}+\mathbf{e}_i :\ \mathbf{N} \in \mathcal{L}_0,\ 1 \le i \le m\}$ *is calculated by (6).*

Theorem 4. *Suppose that* $\mathcal{L}$ *is finite subset of* Z^m *and* Γ *is connected graph with nodes which create a set* $\mathcal{L}$ *and with (4) edges. If elements of the matrix* Θ' *and numbers* $\lambda_1 > 0, \dots, \lambda_m > 0$ *satisfy equalities*

$$\lambda_i\theta'_{ik} = \lambda_k\theta'_{ki},\quad 1 \le k, i \le m \tag{12}$$

then discrete Markov process $\mathbf{N}(t)$ *is ergodic and its limit distribution* $P(\mathbf{N})$, $\mathbf{N} \in \mathcal{L}$, *is calculated by (6).*

2. PROOF OF MAIN RESULTS

Proof of the theorem 1. Denote

$$F_0(\mathbf{N}) = \{\mathbf{N} + \mathbf{e}_i,\ \ 1 \le i \le m\},$$

$$F_i(\mathbf{N}) = \{\mathbf{N} - \mathbf{e}_i,\ \mathbf{N} - \mathbf{e}_i + \mathbf{e}_j,\ \ 1 \le j \le m,\ \ j \ne i\},\ \ 1 \le i \le m;\ \ \mathbf{N} \in Z^m.$$

It is clear that

$$F_i(\mathbf{N}) \bigcap F_j(\mathbf{N}) = \varnothing. \tag{13}$$

As conditions of the theorem 1 are true so in the graph Γ for any $\mathbf{N} \in \mathcal{L}$ there is a set of indexes $I(\mathbf{N}) \subseteq \{0, \dots, m\}$ so that a set of nodes $S(\mathbf{N})$ connected with the node $\mathbf{N}$ by single edges is represented as follows

$$S(\mathbf{N}) = \bigcup_{i \in I(\mathbf{N})} F_i(\mathbf{N}). \tag{14}$$

From (1), (2), (5) obtain

$$\sum_{\mathbf{J} \in F_i(\mathbf{N})} [\Phi(\mathbf{N})L(\mathbf{N}, \mathbf{J}) - L(\mathbf{J}, \mathbf{N})\Phi(\mathbf{J})] = 0,\ 0 \le i \le m,\ \mathbf{N} \in \mathcal{L}. \tag{15}$$

From (15) obtain for $\mathbf{N} \in \mathcal{L}$

$$\sum_{\mathbf{J} \in \mathcal{L}} [\Phi(\mathbf{N})L(\mathbf{N}, \mathbf{J}) - L(\mathbf{J}, \mathbf{N})\Phi(\mathbf{J})] = \sum_{\mathbf{J} \in S(\mathbf{N})} [\Phi(\mathbf{N})L(\mathbf{N}, \mathbf{J}) - L(\mathbf{J}, \mathbf{N})\Phi(\mathbf{J})] = 0. \tag{16}$$

The graph Γ is connected and the conditions (7) and the equalities (16) are true and transition intensities $L(\mathbf{N}, \mathbf{J})$ are bounded by $\lambda + \mu_1 + \ldots + \mu_m$. So the process $\mathbf{N}(t)$, describing an opened queuing network with states set $\mathcal{L}$ and the graph Γ satisfies well known ergodicity theorem for Markov processes [3]. So it is ergodic and its limit distribution $P(\mathbf{N})$ satisfies the formula (6). The theorem 1 is proved.

Proof of the theorem 3. It repeats the proof of the theorem 1 practically word by word. Small changes touche only the inclusion $I(\mathbf{N}) \subseteq \{0, \dots, m\}$ which is replaced by the inclusion $I(\mathbf{N}) \subseteq \{1, \dots, m\}$. And in (15) the inequality $0 \le i \le m$ is replaced by the inequality $1 \le i \le m$.

Proof of the theorem 2. The equalities (6), (8), (9) lead to

$$\Phi(\mathbf{N})\lambda\theta_{0k} - \Phi(\mathbf{N} + \mathbf{e}_k)\mu_k\theta_{k0} = 0,\ \ \mathbf{N},\ \mathbf{N} + \mathbf{e}_k \in \mathcal{L},\ \ 1 \le k \le m, \tag{17}$$

$$\Phi(\mathbf{N})\mu_i\theta_{ik} - \Phi(\mathbf{N} + \mathbf{e}_k - \mathbf{e}_i)\mu_k\theta_{ki} = 0,\ \ \mathbf{N}, \mathbf{N} + \mathbf{e}_k - \mathbf{e}_i \in \mathcal{L},\ \ 1 \le k \ne i \le m. \tag{18}$$

As the formulas (1), (2), (17), (18) are true so the function $\Phi(\mathbf{N})$ satisfies the equations (16). An end of the theorem 2 proof repeats the end of the theorem 1 proof. The theorem 2 is proved.

