ON THE M/G/1 RETRIAL QUEUE WITH FEEDBACK

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In this paper, we investigate the decomposition property of the M/G/1 retrial queue with feedback in case of exponential and general retrial time distributions.

Keywords: retrial queue, feedback, embedded Markov chain, decomposition property.

1. INTRODUCTION: MODEL DESCRIPTION

We consider a single server queueing system with no waiting space at which primary customers arrive according to a Poisson process with rate $\lambda > 0$. An arriving customer receives immediate service if the server is idle, otherwise he leaves the service area temporarily to join the retrial group (orbit). Any orbiting customer produces a Poisson stream of repeated calls with intensity $\theta > 0$ until the time at which he finds the server idle and starts his service. The service times follow a general distribution with distribution function $B(x)$ having finite mean $\frac{1}{\gamma}$ and Laplace-Stieltjes transform $\tilde{B}(s)$. After the customer is served, he will decide either to join the orbit for another service with probability c or to leave the system forever with probability $\bar{c} = 1 - c$. Finally, we admit the hypothesis of mutual independence between all random variables defined above. The state of the system at time t can be described by means of the process $\{C(t), N_o(t), \zeta(t), t \geq 0\}$, where $N_o(t)$ is the number of customers in the orbit, $C(t)$ is the state of the server at time t. We have that $C(t)$ is 0 or 1 depending on whether the server is idle or busy. If $C(t) = 1$, $\zeta(t)$ represents the elapsed service time of the customer being served.

2. EMBEDDED MARKOV CHAIN AND ERGODICITY CONDITION

Let ξ_n be the time when the server enters the idle state for the n-th time. The sequence of random variables $\{q_n = N_o(\xi_n^+), n \geq 1\}$ forms a Markov chain.

Consider the following fundamental equation

$$q_{n+1} = q_n - \delta_{q_n} + \nu_{n+1} + u. \tag{1}$$

The random variable ν_{n+1} represents the number of primary customers arriving at the system during the $(n + 1)$st service time interval. It does not depend on events which have occurred before the beginning of the $(n + 1)$-st service. Its distribution is given by $k_i = P(\nu_{n+1} = i) = \int\limits_0^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} dB(x)$ with generating function $\sum\limits_{i=0}^\infty k_i z^i = \tilde{B}(\lambda - \lambda z)$ [2]. The

Bernoulli random variable δ_{q_n} is equal to 1 if the $(n+1)$st served customer is an orbiting customer, or to 0 if this customer is a primary one. Its conditional distribution is given by $P(\delta_{q_n} = 1/q_n = k) = \frac{k\theta}{\lambda+k\theta}$ and $P(\delta_{q_n} = 0/q_n = k) = \frac{\lambda}{\lambda+k\theta}$.The random variable u is 0 or 1 depending on whether the served customer leaves the system or goes to the orbit. We have also that $P(u=0) = \bar{c}$ and $P(u=1) = c$. The one-step transition probabilities of our embedded Markov chain are

$$\begin{aligned} r_{km} &= P(q_{n+1} = m/q_n = k) = \\ &= k_{m-k}\frac{\lambda}{\lambda+k\theta}\bar{c} + k_{m-k+1}\frac{k\theta}{\lambda+k\theta}\bar{c} + k_{m-k-1}\frac{\lambda}{\lambda+k\theta}c + k_{m-k}\frac{k\theta}{\lambda+k\theta}c. \end{aligned} \quad (2)$$

Note that $r_{km} \neq 0$ for $k = 0, 1, \ldots, m+1$.

Theorem 1. *The embedded Markov chain* $\{q_n, n \geq 1\}$ *is ergodic iff*

$$\rho = \frac{\lambda}{\gamma} + c < 1.$$

Proof. From (1), we can see that $\{q_n, n \geq 1\}$ is an irreductible and aperiodic Markov chain. By applying Foster's criterion and Kaplan's condition, we demonstrate that $\{q_n, n \geq 1\}$ is ergodic if and only if $\rho = \frac{\lambda}{\gamma} + c < 1$.

