MULTISERVER QUEUEING SYSTEM WITH COMPETITION BETWEEN SERVERS

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The queueing systems with the competition between servers or the customers are widely used in modern data transmission and mobile telephone networks [1]. But analytical investigation of the competition influence on the queueing systems characteristics was not made. In this subsection mathematical model of multiserver queueing system with the competition between servers is constructed. This system is compared with the classical multiserver queueing system in terms of their abilities to handle and the distribution tails of the stationary waiting times.

Keywords: queueing system, ability to handle, competition between servers.

1. FORMULATION OF PROBLEM

The queueing systems with the competition between servers or the customers are widely used in modern data transmission and mobile telephone networks [1]. But analytical investigation of the competition influence on the queueing systems characteristics was not made. In this subsection mathematical model of multiserver queueing system with the competition between servers is constructed. This system is compared with the classical multiserver queueing system in terms of their abilities to handle and the distribution tails of the stationary waiting times.

Consider the multiserver queueing system $M|G|m|\infty$ with m servers, the input flow $0 = t_1 \leq t_2 = t_1 + \xi_1 \leq t_3 = t_2 + \xi_2 \leq \ldots$ and the serving times $\eta_1, \eta_2, \ldots$ The random sequences $\{\eta_1, \eta_2, \ldots\}$, $\{\xi_1, \xi_2, \ldots\}$ are independent, consist of independent r.v.'s and have d.f.'s $G(t)$, $H(t)$, concentrated on $[0, \infty)$, with the tails

$$P(\eta_k > t)=1 - G(t)=\overline{G}(t), P(\xi_k > t)= \exp(-\lambda t),\ k \geq 0. \tag{1}$$

Denote this system by A_m and suppose that it is empty at the moment t_1 of the first customer arrival.

Consider multiserver queueing system $GI|GI|m|\infty$ with m servers, input flow $0 = t_1 \leq t_2 = t_1 + \xi_1 \leq t_3 = t_2 + \xi_2 \leq \ldots$ and serving times $\eta_1, \eta_2, \ldots$ The random sequences $\{\eta_1, \eta_2, \ldots\}$, $\{\xi_1, \xi_2, \ldots\}$ are independent and consist of independent r.v.'s with d.f.'s $G(t)$, $H(t)$ concentrated on $[0, \infty)$ and with the tails

$$P(\eta_k > t)=1 - G(t)=\overline{G}(t), P(\xi_k > t)=1 - H(t)=\overline{H}(t),\ k \geq 0. \tag{2}$$

Denote this system by A_m and suppose that it is empty at the moment t_1 of the first customer arrival.

On a base of the system A_m define the multiserver queueing system with the competition between the servers B_m as follows. Suppose that the first customer arrives in empty system B_m at the moment t_1 and asks all m servers how long do they would serve it. The customer receives the information on its possible serving times $\eta_1^{(1)},\dots,\eta_1^{(m)}$ at the servers. Then it chooses the server with the minimal serving time $\zeta_1 = \min(\eta_1^{(1)},\dots,\eta_1^{(m)})$. During the first customer serving all other servers do not work. The second customer arrives into the system B_m at the moment t_2 and receives the information on its possible serving times $\eta_2^{(1)},\dots,\eta_2^{(m)}$ and so on. R.v.'s $\eta_i^{(j)},\ j = 1,\dots,m,\ i \geq 1$ are independent and have common d.f. $G(t)$. So the system B_m may be considered as the oneserver queueing system $GI|GI|1|\infty$ with the Poisson input flow $0 = t_1 \leq t_2 = t_1 + \xi_1 \leq t_3 = t_2 + \xi_2 \leq \dots$ and with i.i.d. serving times $\zeta_1, \zeta_2, \dots,$

$$P(\zeta_1 > x) = \overline{G}^m(x), \quad x \geq 0. \tag{3}$$

Suppose that $w_n^A,\ w_n^B,\ n \geq 0$ are the waiting times of the n–th customer in the systems A_m, B_m, correspondingly. Denote by

$$\overline{W}_m^A(x) = \lim_{n\to\infty} P(w_n^A > x),\ \ \overline{W}_m^B(x) = \lim_{n\to\infty} P(w_n^B > x)$$

the tails limit distributions of the waiting times in the systems $A_m,\ B_m$. Fix d.f. $G(t)$ and designate by

$$\lambda_m^A = \sup\{1/M\xi_1 : \lim_{x\to\infty} \overline{W}_m^A(x) = 0\}, \qquad \lambda_m^B = \sup\{1/M\xi_1 : \lim_{x\to\infty} \overline{W}_m^B(x) = 0\}$$

the maximal abilities to handle of the systems $A_m,\ B_m$. Our problem is to make the asymptotic analysis for $m \to \infty$ of

$$K_m = \frac{\lambda_m^A}{\lambda_m^B}$$

and to analyze the asymptotic of the function $\overline{W}_m^B(x),\ x \to \infty$ when d.f. $G(x)$ is regular varying.

