A MULTI-SERVER MARKOVIAN QUEUEING MODEL WITH PRIMARY AND SECONDARY SERVICES

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We study a multi-server queueing model in which arrivals occur according to a Markovian arrival process. An arriving customer either (a) is lost due to all main servers being busy; or (b) enters into service with one of the main servers and leaves the system (as a satisfied *primary* customer); (c) enters into service with one of the main servers, gets service in self-service mode, and being impatient to get a final service with one of the main servers, may leave the system (as a dissatisfied *secondary* customer); or (d) enters into service with one of the main servers, gets service in self-service mode, becomes successful in getting a final service from one of the main servers, and departs (as a satisfied *secondary* customer). This queueing model is studied in steady state and some selected performance measures are derived.

Keywords: Markovian Arrival Process, retrials, loss, multi-server queue, tandem, asymptotically quasi-toeplitz Markov chain.

1. INTRODUCTION AND MODEL DESCRIPTION

The motivation for studying this queuing model came from a real-life application experienced by one of the authors. The laptop computer owned by this author had a problem for which the computer manufacturer was contacted. He had to wait before a technical expert came on line. The author went over the problem with the expert who then advised to do a scanning test on the hard drive first and get back with the test results. The test indicated a problem with the hard drive and hence the author had to call back the company with the test results. With a few tries the author was successful in reaching an expert and after analyzing the test results the author was sent a replacement hard drive (as the Laptop was still under warranty). Thus, the author has to go through a primary service, a secondary (self) service and another primary service before the problem was solved. In this paper we consider a queueing system that closely models the above mentioned real-life application. The basic assumptions of the model are summarized below.

Model Description

- Arrivals occur according to a Markovian arrival process (MAP) which is defined by the underlying continuous time Markov chain $\{\nu_t\}_{t\geq 0}$, on the state space $\{0, 1, \dots, W\}$

with the matrices of transition intensities given by D_0 and D_1 of dimension $W + 1$. These arrivals are referred to as **primary** customers. Any arrival finding an idle server gets into service immediately; otherwise the arrival is considered lost.

- The service system is divided into two groups: **main** and **self-service**. In the main group there are N homogeneous exponential servers. There is no bound on the number of customers in the self-service system.
- The primary customers are served in the main system at a rate $\mu_1, 0 < \mu_1 < \infty$.
- Upon completion of a service in the main system, a primary customer may move to the self-service system with probability $\eta, 0 < \eta \le 1$; and with probability $\bar{\eta} = 1 - \eta$, the primary customer will leave the system. That is, with probability η a primary customer needs to do a self-test.
- The service time of a customer in the self-service system is assumed to be exponentially distributed with parameter $\kappa, 0 < \kappa < \infty$. Upon completion of a service in this system, the customers need to get served again in the main system. These customers will be referred to as **secondary** customers.
- A secondary customer finding an idle server in the main system will get into service immediately. However, if all the main servers are busy, then this customer will try to get into an *orbit* of finite buffer size M from where the customers will try to access the main system. If at the time of getting into the orbit the buffer is full, the secondary customer will be lost with probability $q, 0 \le q \le 1$, and with probability $1 - q$ the customer will wait for an exponential amount of time with parameter κ before trying to get into the orbit.
- While in orbit, each (secondary) customer will try to access a free server in the main system at random times that are exponentially distributed with parameter α, $0 < \alpha < \infty$. That is, if there are m customers in the orbit then the duration of a retrial time is exponential with parameter $m\alpha$.
- The service time of a secondary customer in the main system is assumed to be exponentially distributed with parameter $\mu_2, 0 < \mu_2 < \infty$. Upon completing a service in the main system, the secondary customer will leave the system.

2. MARKOV PROCESS DESCRIPTION

In this section we describe the queueing model outlined in section 1 as a continuous-time Markov chain. First, we define the following quantities.

- i_t = Number of customers in the self-service system at time $t, i_t \ge 0$.
- m_t = Number of secondary customers in the orbit at time $t, 0 \le m_t \le M$.
- n_t = Number of busy (main) servers at time $t, 0 \le n_t \le N$.
- k_t = Number of secondary customers in the main service group at time $t, 0 \le k_t \le n_t$.
- ν_t = Phase of the arrival process at time $t, 0 \le \nu_t \le W$.

