CONTINUOUS-TIME ASYMPTOTICALLY QUASI-TOEPLITZ MARKOV CHAINS

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We introduce into consideration and study continuous-time multi-dimensional asymptotically quasi-toeplitz Markov chains. Sufficient conditions for stability are proved and algorithm for calculation of the steady state probabilities is presented.

Keywords: regular multi-dimensional continuous-time Markov chain, ergodicity, algorithm, asymptotically quasi-toeplitz Markov chain.

1. INTRODUCTION

In modeling modern communication networks, it is typical that the problem of performance evaluation and capacity planning can be solved only by means of analysis of multi-dimensional stochastic processes. So, essential attention of many researchers is paid to discovering and investigating classes of multi-dimensional processes where it is possible to get constructive results relating to stability conditions establishment and ergodic distribution calculation. A seminal role in these developments is played by M. Neuts, see, e.g., his books [6, 7]. He introduced into considerations and studied so called *Quasi-Birth-and Death Processes, $M/G/1$ and $G/M/1$ type Markov chains.* Among other researchers who were successful in development of investigation of the classes of multi-dimensional processes we can mention names by G. Basharin, P. Bocharov, V. Naumov, W. Ramaswami, D. Lucantoni, S. Chakravarthy, G. Latouche, W. Grassmann, D. Heyman, R. Gail, B. Taylor, S. Hantler and others. The achieved results allow to implement exhaustive analysis of a wide range of stochastic models, queueing systems in particular.

However, most part of results is related to the space homogeneous processes. More complicated case when the processes are inhomogeneous in space is investigated in far less extent. We can mention, e.g. the paper by Bright and Taylor [3] where the so called *Level Dependent Quasi-Birth-and Death Processes (LDQBD)* are investigated. In that paper, the algorithm for calculation of the steady state distribution for *LDQBD* processes is presented. Note, that the authors by [3] do not impose any assumptions about the existence of limits for parameters of *LDQBD*. So, they do not try to investigate conditions for the stationary distribution existence and assume in advance that the process is ergodic. The second shortage of this paper is consideration of only three-block diagonal generator. Thus, authors by [3] solved, in some extent, the problem of expanding M. Neuts' results relating to the Quasi-Birth-and Death processes to the level dependent case.

The work to extend the results by M. Neuts relating to the $M/G/1$ type Markov chains to the level dependent case was done by the authors of a present paper in [2, 4, 5]. Notion of asymptotically quasi-toeplitz Markov chain (*AQTMC*) is introduced and constructive stability conditions are proved there. Algorithm for calculating the steady state distribution is presented. Motivation of such a generalization is quite clear. Retrial queueing models describe many important processes in applications, in telecommunications in particular. The $BMAP/G/1$ retrial queueing model and many its generalizations were successfully investigated since year 2000 by means of the *AQTMC* while no other approaches for investigation of such queueing models are offered in literature yet.

Unfortunately, till now results are obtained only for discrete-time Markov chains. But many multiserver queues (see, e.g., paper [2] where the $BMAP/PH/N$ retrial queue is under study) are described by continuous-time Markov chains. So, its investigation by means of *AQTMC* requires the intermediate reduction of the original continuous-time Markov chain to auxiliary discrete-time Markov chain. This reduction is just technical and it is desirable to avoid it. The present paper has a goal to cancel necessity of reduction to the discrete-time Markov chain in analysis of continuous-time Markov chains. Presented results constitute the background for direct investigation of a level dependent continuous-time Markov chain.

The paper is organized as follows. Section 2 contains the formal definition of multi-dimensional continuous-time asymptotically quasi-toeplitz Markov chain. In section 3, the jump chain for the continuous-time Markov chain is built up and it is shown that the jump chain belongs to the class of multi-dimensional discrete-time asymptotically quasi-toeplitz Markov chain previously studied in [2, 4, 5]. Based on this conclusion, stability condition is presented in section 4 and in section 5 an algorithm for calculating the stationary distribution is given in terms of generator of the multi-dimensional continuous-time asymptotically quasi-toeplitz Markov chain. Section 6 concludes the paper.

