ANALYSIS OF ONE MARKOV QUEUEING NETWORK WITH INCOMES

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Investigation of the probabilistic model of incomes change in the banking network is presented. The closed Markov queueing network applies as model. Income ratings depend on time.

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1. INTRODUCTION

Let's examine closed exponential queueing network (QN), which consists of peripheral queueing systems (QS) $S_1, S_2, \ldots, S_{n-1}$ and central QS S_n, all QS are single-line [1]. The state of network is described by the vector $k(t) = (k, t) = (k_1, k_2, \ldots, k_n, t)$, where k_i – the count of requests in the system S_i, $i = \overline{1,n}$. Let μ_i – service rate of request by system S_i, $i = \overline{1,n}$, p_{ni} – the probability of entering of requests from system S_n into system S_i, $i = \overline{1,n-1}$, $\sum\limits_{i=1}^{n-1} p_{ni} = 1$.

2. ANALYSIS OF CENTRAL QUEUEING SYSTEM'S INCOMES

Through $v_n(k,t)$ let us designate the complete expected income, which the system S_n obtains in time t, if at the initial moment of time network was in state k. Let's assume that the system S_n earns $r_n(k)$ conventional units for the unit of time during entire period of its stay in the state k. When QN accomplishes passage from state (k,t) to state $(k-I_i+I_n, t+\Delta t)$, it brings income to QS S_n in the size $r(k-I_i+I_n)$, and when QN accomplishes passage from state (k,t) to state $(k+I_i-I_n, t+\Delta t)$, then brings income in the size $(-R(k+I_i-I_n, t))$, where I_i – n-vector with zero components except i-th component equals to 1. Let's notice that $r_n(k)$, $r(k,t)$ and $R(k,t)$ has different dimensions. During period Δt QN can stay in state (k,t) or can transfer to state $(k-I_i+I_n, t+\Delta t)$ or state $(k+I_i-I_n, t+\Delta t)$. If it is remaining in state (k,t) than income of QS S_n will be $r_n(k)\Delta t$ plus expected income $v_n(k,t)$ which QN will earn in remaining t units of time. The probability of such event is $1 - \sum\limits_{i=1}^{n} \mu_i u(k_i)\Delta t$, where $u(k_i) = \begin{cases} 1, & k_i > 0 \\ 0, & k_i = 0 \end{cases}$. If network transfers into state $(k - I_i + I_n, t + \Delta t)$ during period Δt with the probability equals $\mu_i u(k_i)\Delta t$ then income of QS S_n will be $r(k - I_i + I_n, t)$ plus expected income $v_n(k - I_i + I_n, t)$, which QS will earn in remaining time, if at the initial moment of time network was in state $(k - I_i + I_n)$. If network transfers into state $(k + I_i - I_n, t + \Delta t)$ during period Δt with the probability equals $\mu_n p_{ni} u(k_n)\Delta t$ then income will be $-R(k + I_i - I_n, t)$ plus expected income of QN for remaining time, if at the initial moment network was in state $(k + I_i - I_n)$, $i = \overline{1, n-1}$. Then it is possible to get a set of difference equations

$$v_n(k, t + \Delta t) = \left(1 - \sum_{i=1}^{n} \mu_i u(k_i)\Delta t\right)[r_n(k)\Delta t + v_n(k, t)] +$$

$$+ \sum_{i=1}^{n-1} \mu_i u(k_i)\Delta t\,[r(k - I_i + I_n, t) + v_n(k - I_i + I_n, t)] +$$

$$+ \mu_n u(k_n)\Delta t \sum_{i=1}^{n-1} p_{ni}\,[-R(k + I_i - I_n, t) + v_n(k + I_i - I_n, t)]$$

for incomes of central QS S_n. From it follows a set of difference-differential equations

$$\frac{dv_n(k, t)}{dt} = r_n(k) - \sum_{i=1}^{n} \mu_i u(k_i) v_n(k, t) + \sum_{i=1}^{n-1} \mu_i u(k_i) r(k - I_i + I_n) -$$

$$- \mu_n u(k_n) \sum_{i=1}^{n-1} p_{ni} R(k + I_i - I_n) + \sum_{i=1}^{n-1} \mu_i u(k_i) v_n(k - I_i + I_n, t) +$$

$$+ \mu_n u(k_n) \sum_{i=1}^{n-1} p_{ni} v_n(k + I_i - I_n, t),$$

which can be transformed to

$$\frac{dv_n(k, t)}{dt} = r_n(k) - \sum_{i=1}^{n} \mu_i u(k_i) v_n(k, t) +$$

$$+ \sum_{i=1}^{n-1} [\mu_i u(k_i) r(k - I_i + I_n, t) - \mu_n u(k_n) p_{ni} R(k + I_i - I_n, t)] + \quad (1)$$

$$+ \sum_{i=1}^{n-1} [\mu_i u(k_i) v_n(k - I_i + I_n, t) + \mu_n u(k_n) p_{ni} v_n(k + I_i - I_n, t)].$$