The process $\{\xi_t\}_{t\ge0} = \{i_t, m_t, n_t, k_t, \nu_t\}_{t\ge0}$ is clearly a continuous-time Markov chain. First note that the dimension of the state space of the process $\{n_t, k_t\}_{t\ge0}$ is given by $K = 0.5(N + +1)(N + 2)$. In the following $\mathbf{e}$ will denote a column vector (of appropriate dimension) of

1's and I an identity matrix (of appropriate dimension). When needed we will identify the dimension of this matrix with a suffix. For example, I_r will denote an identity matrix of dimension r. The symbol $\otimes$ denotes the Kronecker product of matrices. For details and properties on Kronecker products we refer the reader to [5]. We now define a number of auxiliary matrices for use in the sequel.

- By $\Delta(C_1, \cdots, C_n)$ we denote the diagonal (block) matrix whose i^{th} diagonal (block) element is given by C_i.
- $O_{a\times b}$ will denote a rectangular matrix of dimension $a\times b$ with all entries equal to zero. We will denote the case when $a = b$ by O or O_a. The latter symbol will be used when appropriate.
- $E_{a\times(a+1)}$ is a rectangular matrix of dimension $a \times (a+1)$ which in partitioned form is defined by

$$E_{a\times(a+1)} \stackrel{def}{=} \begin{pmatrix} O_{a\times 1} & I_a \end{pmatrix}.$$

- $\tilde{E}$ is a square matrix of order K which in partitioned form is defined as

$$\tilde{E} \stackrel{def}{=} \begin{pmatrix} O_1 & E_{1\times 2} & & & \\ & & E_{2\times 3} & & \\ & & & \ddots & \\ & & & & E_{N\times(N+1)} \\ & & & & O_{N+1} \end{pmatrix}.$$

- $\bar{E}$ is a square matrix of order K whose only nonzero (block) entry is the last diagonal block. That is,

$$\bar{E} \stackrel{def}{=} \begin{pmatrix} O_{K-N-1} & O_{(K-N-1)\times(N+1)} \\ O_{(N+1)\times(K-N-1)} & I_{N+1} \end{pmatrix}.$$

- $\hat{E}^{(r)}$, for $r = 1, 2$, is a square matrix of order K which in partitioned form is defined as

$$\hat{E}^{(r)} \stackrel{def}{=} \begin{pmatrix} O_1 & & & & \\ H_1^{(r)} & & & & \\ & H_2^{(r)} & & & \\ & & \ddots & & \\ & & & H_N^{(r)} & O_{N+1} \end{pmatrix},$$

  where the matrices $H_n^{(r)}$ of dimension $(n+1)\times n$ are defined by

$$H_n^{(1)} = \begin{pmatrix} n & & & \\ & n-1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}, H_n^{(2)} = \begin{pmatrix} 0 & & & \\ 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}, 1 \le n \le N.$$

- A sequence $C_k^{(n)}$ of square matrices of order $\bar{W} = W + 1$ defined as

$$C_k^{(n)} \stackrel{def}{=} D_0 - [k\mu_2 + (n-k)\mu_1]I + \delta_{n,N}D_1, 0 \le k \le n, 0 \le n \le N,$$

  where $\delta_{n,N}$ is the Kronecker delta.

• $C = \Delta(C^{(0)}, \cdots, C^{(N)})$, where $C^{(n)} = \Delta(C_0^{(n)}, \cdots, C_n^{(n)})$, for $0 \le n \le N$.

Lemma 1. *The Markov chain $\{\xi_t\}_{t\ge0}$ has the generator A, given in partitioned form by*

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & \\ & A_{2,1} & A_{2,2} & A_{2,3} & \\ & & A_{3,2} & A_{3,3} & \\ & & & & \ddots \end{pmatrix}, \tag{1}$$

where the square matrices $A_{i,j}$ of dimension $(M+1)K\bar{W}$ are further partitioned into smaller blocks of dimension $K\bar{W}$ as $A_{i,j} = \left(A_{i,j}(m, m')\right), 0 \le m, m' \le M$, and $i, j \ge 0$. These smaller nonzero block matrices are given as follows.