2. DEFINITION OF MULTI-DIMENSIONAL CONTINUOUS-TIME ASYMPTOTICALLY QUASI-TOEPLITZ MARKOV CHAIN

Let $\xi_t = \{i_t, x_t\}, t \geq 0$ be a regular irreducible continuous-time Markov chain. We assume that the process i_t takes values in denumerable set. Without loss of generality, we suppose that $i_t \in \{0, 1, \ldots\}$. When the state of the process $i_t, t \geq 0$ is equal to $i, i \geq 0$, the process $x_t, t \geq 0$ takes values in finite set X_i of finite-dimensional vectors. Note that the vectors belonging to the set X_i can have different dimensionallities. We also assume that there are the nonnegative integer i^* and the set X of finite-dimensional vectors such that $X_i = X$ when $i > i^*$. Thus, the phase space S of the Markov chain $\xi_t, t \geq 0$ has the following form:

$$S = \{(i, x), x \in X_i, i = 0, 1, \ldots, i^*; (i, x), x \in X, i > i^*\}.$$

In what follows we denote the number of vectors in the set X_i as $K_i, K_i \geq 0, i = 0, 1, \ldots, i^*$, and the number of vectors in the set X as $K, K > 0$.

Enumerate the states of the chain $\xi_t, t \geq 0$ as follows. Put the states (i, x) in ascending order of the component i and then, for fixed i, arrange the states (i, x), $x \in X_i$, in any suitable manner (as a rule, the lexicographic order is efficient), $i \geq 0$.

Write the generator A of the chain in the block form $A = (A_{i,l})_{i,l} \geq 0$, where $A_{i,l}$ is the $K_i \times K_l$ matrix formed by intensities $a_{(i,x);(l,z)}$ of the chain $\xi_t, t \geq 0$ transition from the state $(i, x), x \in X_i$, to the state $(l, z), z \in X_l$. The diagonal entries of the matrix $A_{i,i}$ are defined as $a_{(i,x);(i,x)} = -\sum\limits_{z \in \bigcup\limits_{l=0}^{\infty} X_l \setminus \{x\}}^{\infty} a_{(i,x);(i,z)}$. Note that the matrix $A_{i,l}$ is a K-size square matrix for $i, l \geq i^*$.

Introduce the notation R_i for the diagonal matrix having the values $-a_{(i,x);(i,x)}$ as its diagonal entries, $i \geq 0$.

Definition 1. *Regular irreducible continuous-time Markov chain $\xi_t, t \geq 0$ is called asymptotically quasi-toeplitz Markov chain if*

1^0. $A_{i,l} = 0$ *for* $l < i - 1, i > 0$.

2^0. *There is a matrix R such that*

$$\lim_{i \to \infty} R_i^{-1} = R. \tag{1}$$

3^0. *There are integers* $i_0, k_0 \geq 0$ *such that the matrices* $A_{i,i+k}$ *do not depend on i when* $i \geq i_0, k \geq k_0$.

4^0. *There exist the limits*

$$\lim_{i \to \infty} R_i^{-1} A_{i,i+k}, k = -1, 0, \dots, k_0 - 1. \tag{2}$$

5^0. *Jump chain (see [1])* $\xi_n, n \geq 1$ *of the process* $\xi_t, t \geq 0$ *is non-periodic.*

Investigation of the Markov chain $\xi_t, t \geq 0$ is based on the analysis of its jump chain.

3. JUMP CHAIN

The jump chain $\xi_n = \{i_n, x_n\}, n \geq 1$ for the process $\xi_t, t \geq 0$ has state space S and the transition probability matrices $P_{i,l}, i, l \geq 0$ defined as follows

$$P_{i,l} = \begin{cases} 0, & l < i - 1, i > 0; \\ R_i^{-1} A_{i,l}, & l \geq max\{0, i - 1\}, l \neq i; \\ R_i^{-1} A_{i,i} + I, & l = i, i \geq 0. \end{cases} \tag{3}$$

The following statement holds true.

Lemma 1. *Markov chain $\xi_n, n \geq 1$ belongs to class of asymptotically quasi-toeplitz discrete-time Markov chains.*

Proof. Show that the $\xi_n, n \geq 1$ satisfies to definition of asymptotically quasi-toeplitz discrete-time Markov chains given in [5].

The chain $\xi_n, n \geq 1$ is irreducible (since the chain $\xi_t, t \geq 0$ is irreducible) and non-periodic (by point 5^0 of definition 1).

Then, we should verify that conditions (*i*), (*ii*) and (*iii*) of definition 1 in [5] are satisfied.

Condition (*i*) holds good because $P_{i,l} = 0$, $l < i - 1, i > 0$ by (3).