2. PRELIMINARY INFORMATION

Suppose that $\mathcal{L}$ is the class of slowly varying functions, define

$$\mathcal{R}(a) = \{x^a l(x),\ \ l(x) \in \mathcal{L}\},\ \ -\infty < a < \infty.$$

In [2], [3] the Karamata theorem is proved: suppose that $l(x) \in \mathcal{L}$, then for $a < -1$,

$$\int_x^\infty t^a l(t)dt \sim -(a+1)^{-1} x^{a+1} l(x),\ \ x \to \infty. \tag{4}$$

For the system $M|G|1|\infty$ with the Poisson input flow (with the intensity λ) if the condition $\overline{G} \in \mathcal{R}(a)$ and the equality

$$\overline{G}_I(x) = \int_x^\infty \overline{G}(y)dy$$

are true then the Embrechts-Veraverbeke formula [4]

$$\overline{W}(x) \sim \frac{\lambda}{1 - \lambda M\eta_1}\overline{G}_I(x),\ x \to \infty$$

is true.

Consider now the Kiefer-Wolfowitz chain [5] $(w_{n,1}, w_{n,2}, \ldots, w_{n,m})$, $n \geq 0$, describing multiserver queueing system A_m. Here $w_{n,i}$ is the interval between the moment t_n and the moment when i servers become free of the 1-st,...,$(n-1)$-th customers of input flow. Necessary and sufficient condition of the Kiefer-Wolfowitz chain ergodicity (and so the existence of the limit distribution $\lim_{n\to\infty} P(w_{n,1} > t)$) is the inequality

$$M\eta_1 < \frac{m}{\lambda}. \tag{5}$$

So the formula

$$\lambda_m^A = \frac{m}{\int_0^\infty \overline{G}(t)dt} \tag{6}$$

takes place and accordingly to the equality (6) obtain

$$K_m = \frac{m\int_0^\infty \overline{G}^m(t)dt}{\int_0^\infty \overline{G}(t)dt}. \tag{7}$$

In [6, theorem 6] it is proved that **Theorem 1.** *Suppose that in the system A_m for some $a > 1$ d.f. $\overline{G}(x) \in \mathcal{R}(-a)$ and the ergodicity condition (5) is true then the function $\overline{W}_m^A(x) \in \mathcal{R}(-ma + m)$.*

3. MAIN RESULTS AND PROVES

Theorem 2. *Suppose that for some $a > 0,\ b > 0$:*

$$G(x) \sim ax^b,\quad x \to 0 \tag{8}$$

then in the system B_m for $m \to \infty$ the following formula is true:

$$\text{if } 0 < b \leq 1, \text{ then } K_m = O(m^{1-1/b}), \tag{9}$$

$$\text{if } b \geq 1, \text{ then } 1/K_m = O(1/m^{1-1/b}). \tag{10}$$

Proof. Suppose that $0 < b \leq 1$, $a > \ > 0$ are fixed. Then there exist the positive numbers α, m_0 so that for $0 < t < 1/m_0$ the inequality $G(t) \geq \alpha t^b$ is true and so $\overline{G}(t) \leq \exp(-\alpha t^b)$. Suppose that $m > m_0$ and denote

$$J_m = \int_0^\infty m\overline{G}^m(t)dt = U_m + V_m,\quad U_m = \int_0^{1/m} m\overline{G}^m(t)dt,$$

$$V_m = \int_{1/m}^\infty m\overline{G}^m(t)dt.$$

Estimate the quantity U_m for $m \to \infty$:

$$U_m \le \int_0^{1/m} m\exp(-\alpha m t^b)dt = \int_0^{1/m} m\exp(-((\alpha m)^{1/b}t)^b)\frac{dt(\alpha m)^{1/b}}{(\alpha m)^{1/b}} =$$
$$= \frac{m}{(\alpha m)^{1/b}}\int_0^{(\alpha m)^{1/b}/m}\exp(-v^b)dv \sim \frac{m}{(\alpha m)^{1/b}}\int_0^{\infty}\exp(-v^b)dv = O(m^{1-1/b}).$$

Estimate now the quantity V_m for $m \to \infty$:

$$V_m \le m\overline{G}^{m-1}(1/m)\int_{1/m}^{\infty}\overline{G}(t)dt \le$$
$$\le m\exp(-\alpha(m-1)m^{-b})\int_0^{\infty}\overline{G}(t)dt = o(m^{1-1/b}).$$

Then $J_m = O(m^{1-1/b})$ for $m \to \infty$ and

$$K_m = J_m/J_1 = O(m^{1-1/b}),\ \ m \to \infty.$$

The formula (9) is proved.

