ON THE LAW OF THE ITERATED LOGARITHM IN OPEN QUEUEING NETWORKS

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We investigated an open queueing network model in heavy traffic. The law of the iterated logarithm for the queue length of customers in an open queueing network has been proved.

Keywords: open queueing network, heavy traffic, iterated logarithm law, customers queue length.

1. INTRODUCTION

First of all note that one can apply the law of the iterated logarithm to the waiting time of a customer, virtual waiting time of a customer, and the queue length of customers to get more important probabilistic characteristics of the queueing theory in heavy traffic (see, for example, [2], [3], [4], [7]). But there are only several papers on the theory of open queueing networks in heavy traffic (see, for example, [6]) and no proof of the theorems on laws of the iterated logarithm for the main probabilistic characteristics of an open queueing network in heavy traffic.

So in this paper, we prove the theorem on the law of the iterated logarithm for the queue length of customers in the queueing network. The service discipline is "first come, first served" (FCFS). We consider open queueing networks with the FCFS service discipline at each station and general distributions of interarrival and service times. The queueing network we studied has k single server stations, each of which has an associated infinite capacity waiting room. Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables.

The basic components of the queueing network are arrival processes, service processes, and routing processes. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, \ldots, k$; defined on the probability space. Random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have support in $\{0, 1, 2, \ldots, k\}$. We define $\mu_j = \left(M\left[S_n^{(j)}\right]\right)^{-1} > 0$,

$\sigma_j = D\left(S_n^{(j)}\right) > 0$ and $\lambda_j = \left(M\left[z_n^{(j)}\right]\right)^{-1} > 0$, $a_j = D\left(z_n^{(j)}\right) > 0$, $j = 1, 2, \ldots, k$; with all of these terms assumed finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right) > 0$, $j = 1, 2, \ldots, k$. In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network. The matrix P is called a routing matrix.

To construct renewal processes generated by the interarrival and service times, we assume

$$z_j(0) = 0, z_j(l) = \sum_{m=1}^{l} z_m^{(j)}, S_j(0) = 0, S_j(l) = \sum_{m=1}^{l} S_m^{(j)}, l \geq 1, j = 1, 2, \ldots, k.$$

We now designate $A_j(t) = \max\left(l \geq 0 : z_j(l) \leq t\right)$ and $X_j(t) = \max\left(l \geq 0 : S_j(l) \leq t\right)$, and denote $\tau_j(t)$ as the total number of customer service departure from the jth station of the network until time t, $\tilde{\tau}_j(t)$ as the total number of customer arrival at the jth station of the network until time t, $\tau_{ij}(t)$ as the total number of customers after service that depart from the ith station of the network and arrive at the jth station of the network until time t, and $p_{ij}^t = \frac{\tau_{ij}(t)}{\tau_i(t)}$ as a part of the total number of customers which, after service at the ith station of the network, arrive at the jth station; $i, j = 1, 2, \ldots, k$ and $t > 0$.

First let us define $Q_j(t)$ as the queue length of customers at the jth station of the queueing network in time t; $\hat{\sigma}_j^2 = \left(\lambda_j\right)^3 \cdot Dz_n^{(j)} + \sum_{i=1}^{k} (\mu_i)^3 \cdot DS_n^{(i)} \cdot \left(p_{ij}\right)^2 + \left(\mu_j\right)^3 \cdot DS_n^{(j)} > 0$, $\hat{\beta}_j = \lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} - \mu_j > 0$, $j = 1, 2, \ldots, k$.

We assume that the following condition is fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \quad j = 1, 2, \ldots, k. \tag{1}$$

Note that this condition quarantees that, with probability one there exists a queue length of customers and this queue length of customers is constantly growing.

2. MAIN RESULT

One of the results of the paper is a theorem on the law of the iterated logarithm for the queue length of customers in an open queueing network.

Theorem. *If conditions (1) are fulfilled, then*

$$P\left(\overline{\lim_{t\to\infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1\right) = P\left(\underline{\lim_{t\to\infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1\right) = 1.$$

First define $\hat{X}_j(t) = \sum_{i=1}^{k} X_i(t) \cdot p_{ij} + A_j(t) - X_j(t)$, $w_j(t) = X_j(t) \cdot |p_{ij}^t - p_{ij}|$, $j = 1, 2, \ldots, k$ and $t > 0$.