$$A_{i,i-1}(m, m) = i\kappa(\tilde{E} \otimes I_{\bar{W}}),\ i \ge 1, 0 \le m \le M-1, \tag{2}$$

$$A_{i,i-1}(M, M) = i\kappa(\tilde{E} + q\bar{E}) \otimes I_{\bar{W}},\ i \ge 1, \tag{3}$$

$$A_{i,i-1}(m, m+1) = i\kappa(\bar{E} \otimes I_{\bar{W}}),\ i \ge 1, 0 \le m \le M-1, \tag{4}$$

$$A_{i,i+1}(m, m) = \eta\mu_1(\hat{E}^{(1)} \otimes I_{\bar{W}}),\ i \ge 0, 0 \le m \le M, \tag{5}$$

$$A_{i,i}(m, m-1) = m\alpha(\tilde{E} \otimes I_{\bar{W}}),\ i \ge 0, 0 \le m \le M, \tag{6}$$

$$A_{i,i}(m, m) = C + \bar{\eta}\mu_1(\hat{E}^{(1)} \otimes I_{\bar{W}}) + \mu_2(\hat{E}^{(2)} \otimes I_{\bar{W}}) + \tilde{E} \otimes D_1 - (m\alpha + i\kappa)I, 0 \le m \le M-1, \tag{7}$$

$$\begin{aligned} A_{i,i}(M, M) = {} & C + \bar{\eta}\mu_1(\hat{E}^{(1)} \otimes I_{\bar{W}}) + \mu_2(\hat{E}^{(2)} \otimes I_{\bar{W}}) + \tilde{E} \otimes D_1 - \\ & -M\alpha(I - \bar{E}) \otimes I_{\bar{W}} - i\kappa(I - \bar{E}) \otimes I_{\bar{W}} - qi\kappa(\bar{E} \otimes I_{\bar{W}}),\ i \ge 0. \end{aligned} \tag{8}$$

Proof: Follows immediately by looking at all possible transitions of the Markov chain.

3. STEADY STATE ANALYSIS

In this section we will perform the steady state analysis of the model under study. First we will show that the queueing model described in section 1 is always stable. Next, we will calculate the steady state probabilities and list some selected system performance measures. Observe that the Markov chain $\{\xi_t\}_{t\ge0}$ is a level dependent quasi-birth and death process and the steady state analysis of such processes has been done in the literature (see e.g. [2]. However, here we will use the results for the multi-dimensional asymptotically quasi-toeplitz Markov chains (*AQTMC*) as presented in [1, 3]. This approach allows us to prove the stability condition and also establish an efficient algorithm for the computation of the steady-state probability vector.

Let R_i be the diagonal matrix obtained by taking the diagonal entries of the matrix $A_{i,i}, i \ge 0$. It is well-known that the block matrix P having the structure like (1) with blocks $P_{i,l} = R_i^{-1}A_{i,l}, l = i-1, i+1$ and $P_{i,i} = R_i^{-1}A_{i,i} + I, i \ge 0$, is the one-step transition probability matrix of the *embedded* (or *jump*) discrete-time Markov chain, $\{\tilde{\xi}_t\}_{t\ge0}$, corresponding to the original continuous-time Markov chain $\{\xi_t\}_{t\ge0}$. It is easy to verify that: (i) this discrete-time

Markov chain is $AQTMC$ (see [1, 3]) and (ii) if this chain is positive recurrent then the Markov chain $\{\xi_t\}_{t\geq 0}$ is also ergodic.

It is easy to see that the limits given by $Y_k = \lim_{i\to\infty} P_{i,i+k-1}$, for $k = 0, 1, 2$, exist for all values of the system parameters for the model under study. Specifically the expressions for these limits derived by considering two cases: (a) $q > 0$ and (b) $q = 0$, are given in the following two lemmas. In the following the matrices Y_k will be displayed in block partitioned form and we denote the $(m, r)^{th}$ block of Y_k by $Y_k(m, r)$.

Lemma 2. *For $q > 0$ we have $Y_1 = Y_2 = O$, and the nonzero blocks of the matrix Y_0 are given by*

$$Y_0(m,r) = \begin{cases} \tilde{E} \otimes I_{\bar{W}}, r = m, 0 \leq m \leq M-1, \\ (\tilde{E} + \bar{E}) \otimes I_{\bar{W}}, r = m = M, \\ \bar{E} \otimes I_{\bar{W}}, r = m+1, 0 \leq m \leq M-1. \end{cases} \tag{9}$$

Proof: Follows immediately from the definition of the limits.