Proof of the theorem 4. The equalities (6), (12) lead to the formulas (18) so the function $\Phi(\mathbf{N})$ satisfies the equations (16). An end of the theorem 4 proof repeats the end of the theorem 1 proof. The theorem 4 is proved.

Remark 1. If in the conditions of the theorems 1 – 4 $\lambda_1 < \mu_1, \dots, \lambda_m < \mu_m$ then in these theorems, the set may be infinite.

Remark 2. In the theorem 2 (the theorem 4) new results are obtained not by special choice of the graph Γ (it is arbitrary here) but by a replacement of Jackson motion equations (5) (Gordon-Newell motion equations(11)) by modified motion equations (8), (9) (modified motion equations (12)).

Remark 3. It is clear from the proof that the theorem 1 is true for any connected graph Γ satisfying the following condition. For any node **N** of the graph Γ there is a set of indexes $I(\mathbf{N}) \subseteq \{0, \dots, m\}$ so that $S(\mathbf{N})$ may be represented in the form (14). It is easy to prove that the conditions of the theorem 1 describe all graphs Γ which satisfy this condition. Analogous statement is true for the theorem 3.

3. ALGORITHMS OF ROUTE MATRIX CONSTRUCTION

At first consider the opened network. Suppose that $\lambda > 0,\ \theta_{0k} > 0,\ 1 > \theta_{k0} > 0,\ 1 \le k \le m$ are fixed. Using the formulas (8) it is possible to define $\lambda_1, \dots, \lambda_m$ and $\theta_{ki},\ 1 \le k, i \le m$ which are solutions of the following problem.
Denote

$$\Lambda_k = \lambda_k(1 - \theta_{k0}), \tag{19}$$

then from (8)

$$\Lambda_k = \frac{\lambda\theta_{0k}(1 - \theta_{k0})}{\theta_{k0}}, \quad 1 \le k \le m. \tag{20}$$

Let

$$\pi_{ki} = \theta_{ki}/(1 - \theta_{k0}), \tag{21}$$

so

$$\sum_{i=1}^{m} \pi_{ki} = 1, \quad 1 \le k \le m. \tag{22}$$

Denote

$$C_{ki} = \Lambda_k \pi_{ki}, \quad 1 \le k, i \le m. \tag{23}$$

As the formulas (9), (22) are true so the matrix $C = (C_{ki})_{k,i=1}^{m}$ is a permissible solution of the transportation problem

$$\sum_{i=1}^{m} C_{ki} = \Lambda_k, \quad C_{ki} = C_{ik} > 0, \quad 1 \le k, i \le m. \tag{24}$$

So using fixed numbers $\lambda > 0,\ \theta_{0k} > 0,\ 1 > \theta_{k0} > 0,\ 1 \le k \le m$ find by the formulas (19) $\Lambda_k,\ 1 \le k \le m$. Then using $\Lambda_k,\ 1 \le k \le m$ define a permissible solution of the transportation problem (24) and from the formulas (21), (23) find $\pi_{ki},\ \theta_{ki},\ 1 \le k, i \le m$.

Consider now a search of permissible solutions of the transportation problem (24). Renumber nodes $1, \dots, m$ so that $\Lambda_1 \le \Lambda_2, \dots, \Lambda_1 \le \Lambda_m$. Choose $C_{11}, \dots, C_{1m}$ from the conditions

$$\sum_{k=1}^{m} C_{1k} = \Lambda_1, \quad C_{11} > 0, \dots, C_{1m} > 0$$

and let

$$C_{k1} = C_{1k}, \quad k = 1, \ldots, m.$$

Redefine now $\Lambda_2, \ldots, \Lambda_m$ by the following formulas:

$$\Lambda_2 := \Lambda_2 - C_{21}, \ldots, \Lambda_m := \Lambda_m - C_{m1}.$$

Now it is necessary to define $C_{ki}, \quad 2 \le k, i \le m$ from

$$\sum_{i=2}^{m} C_{ki} = \Lambda_k, \quad C_{ki} = C_{ik} > 0, \quad 2 \le k, i \le m. \tag{25}$$

Apply the same procedure but to the problem (25) and so on. Resulting algorithm is a modification of well known North-West Angle algorithm [4] of transportation problem (24) solution. The only difference is in additional condition that the matrix C is symmetric.

The case of the closed network may be considered analogously.

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REFERENCES

1. *Jackson J. R.* Networks of Waiting Lines // Oper. Research. 1957. V. 5. № 4. P. 518–521.
2. *Gordon K. D., Newell G. F.* Closed Queuing Systems with Exponential Servers// Oper. Research. 1967. V. 15. № 2. P. 254–265.
3. *Ivchenko G. I., Kashtanov V. A., Kovalenko I. N.* Queuing Theory. M.: Vischaija shkola, 1982. 256 p. (in Russian).
4. *Gabasov R., Kirillova F. M.* Linear Programming Methods. Mn.: BSU, 1978. Part II. 239 p. (in Russian).