By definition of the queue length of customers at the stations of the network, we get that

$$Q_j(t) = \tilde{\tau}_j(t) - \tau_j(t) = \tilde{\tau}_j(t) - X_j(t) + X_j(t) - \tau_j(t) \le \tilde{\tau}_j(t) - X_j(t) + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)) =$$

$$= \sum_{i=1}^{k} \tau_i(t) \cdot p_{ij}^t + A_j(t) - X_j(t) + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)) \le$$

$$\le \sum_{i=1}^{k} X_i(t) \cdot p_{ij}^t + A_j(t) - X_j(t) + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)) \le \sum_{i=1}^{k} X_i(t) \cdot p_{ij} + A_j(t) - X_j(t) +$$

$$+ \sum_{i=1}^{k} X_i(t) \cdot |p_{ij}^t - p_{ij}| + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)) = \hat{X}_j(t) + \sum_{i=1}^{k} w_i(t) + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)),$$

$$j = 1, 2, \ldots, k; t > 0 \Rightarrow \quad (2)$$

$$\Rightarrow Q_j(t) \le \hat{X}_j(t) + \sup_{0\le s\le t}(X_j(s) - \tau_j(s)) + \sum_{i=1}^{k} w_i(t), \ j = 1, 2, \ldots, k \text{ and } t > 0. \quad (3)$$

Let us fix $\varepsilon > 0$ as a small constant. So from (3) we see that for $\varepsilon > 0$

$$P\left(\frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + 3\varepsilon\right) \le P\left(\frac{\hat{X}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + \varepsilon\right) +$$

$$+ P\left(\frac{\sup_{0\le s\le t}(X_j(s) - \tau_j(s))}{\hat{\sigma}_j \cdot a(t)} > \varepsilon\right) + P\left(\frac{\sum_{i=1}^{k} w_i(t)}{\hat{\sigma}_j \cdot a(t)} > \varepsilon\right) \le P\left(\frac{\hat{X}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + \varepsilon\right) +$$

$$+ P\left(\frac{\sup_{0\le s\le t}(X_j(s) - \tau_j(s))}{\hat{\sigma}_j \cdot a(t)} > \varepsilon\right) + \sum_{i=1}^{k} P\left(\frac{w_i(t)}{\hat{\sigma}_j \cdot a(t)} > \frac{\varepsilon}{k}\right), \ j = 1, 2, \ldots, k \text{ and } t > 0 \Rightarrow \quad (4)$$

$$\Rightarrow P\left(\overline{\lim_{t\to\infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + 3\varepsilon\right) \le P\left(\overline{\lim_{t\to\infty}} \frac{\hat{X}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + \varepsilon\right) +$$

$$+ P\left(\overline{\lim_{t\to\infty}} \frac{\sup_{0\le s\le t}(X_j(s) - \tau_j(s))}{\hat{\sigma}_j \cdot a(t)} > \varepsilon\right) + \sum_{i=1}^{k} P\left(\overline{\lim_{t\to\infty}} \frac{w_i(t)}{\hat{\sigma}_j \cdot a(t)} > \frac{\varepsilon}{k}\right), \ j = 1, 2, \ldots, k. \quad (5)$$

Applying the results of [5], we see that the second and third terms in (5) converge to zero.

Using the law of the iterated logarithm in renewal processes (see, for example, [1]), we obtain for $\varepsilon > 0$

$$P\left(\overline{\lim_{t\to\infty}} \frac{\hat{X}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} > 1 + \varepsilon\right) = 0, \ j = 1, 2, \ldots, k. \quad (6)$$

From this we achieve that for $\varepsilon > 0$

$$P\left(\overline{\lim_{t\to\infty}}\frac{Q_j(t)-\hat{\beta}_j\cdot t}{\hat{\sigma}_j\cdot a(t)} > 1+\varepsilon\right) = 0, \ j = 1, 2, \ldots, k. \tag{7}$$