In particular case if $r_n(k)$, $r(k, t)$, $R(k, t)$ are independent of state of the network and equal r_n, $r(t)$ and $R(t)$ accordingly then

$$\frac{dv_n(k, t)}{dt} = r_n - \mu_n u(k_n) R(t) + r(t) \sum_{i=1}^{n-1} \mu_i u(k_i) - \sum_{i=1}^{n} \mu_i u(k_i) v_n(k, t) +$$

$$+ \sum_{i=1}^{n-1} [\mu_i u(k_i) v_n(k - I_i + I_n, t) + \mu_n u(k_n) p_{ni} v_n(k + I_i - I_n, t)]. \quad (2)$$

3. ANALYSIS OF PERIPHERAL SYSTEMS AND NETWORK'S INCOMES

Let's examine incomes of peripheral systems S_i, $i = \overline{1, n-1}$. Through $v_i(k,t)$ let's designate the complete expected income, which the peripheral system S_i obtains in time t if at the initial moment of time network was in state k, $i = \overline{1, n-1}$. It is clear that when the network transfers from state (k,t) to state $(k - I_i + I_n, t + \Delta t)$ it earns $(-r(k - I_i + I_n, t))$ for QS S_i, and when transfers from state (k,t) to state $(k + I_i - I_n, t + \Delta t)$ it earns $R(k + I_i - I_n, t)$. Let's assume that QS earns $r_i(k)$ for QS S_i during a conventional unit of time for all time that QN was in state k, $i = \overline{1, n-1}$. Then we will get

$$v_i(k, t + \Delta t) = \left(1 - \sum_{j=1}^{n} \mu_j u(k_j)\Delta t\right)[r_i(k)\Delta t + v_i(k,t)] +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \mu_j u(k_j)\Delta t v_i(k - I_j + I_n, t) + \mu_n u(k_n)\Delta t \sum_{\substack{j=1 \\ j \neq i}}^{n-1} p_{nj} v_i(k + I_j - I_n, t) =$$

$$= \left(1 - \sum_{j=1}^{n} \mu_j u(k_j)\Delta t\right)[r_i(k)\Delta t + v_i(k,t)] - \mu_i u(k_i) r(k - I_i + I_n)\Delta t +$$

$$+ \mu_n u(k_n) p_{ni} R(k + I_i - I_n)\Delta t + \sum_{j=1}^{n-1} \mu_j u(k_j)\Delta t v_i(k - I_j + I_n, t) +$$

$$+ \mu_n u(k_n)\Delta t \sum_{j=1}^{n-1} p_{nj} v_i(k + I_j - I_n, t),$$

from it follows

$$\frac{dv_i(k,t)}{dt} = r_i(k) - \sum_{j=1}^{n} \mu_j u(k_j) v_i(k,t) +$$

$$+ [\mu_n u(k_n) p_{ni} R(k + I_i - I_n) - \mu_i u(k_i) r(k - I_i + I_n)] + \qquad (3)$$

$$+ \sum_{j=1}^{n-1} \left[\mu_j u(k_j) v_i(k - I_j + I_n, t) + \mu_n u(k_n) p_{nj} v_i(k + I_j - I_n, t)\right], \; i = \overline{1, n-1}.$$

If $r_i(k)$, $r(k,t)$, $R(k,t)$ are independent of the network state then

$$\frac{dv_i(k,t)}{dt} = r_i + [\mu_n u(k_n) p_{ni} R - \mu_i u(k_i) r] - \sum_{j=1}^{n} \mu_j u(k_j) v_i(k,t) +$$

$$+\sum_{j=1}^{n-1}\left[\mu_j u(k_j) v_i(k - I_j + I_n, t) + \mu_n u(k_n) p_{nj} v_i(k + I_j - I_n, t)\right],\ i = \overline{1, n-1}. \quad (4)$$