Let us show that the series $\sum_{k=0}^{\infty} P_{i,i+k-1}$ converges uniformly in the region $i \geq max\{i^*+1, i_0\}$ what guarantees the fulfillment of condition (*ii*).

This series is represented in the following form:

$$\sum_{k=0}^{\infty} P_{i,i+k-1} = \sum_{k=0}^{k_0} P_{i,i+k-1} + R_i^{-1} \sum_{k=k_0+1}^{\infty} A_{i,i+k-1}.$$

The uniform convergence of the series stems from the independence of $\sum_{k=k_0+1}^{\infty} A_{i,i+k-1}$ of i for large values of i and the uniform boundedness of the matrices R_i^{-1}. The first fact ensues from point 3^0 of definition 1 while the second one ensues from point 2^0 of definition 1.

To check condition (*iii*), consider the following limits:

$$Y_k = \lim_{i\to\infty} P_{i,i+k-1} = \lim_{i\to\infty} R_i^{-1} A_{i,i+k-1}, k = 0, 2, 3, \ldots, \quad (4)$$

$$Y_1 = \lim_{i\to\infty} P_{i,i} = \lim_{i\to\infty} R_i^{-1} A_{i,i} + I.$$

For $k < k_0$, limits (4) exist according to point 4^0 of definition 1. For $k \geq k_0$, the existence of limits (4) follows from points 2^0 and 3^0 of definition 1.

The matrices $Y_k, k \geq 0$ are substochastic since they are the limits of the sequences of the substochastic matrices.

Equations (4) and the uniform convergence of the series $\sum_{k=0}^{\infty} P_{i,i+k-1}$ imply the relations

$$Y \stackrel{def}{=} \lim_{i\to\infty} \sum_{k=0}^{\infty} P_{i,i+k-1} = \sum_{k=0}^{\infty} \lim_{i\to\infty} P_{i,i+k-1} = \sum_{k=0}^{\infty} Y_k. \quad (5)$$

Because all elements of the sequence $\sum_{k=0}^{\infty} P_{i,i+k-1}$, $i \geq 0$ are stochastic matrices, its limit Y is stochastic one as well. According to (5), the matrix $\sum_{k=0}^{\infty} Y_k$ is also stochastic.

So, the matrices $Y_k, k \geq 0$ defined by (4) exist, they are substochastic while their sum is the stochastic matrix. It means that condition (*iii*) is satisfied.

Thus, we have proven that the Markov chain $\xi_n, n \geq 1$ satisfies to definition 1 in [5] and, consequently, it is the asymptotically quasi-toeplitz Markov chain. □

4. ERGODICITY CONDITION

Further investigation of the chain $\xi_t, t \geq 0$ is based on the results for multi-dimensional asymptotically quasi-toeplitz discrete-time Markov chain given in [5].

Introduce the notation $Y(z) = \sum_{k=0}^{\infty} Y_k z^k$, $|z| \leq 1$ for the generating function of the matrices $Y_k, k \geq 1$ defined by (4). The following two theorems give the stability conditions for the Markov chain $\xi_t, t \geq 0$ in terms of generating function $Y(z)$. We distinguish the cases of irreducible and reducible matrix $Y(1)$.

Theorem 1. *Let $Y(1)$ be an irreducible matrix and $Y'(1) < \infty$. Suppose that the series $\sum_{k=1}^{\infty} kA_{i,i+k-1}\mathbf{e}$ converges for $i = \overline{0, i^*}$ and the series $\sum_{k=1}^{\infty} kA_{i,i+k-1}$ converges for any $i > i^*$. Then the sufficient condition for the Markov chain $\xi_t, t \geq 0$ ergodicity is the fulfillment of the following inequality:*

$$[det(zI - Y(z))]'_{z=1} > 0. \tag{6}$$

Proof. First, show that inequality (6) is the sufficient condition for ergodicity of the jump chain $\xi_n, n \geq 1$ conditionally the hypothesis of theorem 1 hold.