Also note that

$$Q_j(t) \geq \tilde{\tau}_j(t) - X_j(t) = \sum_{i=1}^{k}\tau_i(t)\cdot p_{ij}^t + A_j(t) - X_j(t) = \sum_{i=1}^{k}X_i(t)\cdot p_{ij}^t + A_j(t) - X_j(t) +$$
$$+\sum_{i=1}^{k}(\tau_i(t)-X_i(t))\cdot p_{ij}^t = \sum_{i=1}^{k}X_i(t)\cdot p_{ij} + A_j(t) - X_j(t) + \sum_{i=1}^{k}X_i(t)\cdot(p_{ij}^t - p_{ij}) =$$
$$= \hat{X}_j(t) + \sum_{i=1}^{k}X_i(t)\cdot(p_{ij}^t - p_{ij}) + \sum_{i=1}^{k}(\tau_i(t)-X_i(t))\cdot p_{ij}^t \geq \hat{X}_j(t) - \sum_{i=1}^{k}X_i(t)\cdot|p_{ij}^t - p_{ij}| -$$
$$-\sum_{i=1}^{k}(X_i(t)-\tau_i(t))\cdot p_{ij}^t \geq \hat{X}_j(t) - \sum_{i=1}^{k}w_i(t) - \sum_{i=1}^{k}(X_i(t)-\tau_i(t)) \geq \hat{X}_j(t) - \sum_{i=1}^{k}w_i(t) -$$
$$-\sup_{0\leq s\leq t}\sum_{i=1}^{k}(X_i(t)-\tau_i(t)) \geq \hat{X}_j(t) - \sum_{i=1}^{k}w_i(t) - \sum_{i=1}^{k}\sup_{0\leq s\leq t}(X_i(t)-\tau_i(t)), \ j = 1, 2, \ldots, k; t > 0. \tag{8}$$

From this it follows that

$$Q_j(t) \geq \hat{X}_j(t) - \sum_{i=1}^{k}w_i(t) - \sum_{i=1}^{k}\sup_{0\leq s\leq t}(X_i(t)-\tau_i(t)), \ j = 1, 2, \ldots, k \text{ and } t > 0. \tag{9}$$

Similarly as in (4) we have

$$P\left(\frac{Q_j(t)-\hat{\beta}_j\cdot t}{\hat{\sigma}_j\cdot a(t)} < 1-2\varepsilon\right) \leq P\left(\frac{\hat{X}_j(t)-\hat{\beta}_j\cdot t}{\hat{\sigma}_j\cdot a(t)} < 1-\varepsilon\right) +$$
$$+P\left(\frac{\sum\limits_{i=1}^{k}w_i(t) + \sum\limits_{i=1}^{k}\sup\limits_{0\leq s\leq t}(X_i(s)-\tau_i(s))}{\hat{\sigma}_j\cdot a(t)} > \varepsilon\right) \leq P\left(\frac{\hat{X}_j(t)-\hat{\beta}_j\cdot t}{\hat{\sigma}_j\cdot a(t)} < 1-\varepsilon\right) +$$
$$+\sum_{i=1}^{k}P\left(\frac{w_i(t)}{\hat{\sigma}_j\cdot a(t)} > \frac{\varepsilon}{2k}\right) + \sum_{i=1}^{k}P\left(\frac{\sup\limits_{0\leq s\leq t}(X_i(s)-\tau_i(s))}{\hat{\sigma}_j\cdot a(t)} > \frac{\varepsilon}{2k}\right), \ j = 1, 2, \ldots, k \tag{10}$$

and $t > 0$.

Again, applying the law of the iterated logarithm in renewal processes (see [1]), we have for $0 < \varepsilon < 1$ that

$$P\left(\overline{\lim_{t\to\infty}}\frac{\hat{X}_j(t)-\beta_j\cdot t}{\hat{\sigma}_j\cdot a(t)} < 1-\varepsilon\right) = 0, \ j = 1, 2, \ldots, k. \tag{11}$$

Now using (10), (11), and the lemma of [5], we obtain for $0 < \varepsilon < 1$ that

$$P\left(\overline{\lim_{t\to\infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) = 0, \ j = 1, 2, \ldots, k. \tag{12}$$

So, since $\varepsilon > 0$ is free, we get from (7) and (12) that $P\left(\overline{\lim_{t\to\infty}} \frac{Q_j(t)-\hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1\right) = 1$, $j = 1, 2, \ldots, k$.

The proof that $P\left(\underline{\lim_{t\to\infty}} \frac{Q_j(t)-\hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1\right) = 1$, $j = 1, 2, \ldots, k$ is similar to the proof of (13).

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