From (3) follows that total income for peripheral QS satisfies a set of equations

$$\frac{dv(k,t)}{dt} = \sum_{i=1}^{n-1} r_i(k) - \sum_{j=1}^{n} \mu_j u(k_j) v(k,t) +$$

$$+\sum_{i=1}^{n-1}\left[\mu_n u(k_n) p_{ni} R(k + I_i - I_n, t) - \mu_i u(k_i) r(k - I_i + I_n, t)\right] + \quad (5)$$

$$+\sum_{j=1}^{n-1}\left[\mu_j u(k_j) v(k - I_j + I_n, t) + \mu_n u(k_n) p_{nj} v(k + I_j - I_n, t)\right],$$

and total income of all systems of the network $\Theta(k,t) = v(k,t) + v_n(k,t)$, as it follows from (1), (5), satisfies a set

$$\frac{d\Theta(k,t)}{dt} = \sum_{i=1}^{n} r_i(k) - \sum_{i=1}^{n} \mu_i u(k_i) \Theta(k,t) +$$

$$+\sum_{i=1}^{n-1}\left[\mu_i u(k_i) \Theta(k - I_i + I_n, t) + \mu_n u(k_n) p_{ni} \Theta(k + I_i - I_n, t)\right]. \quad (6)$$

Let us note that the total income of network $\Theta(k,t)$ does not depend on $r(k,t)$, $R(k,t)$. It was to be expected this from the law of conservation of money mass in the closed network (if Central QS receives incomes then peripheral QS suffer losses and vice versa), entire money mass in the closed network grows only due to an increase in the percentages of the money mass, which is stored in the systems of network, which is reflected by value $\sum_{i=1}^{n} r_i(k)$.

4. ABOUT RESOLVING A SET OF EQUATIONS FOR INCOMES

After numbering states of the queueing network $1, 2, \ldots, l$ consecutively, set of equations for incomes (2), (4), (6) can be presented in matrix form

$$\frac{dV_i(t)}{dt} = Q_i(t) + AV_i(t), \quad (7)$$

where $V_i^T(t) = (v_i(1,t), \ldots, v_i(l,t))$ – QS S_i incomes vector at the moment of time t depending on states of QN at the initial moment, $i = \overline{1,n}$, $V_{n+1}^T(t) = (\Theta_i(1,t), \ldots, \Theta_i(l,t))$ – QN total incomes vector. For its solution it is possible to use operating method – we need to assign a vector of initial conditions $V_i(0)$. Let $U_i(s)$ – vector of Laplace transformations of incomes $v_i(j,t)$, $j = \overline{1,l}$, $G_i(S)$ – Laplace transformations of $Q_i(t)$. Then $sU_i(s) - V_i(0) = G_i(s) + AU_i(s)$ or $(sI - A)U_i(s) = G_i(s) + V_i(0)$, where I – identity matrix. So we can find $U_i(s)$:

$$U_i(s) = (sI - A)^{-1} G_i(s) + (sI - A)^{-1} V_i(0). \quad (8)$$

Vector of incomes $V_i(t)$ can be find with the help of inverse transformation of (8). If $H(t)$ is inverse Laplace transformation of $(sI - A)^{-1}$ then inverse transformation from (8) makes

$$V_i(t) = H(t) * Q_i(t) + H(t)V_i(0), \ i = \overline{1, n+1}, \tag{9}$$

where $H(t) * Q_i(t)$ – convolution of functions $H(t)$ and $Q_i(t)$, $H(t) * Q_i(t) = \int_0^t H(u) \times Q_i(t-u)du$. In that way we can produce a relation for estimation of incomes of central and peripheral QS and whole QN at any moment of time t.

Case when $r(k, t)$ and $R(k, t)$ don't depend on on time examined in paper [1].

Notes. Calculations of different examples showed that it is possible to find solution of equations sets of type (1), (5), (7) by operating method successfully when queueing networks has relatively small number of states. It is because, in particular case, the calculation of inverse matrices $(sI - A)^{-1}$, its decomposition and inverse Laplace transformation finding with the aid of Mathematica. Therefore it is expected to use this method for the investigation of banking network models which consist of Central bank with small number of its branches.

REFERENCES

1. *Matalytsky M. A., Pankov A. V.* Probabilistic analysis of incomes in banking networks // Vestnik BSU. Ser. 1. Physics, mathematics, computer science. 2004. № 2. P. 86–91.