Suppose that $a > 0,\ b \ge 1$ are fixed. Then there exist the positive numbers α, m_0 so that for $0 < t < 1/m_0$ the inequality $G(t) \le \alpha t^b$ is true and so $\overline{G}(t) \ge 1 - \alpha t^b$. Put $m > m_0$ and obtain

$$J_m \ge \int_0^{1/m^{1/b}} m(1-\alpha/m)^m dt \sim m^{1-1/b}\exp(-\alpha),\ m \to \infty$$

and

$$1/K_m = J_1/J_m = O(1/m^{1-1/b}),\ \ m \to \infty.$$

The formula (10) is proved. The theorem 2 is proved.

Corollary. The theorem 2 shows how the ratio between the abilities to handle in the systems A_m (without the competition) and B_m (with the competition) depends on the parameter b.

Consider the following examples of d.f. $G(x),\ x \ge 0$, satisfying the condition (8): Weibull distribution

$$\overline{G}(x) = \exp(-ax^b),\ \ 0 < b < 1$$

and Burr distribution

$$\overline{G}(x) = (1 + cx^b)^{-a/c},\ \ 0 < a,\ 0 < b \le 1,\ 0 < c < ab.$$

Consider now the asymptotic behavior of the function $\overline{W}_m^B(x)$ for $x \to \infty$.

Theorem 3. *Suppose that in the system B_m^* d.f. $G(x) \in \mathcal{R}(-a), a > 1$ and the ergodicity condition $\lambda b_m < 1$ with $b_m = \int_0^\infty \overline{G}^m(x)dx$ is true. Then*

$$\overline{W}_m^B(x) \sim \frac{\lambda x\overline{G}^m(x)}{(1-\lambda b_m)(ma-1)},\ \ x \to \infty \qquad (11)$$

and consequently the function $\overline{W}_m^B(x) \in \mathcal{R}(-ma+1)$.

Proof. As d.f. $G(t) \in \mathcal{R}(-a)$, $a > 1$ so for some function $l(t) \in \mathcal{L}$ (and consequently $l^m(t) \in \mathcal{L}$) $\overline{G}^m(t) = l^m(t)t^{-ma}$. Using the Karamata theorem for $x \to \infty$ obtain

$$\frac{1}{b_m}\int_x^{\infty}\overline{G}^m(t)dt \sim \frac{l^m(x)x^{-ma+1}}{b_m(ma-1)} = \frac{x\overline{G}^m(x)}{b_m(ma-1)}. \tag{12}$$

Then d.f.

$$1 - \frac{\int_x^{\infty}\overline{G}^m(t)dt}{b_m}$$

is subexponential and from the ergodicity condition $\lambda b_m < 1$ we obtain the Embrechts-Veraberbeke formula

$$\overline{W}_m^B(x) \sim \frac{\lambda b_m \int_x^{\infty}\overline{G}^m(t)dt}{b_m(1-\lambda b_m)}, \quad x \to \infty. \tag{13}$$

Put the formula (12) into (13) and obtain the formula (11). The theorem 3 is proved.

Remark. The theorems 1, 3 comparison shows that in the system B_m with the competition the tail of the stationary waiting time distribution is lighter then in the system A_m without the competition.

REFERENCES

1. *Anastasi G., Lenzini L., Mingozzi E.* et al. MAC Protocols for Wideband Wireless Local Access: Evolution Toward Wireless ATM // IEEE Personal Communicatuions. 1998. V. 5. № 5. P. 53–64.
2. *Embrechts P., Klüppelberg C., Mikosch T.* Modelling Extremal Events. Berlin: Springer, 1997.
3. *Tsitsiashvili G. Sh., Markova N. V.* Asymptotical characteristics of input flows in queueing networks // Far Eastern Mathematical J. Vladivostok, 2003. V. 4. № 1. P. 36–43.
4. *Embrechts P., Veraverbeke N.* Estimates for the probability of ruin with special emphasis on the possibility of large claims // Insurance: Math. and Econom. 1982. V. 1. P. 55–72.
5. *Kiefer J., Wolfowitz J.* On the theory of queues with many servers // Trans. Amer. Math. Soc. 1955. V. 78. P. 147–161.
6. *Whitt W.* The impact of a heavy-tailed service-time distribution upon the M/GI/s waiting-time distribution // Queueing Systems. 2000. V. 36. P. 71–87.