Lemma 3. *For $q = 0$, the nonzero blocks of the matrices $Y_k, k = 0, 1, 2$, are given by*

$$Y_0(m,r) = \begin{cases} \tilde{E} \otimes I_{\bar{W}}, r = m, 0 \leq m \leq M, \\ \bar{E} \otimes I_{\bar{W}}, r = m+1, 0 \leq m \leq M-1, \end{cases} \tag{10}$$

$$Y_1(M,M) = (\bar{E} \otimes I_{\bar{W}})\hat{R}_i^{-1}[C + \bar{\eta}\mu_1(\hat{E}^{(1)} \otimes I_{\bar{W}}) + \mu_2(\hat{E}^{(2)} \otimes I_{\bar{W}}) - M\alpha(I - \bar{E}) \otimes I_{\bar{W}}] + I_{K\bar{W}}, \tag{11}$$

$$Y_2(M,M) = \eta\mu_1(\bar{E} \otimes I_{\bar{W}})\hat{R}_i^{-1}(\hat{I}^{(1)} \otimes I_{\bar{W}}), \tag{12}$$

where $\hat{R}_i^{-1} = R_i^{-1}(M, M)$.

Proof: Follows immediately from the definition of the limits.

Remark. It is easy to verify that (since $q = 0$) the matrix $(\bar{E} \otimes I_{\bar{W}})\hat{R}_i^{-1}$ does not depend on i and so the matrices $Y_k(M, M), k = 1, 2$ also do not depend on i.

In order to establish the stability condition along the lines of [1, 3], we first define the matrix generating function $Y(z) = Y_0 + Y_1 z + Y_2 z^2, |z| \leq 1$.

When $Y(1)$ is a reducible matrix with stochastic irreducible blocks, $Y^{(l)}(1)$, $1 \leq l \leq L$, of its normal form and $Y'(1) < \infty$, the Markov chain $\{\xi_t\}_{t\geq 0}$ is stable if the following conditions are fulfilled (see [1, 3]):

$$\mathbf{x}_l \frac{dY^{(l)}(z)}{dz}\Big|_{z=1} \mathbf{e} < 1, \; 1 \leq l \leq L, \tag{13}$$

where $\mathbf{x}_l$ is the unique solution of the equations

$$\mathbf{x}_l Y^{(l)}(1) = \mathbf{x}_l, \; \mathbf{x}_l \mathbf{e} = 1, \; 1 \leq l \leq L. \tag{14}$$

Now we are ready to establish the following theorem.

Theorem 1. *The Markov chain $\{\xi_t\}_{t\geq 0}$ is stable.*

Proof. In the case when $q > 0$, from Lemma 2 we see that $Y'(z)$ is equal to zero for any z. So, condition in (13) always holds good.

In the case when $q = 0$, it follows from (10)-(12) that the nonzero blocks of the matrix generating function $Y(z)$ are given by

$$Y(z)(m,r) = \begin{cases} \tilde{E} \otimes I_{\bar{W}}, r = m, 0 \le m \le M-1, \\ \bar{E} \otimes I_{\bar{W}}, r = m+1, 0 \le m \le M-1, \\ \tilde{E} \otimes I_{\bar{W}} + Y^{(M,M)}(z), r = m = M, \end{cases} \tag{15}$$

where the square matrix $Y^{(M,M)}(z)$ of order K is such that the only nonzero block entry occurs in the last $(N+1)W$ rows which are given by

$$\left[\begin{array}{ccc} O_{(N+1)\bar{W}\times(K-2N-1)\bar{W}} & R^{-1}(\hat{A}+Bz)z & R^{-1}C^{(N)}z + zI_{(N+1)\bar{W}} \end{array} \right],$$

where the diagonal (block) entries of the diagonal matrix R^{-1} are the $(N+1)\bar{W}$ diagonal entries of the matrix $\hat{R}_i^{-1}$, and the matrices $\hat{A}$ and B are defined by

$$\hat{A} = (\mu_1 \bar{\eta} H_N^{(1)} + \mu_2 H_N^{(2)}) \otimes I_{\bar{W}}, B = \mu_1 \eta (H_N^{(1)} \otimes I_{\bar{W}}).$$