To do this, we use theorem 1 in [5]. According to that theorem we must only prove that the series $\sum_{k=1}^{\infty} kP_{i,i+k-1}\mathbf{e}$ converges for $i = \overline{0, i^*}$, and the series $\sum_{k=1}^{\infty} kP_{i,i+k-1}$ converges for all $i > i^*$ and converges uniformly for large values of i. Convergence of the series $\sum_{k=1}^{\infty} kP_{i,i+k-1}\mathbf{e}$ follows directly from conditions of the theorem under proof. The second series is represented in the following form:

$$\sum_{k=1}^{\infty} kP_{i,i+k-1} = \sum_{k=1}^{k_0} kP_{i,i+k-1} + R_i^{-1} \sum_{k=k_0+1}^{\infty} kA_{i,i+k-1}. \tag{7}$$

The converges of series (7) for $i > i^*$ follows from the convergence of the series $\sum_{k=1}^{\infty} kA_{i,i+k-1}$ for $i > i^*$. The uniform convergence of (7) stems from the independence of $\sum_{k=k_0+1}^{\infty} kA_{i,i+k-1}$ of i and the uniform boundedness of the matrices R_i^{-1} for large value of i. These facts ensue from points 2^0, 3^0 of definition 1.

Thus, we have shown that the fulfillment of inequality (6) is the sufficient condition for the jump chain $\xi_n, n \geq 1$ ergodicity.

Represent the stationary (ergodic) distribution of the chain $\xi_n, n \geq 1$ in the partitioned form $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$ where the row vector $\boldsymbol{\pi}_i$ is the vector of steady state probabilities of the chain $\xi_n, n \geq 1$ corresponding to the state $i_n = i$ of the denumerable component. It is easy verified that row vector $(\mathbf{p}_0, \mathbf{p}_1, \ldots)$, where $\mathbf{p}_i = c\boldsymbol{\pi}_i R_i^{-1}$, $i \geq 0$, c is some constant, satisfies the system of balance equations for the stationary (ergodic) distribution of the process $\xi_t, t \geq 0$. Since the $\xi_t, t \geq 0$ is regular irreducible Markov chain then, according to the Foster theorem [1], the sufficient condition for its ergodicity is distinction of zero of the constant c which has form

$$c = (\sum_{i=0}^{\infty} \boldsymbol{\pi}_i R_i^{-1}\mathbf{e})^{-1}. \tag{8}$$

Taking into account the uniform boundedness of the matrices R_i^{-1} for large values of i we see that the series in the right part of (8) converges to some positive value. So, $0 < c < \infty$, and row vector $(\mathbf{p}_0, \mathbf{p}_1, \ldots)$ is the stationary distribution of the chain $\xi_t, t \geq 0$. □

Corollary 1. *Inequality (6) is equivalent to the following inequality:*

$$\mathbf{y}Y'(1)\mathbf{e} < 1,$$

where the vector $\mathbf{y}$ *is the unique solution to the system of linear algebraic equations*

$$\mathbf{y}Y(1) = \mathbf{y}, \ \ \mathbf{y}\mathbf{e} = 1.$$

Consider now the case of reducible matrix $Y(1)$.

Theorem 2. *Let* $Y(1)$ *be reducible matrix having the matrices* $Y^{(l)}(1), l = \overline{1,m}$ *as irreducible stochastic blocks of its normal form,* $Y^{(l)}(z), l = \overline{1,m}$ *are the generating functions corresponding to these blocks and* $Y'(1) < \infty$.

If the series $\sum_{k=1}^{\infty} kA_{i,i+k-1}\mathbf{e}$ *converges for* $i = \overline{0,i^*}$ *and the series* $\sum_{k=1}^{\infty} kA_{i,i+k-1}$ *converges for any* $i > i^*$ *then the sufficient condition for the Markov chain* $\xi_t, t \geq 0$ *ergodicity is the fulfillment of the inequalities*

$$[det(zI - Y(z))]'_{z=1} > 0, \ l = \overline{1,m}. \tag{9}$$

The proof is implemented by analogy with the proof of theorem 1. But we use theorem 2 instead of theorem 1 from [5] in the proof of jump chain ergodicity condition. □

Corollary 2. *Inequations (9) are equivalent to the following inequations*

$$\mathbf{y}_l \frac{dY^{(l)}(z)}{dz}|_{z=1}\mathbf{e} < 1, \ \ l = \overline{1,m}, \tag{10}$$

where $\mathbf{y}_l$ *is the unique solution to the system of linear algebraic equations*

$$\begin{cases} \mathbf{y}_l Y^{(l)}(1) = \mathbf{y}_l, \\ \mathbf{y}_l\mathbf{e} = 1, \ \ l = \overline{1,m}. \end{cases}$$

To check the ergodicity condition in case of reducible matrix $Y(1)$ we have to reduce this matrix to its normal form. This reduction is associated with some technical difficulties. At the same time, asymptotically quasi-toeplitz continuous-time Markov chains, describing many queueing process, have the reducible matrix $Y(1)$ of specific form what allows to involve only a part of the matrix $Y(1)$ in checking the ergodicity condition. As a rule, this part turns out to be a diagonal block of $Y(1)$ of much smaller size comparing to the size of $Y(1)$. So, the following lemma and theorem can be useful.