Transforming the matrix $Y(1)$ into the normal form (see, e.g., [4]), it is easy to verify that the unique irreducible (stochastic) block of this matrix is

$$Y^{(1)}(1) = \begin{pmatrix} \tilde{R}^{-1}C_N^{(N)} + I_{\bar{W}} & \tilde{R}^{-1}N\mu_2 \\ I_{\bar{W}} & O_{\bar{W}} \end{pmatrix},$$

where the diagonal entries of the diagonal matrix $\tilde{R}^{-1}$ are given by the last $\bar{W}$ diagonal entries of the matrix R^{-1}.

The corresponding block in the matrix $Y(z)$ has the form:

$$Y^{(1)}(z) = \begin{pmatrix} z\tilde{R}^{-1}C_N^{(N)} + zI & \mu_2 z\tilde{R}^{-1}N \\ I & O \end{pmatrix} = \begin{pmatrix} z\tilde{R}^{-1}(D(1) - N\mu_2 I) + zI & \mu_2 z\tilde{R}^{-1}N \\ I & O \end{pmatrix},$$

where $D(1) = D_0 + D_1$ is the generator of the underlying MAP, $\{\nu_t\}_{t\ge0}$. Denote by $\boldsymbol{\theta}$ the stationary probability vector of $D(1)$. This vector satisfies the system:

$$\boldsymbol{\theta} D(1) = \mathbf{0}, \boldsymbol{\theta}\mathbf{e} = 1. \tag{16}$$

It can be verified that the vector $\mathbf{x}_1$, which is the solution of system (14), is calculated as $\mathbf{x}_1 = \boldsymbol{\theta}$ and consequently inequality (13) has the form:

$$\boldsymbol{\theta}(-D(1) + N\mu_2 I_{\bar{W}})\mathbf{e} > 0,$$

and this, upon using (16), reduces to

$$N\mu_2 > 0,$$

which is always true. So, the Markov chain $\{\xi_t\}_{t\ge0}$ is always stable.

Remark. We formally proved the existence of the stationary distribution of the Markov chain $\{\xi_t\}_{t\ge0}$. However, we can intuitively explain this existence. With respect to the primary customers the system is like a multi-server loss system in that an arriving (primary) customer finding all servers is considered lost. With a certain probability a primary customer becomes a secondary customer. When all secondary customers compete (the worst case scenario by taking $q = 0$) with primary customers for an additional service in the main system, the system will be stable as these secondary customers will have a very high probability of occupying a free (main) server once the number in self-service system becomes relatively large.

The stationary distribution: First, we enumerate the states of the Markov chain $\{\xi_t\}_{t\geq 0}$ and the corresponding embedded chain, $\{\tilde{\xi}_t\}_{t\geq 0}$ of section 2, in lexicographic order. Denote by $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \cdots)$ and $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots)$, respectively, the steady state probability vectors of the Markov chains, $\{\xi_t\}_{t\geq 0}$ and $\{\tilde{\xi}_t\}_{t\geq 0}$.

Looking at the relationship between $\{\xi_t\}_{t\geq 0}$ and $\{\tilde{\xi}_t\}_{t\geq 0}$, we observe that $\mathbf{p}$ is related to $\boldsymbol{\pi}$ in the following way:

$$\mathbf{p}_i = \bar{c}\boldsymbol{\pi}_i R_i^{-1}, \quad i \geq 0, \tag{17}$$

where the positive (finite) constant $\bar{c}$ is given by

$$\bar{c} = (\sum_{l=0}^{\infty} \boldsymbol{\pi}_l R_l^{-1}\mathbf{e})^{-1}. \tag{18}$$

The calculation of the stationary vector $\boldsymbol{\pi}$ using $AQTMC$ is discussed in [1, 3] and hence we have the following theorem.

Theorem 2. *The stationary probability vector $\boldsymbol{\pi}$ of embedded Markov chain $\{\tilde{\xi}_t\}_{t\geq 0}$, is calculated as follows:*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0\Phi_i, i \geq 1, \tag{19}$$

where the matrices $\Phi_i, i \geq 1$, are computed as

$$\Phi_0 = I, \Phi_i = \prod_{l=1}^{i} \bar{P}_{l-1,l}(I - \bar{P}_{l,l})^{-1}, i \geq 1, \tag{20}$$

and the matrices $\bar{P}_{i,l}$ are defined by

$$\bar{P}_{l-1,l} = P_{l-1,l}, l \geq 1, \quad \bar{P}_{l,l} = P_{l,l} + P_{l,l+1}G^{(l+1)}, l \geq 0,$$

with the matrices $G^{(k)}$ satisfy the following backward recursion:

$$G^{(k)} = P_{k-1,k} + P_{k,k}G^{(k)} + P_{k,k+1}G^{(k+1)}G^{(k)}, k \geq 0, \tag{21}$$

and the vector $\boldsymbol{\pi}_0$ is the unique solution to the following system of linear equations:

$$\boldsymbol{\pi}_0(I - \bar{P}_{0,0}) = \mathbf{0} \quad \textit{and} \quad \boldsymbol{\pi}_0 \sum_{k=0}^{\infty} \Phi_k\mathbf{e} = 1. \tag{22}$$

Note that the critical aspect of solving for the steady state probability vector $\boldsymbol{\pi}$ lies in the evaluation of $G^{(k)}$ matrices using the recursion in (21). This evaluation can be accomplished (see for example, [1, 3]) by starting with a large k and setting $G^{(k)} = G$, where the matrix G is a solution to the equation

$$G = Y(G). \tag{23}$$

For the model under study, the solution of (23) is described in the following theorem.

Theorem 3. *For the case when $q > 0$, the matrix G is calculated as*

$$G = Y_0 \tag{24}$$

where the matrix Y_0 is defined by formula (9).

For the case when $q = 0$, the matrix G has following structure:

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{25}$$

with G_1 and G_2 calculated as

$$G_1 = Y_0^{(1)}, G_2 = \begin{bmatrix} O_{(N+1)\bar{W}\times(K-N-1)\bar{W}} & -(\tilde{B} + C^{(N)})^{-1}\hat{A}E_{N\bar{W}\times(N+1)\bar{W}} \end{bmatrix}, \tag{26}$$

where the matrix $Y_0^{(1)}$ is obtained from Y_0 by deleting the last $(N + 1)\bar{W}$ rows, and $\tilde{B} = BE_{N\bar{W}\times(N+1)\bar{W}}$.

Once the steady state vector $\boldsymbol{\pi}$ is computed, we can use equations (17) and (18) to evaluate $\mathbf{p}$ of the Markov chain $\{\xi_t\}_{t\geq 0}$.

Selected System Performance Measures: With the knowledge of the steady state probability vector, $\mathbf{p}$, we can compute a variety of system performance measures to study the qualitative behavior of the model. Here we will list a few measures along with their formulas.

- The probability, $P_{loss}^{(1)}$, that a primary customer will be lost is calculated as

$$P_{loss}^{(1)} = \lambda^{-1}\sum_{i=0}^{\infty} \mathbf{p}_i(I_{M+1} \otimes \bar{E} \otimes D_1)\mathbf{e}.$$

- The probability, P_{orbit}, that an arbitrary self-service customer will enter into the orbit is calculated as

$$P_{orbit} = \sum_{i=1}^{\infty} \mathbf{p}_i(\hat{F}_{M+1} \otimes \bar{E} \otimes I_{\bar{W}})\mathbf{e},$$

  where $\hat{F}_{M+1}$ is a square matrix of order $M + 1$ obtained from I_{M+1} by replacing the last diagonal entry with 0.

- The probability, $P_{loss}^{(2)}$, that a secondary customer will be lost is calculated as

$$P_{loss}^{(2)} = q\sum_{i=1}^{\infty} \mathbf{p}_i((I_{M+1} - \hat{F}_{M+1}) \otimes \bar{E} \otimes I_{\bar{W}})\mathbf{e}.$$

- The probability, P_{self}, that an arbitrary primary customer will get into a self-service system is calculated as

$$P_{self} = \eta(1 - P_{loss}^{(1)}).$$

- The rate (or throughput), γ, of satisfied customers leaving the system is given by

$$\gamma = \lambda(1 - P_{loss}^{(1)})(1 - \eta P_{loss}^{(2)}).$$

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