Lemma 2. *Let* $Y(1)$ *be a reducible* $K \times K$ *matrix, which can be represented in the form*

$$Y(1) = \begin{pmatrix} Y_{11}(1) & Y_{12}(1) \\ 0 & Y_{22}(1) \end{pmatrix},$$

where $Y_{11}(1)$ *and* $Y_{22}(1)$ *are the square matrices of sizes* L_1 *and* L_2 *respectively,* $0 \leq L_1 \leq K - 1$, $L_1 + L_2 = K$. *Suppose that all diagonal and under-diagonal entries of the matrix* $Y_{11}(1)$ *are equal to zero.*

Let also $Y_{22}^{(l)}(1), l = \overline{1,m}$ *be irreducible stochastic blocks of normal form* $Y_{22}^{(N)}(1)$ *of the matrix* $Y_{22}(1)$.

Then, these blocks are also irreducible stochastic blocks of normal form $Y^{(N)}(1)$ *of the matrix* $Y(1)$ *and* $Y^{(N)}(1)$ *has no another stochastic blocks.*

Proof. By means of coordinated permutations of rows and columns of the matrix $Y(1)$ we can reduce this matrix to the form

$$\hat{Y} = \begin{pmatrix} Y_{22}(1) & 0 \\ \hat{Y}_{12} & \hat{Y}_{11} \end{pmatrix},$$

where all diagonal and off-diagonal entries of the matrix $\hat{Y}_{11}$ are equal to zero.

Next, reduce the block $Y_{22}(1)$ to its normal form $Y_{22}^{(N)}(1)$ by means of coordinated permutations of the matrix $\hat{Y}$ rows and columns. As the result we get the normal form $Y^{(N)}(1)$ of the matrix $Y(1)$

$$Y^{(N)}(1) = \begin{pmatrix} Y_{22}^{N}(1) & 0 \\ \tilde{Y}_{12} & \hat{Y}_{11} \end{pmatrix}.$$

It is easy to see that all irreducible stochastic diagonal blocks of the matrix $Y^{(N)}(1)$ are contained in the matrix $Y_{22}^{(N)}(1)$ since the diagonal irreducible blocks of the matrix $\hat{Y}_{11}$ are substochastic matrices of 1×1 size, i.e., each of these blocks is the scalar which is equal to zero.□

Theorem 3. *Let $Y(1)$ be a reducible matrix satisfying the conditions of lemma 2, $Y_{22}^{(l)}(1)$, $l = \overline{1,m}$ are the irreducible stochastic blocks of the normal form $Y_{22}^{(N)}(1)$ of the matrix $Y_{22}(1)$.*

Then the sufficient condition for the Markov chain $\xi_t, t \geq 0$ ergodicity is the fulfillment of the inequalities

$$[det(zI - Y_{22}^{(l)}(z))]'_{z=1} > 0, \; l = \overline{1,m}. \tag{11}$$

The proof of the theorem follows from lemma 2 and theorem 2. □

5. ALGORITHM FOR CALCULATING THE STATIONARY DISTRIBUTION

The algorithm will be derived by means the minor modification of the algorithm for calculating the stationary distribution of asymptotically quasi-toeplitz discrete-time Markov chain given in [5]. We will use correspondence (3) between the chain $\xi_t, t \geq 0$ transition rates and the chain $\xi_n, n \geq 1$ transition probabilities as well as the following correspondence between the stationary distributions of these chains:

$$\boldsymbol{\pi}_i = c^{-1}\boldsymbol{p}_i R_i, \; i \geq 0, \tag{12}$$

where the value c is defined by (8).

Putting expressions (3), (12) for $P_{i,l}$, $l \geq max\{0, i-1\}, i \geq 0$ and $\boldsymbol{\pi}_i, i \geq 0$ respectively to formulae (9)–(13) in [5] and using formula (16) in [5] we get the following algorithm for calculating the vectors $\boldsymbol{p}_i$, $i \geq 0$.

- Calculate the matrix G as the minimal nonnegative solution to the matrix equation

$$G = \sum_{k=0}^{\infty} Y_k G^k.$$

- Calculate the matrices $G_i, i = 0, 1, \dots, \tilde{i} - 1$, by using the backward recursion

$$G_i = (-\sum_{n=i+1}^{\infty} A_{i+1,n} G_{n-1} G_{n-2} \dots G_{i+1})^{-1} A_{i+1,i},\ i = 0, 1, \dots, \tilde{i} - 1 \quad (13)$$

  with the boundary condition $G_i = G,\ i \geq \tilde{i}$. Here $\tilde{i}$ is some threshold such that $\tilde{i} \geq i^*$. Its value depends on the convergence rate of the matrices $P_{i,i+k-1}$ to the corresponding matrices $Y_k, k \geq 0$ and the required accuracy of calculations.
- Calculate the matrices $\bar{A}_{i,l},\ l \geq i, i \geq 0$ by the formulae

$$\bar{A}_{i,l} = A_{i,l} + \sum_{n=l+1}^{\infty} A_{i,n} G_{n-1} G_{n-2} \dots G_l,\ \ l \geq i,\ i \geq 0.$$

- Calculate the matrices $F_l,\ l \geq 0$ using the recurrent formulae

$$F_0 = I, F_l = \sum_{i=0}^{l-1} F_i \bar{A}_{i,l} (-\bar{A}_{l,l})^{-1},\ l \geq 1. \quad (14)$$

- Calculate the vector $\boldsymbol{p}_0$ as the unique solution to the system of the linear algebraic equations

$$\begin{cases} \boldsymbol{p}_0(-\bar{A}_{0,0}) = 0, \\ \boldsymbol{p}_0 \sum_{l=0}^{\infty} F_l \mathbf{e} = 1. \end{cases}$$

- Calculate the vectors $\boldsymbol{p}_l,\ l \geq 1$ by

$$\boldsymbol{p}_l = \boldsymbol{p}_0 F_l,\ \ l \geq 1.$$

Note, that inverse matrices in (13), (14) exist and are nonnegative. So, the described algorithm includes operations with nonnegative matrices only and is computationally stable. This algorithm is implemented in computer program and showed its high quality. It is much better comparing to any algorithms based on truncation of the state space of the Markov chain.

6. CONCLUSION

Presented results allow to investigate a wide range of different queueing models described in terms of continuous-time Markov chains. They can be useful for investigation of many queueing models, in particular multi-server retrial queues and controlled queues. After describing the behavior of the queueing model in terms of continuous-time Markov chain, researcher should derive its generator and check its correspondence to definition of continuous-time multi-dimensional asymptotically quasi-toeplitz Markov chain given in our paper. After that our results can be formally used for calculating the steady state distribution of the model. Informal things in investigation, which can be further done by the researcher of a concrete queueing model, are attempts to simplify the stability condition by transformation it into the scalar form and derivation of expressions for performance measures of a queue under study.

REFERENCES

1. *Asmussen S.* Applied Probability and Queues. New York: Springer, 2003.
2. *Breuer L., Dudin A. N., Klimenok V. I.* A retrial $BMAP|PH|N$ system // Queueing Systems. 2002. V. 40. P. 433–457.
3. *Bright L., Taylor P. G.* Calculating the equilibrium distribution in Level Dependent Quasi-Birth-and-Death processes // Communications in Statistics-Stochastic Models. 1995. V. 11. P. 497–525.
4. *Dudin A. N., Klimenok V. I.* A retrial $BMAP/SM/1$ system with linear repeated requests // Queueing Systems. 2000. V. 34. P. 47–66.
5. *Dudin A. N., Klimenok V. I.* Multi-dimensional asymptotically quasi-Toeplitz Markov chains // The XXIV International Seminar on Stability Problems for Stochastic Models. Jurmala, Latvia, September 10–17, 2004. P. 111–118.
6. *Neuts M. F.* Matrix-Geometric Solutions in Stochastic Models – An Algorithmic Approach. Johns Hopkins University Press, 1981.
7. *Neuts M. F.* Structured Stochastic Matrices of $M/G/1$ Type and Their Applications. New York: Marcel Dekker, 1989.