Now, we find the stationnary distribution $d_k = \lim_{n\to\infty} P(q_n = k)$. With the help of the generating functions $D(z) = \sum_{k=0}^{\infty} d_k z^k$ and $L(z) = \sum_{k=0}^{\infty} \frac{d_k}{\lambda+k\theta} z^k$, from (2), we obtain ($\rho < 1$)

$$D(z) = \frac{(1-\rho)(\bar{c}+cz)\tilde{B}(\lambda-\lambda z)(1-z)}{(\bar{c}+cz)\tilde{B}(\lambda-\lambda z)-z}\exp\left\{\frac{\lambda}{\theta}\int_1^z \frac{1-(\bar{c}+cu)\tilde{B}(\lambda-\lambda u)}{(\bar{c}+cu)\tilde{B}(\lambda-\lambda u)-u}du\right\}. \quad (3)$$

□

3. STOCHASTIC DECOMPOSITION PROPERTY

Consider the expression (3). It is easy to see that the right hand part can be decomposed into two factors

$$D(z) = \left[\frac{(1-\rho)\tilde{B}(\lambda-\lambda z)(1-z)}{(\bar{c}+cz)\tilde{B}(\lambda-\lambda z)-z}\right]\cdot\left[(\bar{c}+cz)\exp\left\{\frac{\lambda}{\theta}\int_1^z \frac{1-(\bar{c}+cu)\tilde{B}(\lambda-\lambda u)}{(\bar{c}+cu)\tilde{B}(\lambda-\lambda u)-u}du\right\}\right].$$

The first factor is the generating function for the number of customers in the M/G/1 queueing system with Bernoulli feedback [5]; the remaining one is the generating function for the number of customers in the retrial queue with feedback given that the server is idle.

Now, we extend this result for general retrial time distribution. In the first time, we introduce some notations. Let ξ_n be the time when the server enters the idle state for the n-th time; ς be the time at which the n-th fresh customer arrives at the server; X_i^n be the time elapsed since the last attempt made by the i-th customer in the orbit until instant ξ_n^+; $q_n = N_o(\xi_n^+)$ be the number of customers in the orbit at instant ξ_n^+. We assume that the system is in steady state. Let $q = \lim_{n\to\infty} q_n$; $X_i = \lim_{n\to\infty} X_i^n$. When $q > 0$, we have a vector $X = (X_1, X_2, ..., X_q)$. We denote by $f_q(x_1, x_2, ..., x_q) = f_q(x)$ the joint density function of q and X. Define

$$r_{ij} = \lim_{n\to\infty} P(C(\varsigma_n^-) = i, N_o(\varsigma_n^-) = j) \qquad i = 0,1 \qquad j = 0,1,2,...;$$

$$d_k = \lim_{n\to\infty} P(q_n = k) \qquad k = 0,1,2,... \qquad d_k = \int_0^\infty f_k(x)dx \qquad k = 1,2,...;$$

$$D(z) = \sum_{k=0}^{\infty} d_k z^k \text{ and } \qquad R_i(z) = \sum_{j=0}^{\infty} r_{ij} z^j \qquad i = 0,1.$$

Consider the following fundamental equation for our embedded Markov chain $\{q_n, n \geq 1\}$

$$q_{n+1} = q_n - \delta(q_n; X^n) + v_{n+1} + u,$$

where $\delta(q_n; X^n)$ is 1 or 0 depending on whether the $(n+1)$st served customer is an orbiting customer or a primary one. When $q_n = 0$, $P(\delta(0; X^n) = 0) = P(\delta(0) = 0) = 1$. Since the random variables v_{n+1}, $q_n - \delta(q_n; X^n)$ and u are mutually independent, we have

$$E[z^{q_{n+1}}] = E[z^{q_n - \delta(q_n; X^n)}]E[z^{v_{n+1}}]E[z^u].$$

Let $n \to \infty$. We find

$$D(z) = E[z^{q-\delta(q;X)}]\tilde{B}(\lambda - \lambda z)(\bar{c} + cz). \tag{4}$$

Using the rule of conditional expectation, we obtain

$$\begin{aligned} E\left[z^{q-\delta(q;X)}\right] &= \sum_{j=0}^{\infty} \int_0^\infty f_j(x) E\left[z^{j-\delta(j;x)}\right] dx = \qquad (5) \\ &= \sum_{j=0}^{\infty} \int_0^\infty f_j(x)\left[z^j P(\delta(j;x) = 0) + z^{j-1}(1 - P(\delta(j;x) = 0))\right] dx = \\ &= \frac{1}{z}\sum_{j=0}^{\infty} z^j d_j + \left(1 - \frac{1}{z}\right)\sum_{j=0}^{\infty} z^j \int_0^\infty f_j(x) P(\delta(j;x) = 0)\, dx. \end{aligned}$$

Consider $\int_0^\infty f_j(x) P(\delta(j;x) = 0)\, dx$. This is the probability that an arriving customer finds j customers in the orbit and no customer at the server. This event takes place if and only if the last served customer leaves j customers in the orbit, he still did not decide to join the orbit or to leave the system and the new arrival occurs before any of the j orbiting customers retry for service. Therefore, $r_{oj} = \int_0^\infty f_j(x) P(\delta(j;x) = 0)\, dx$.

We can rewrite (5) as

$$E\left[z^{q-\delta(q;X)}\right] = \frac{1}{z}D(z) + \left(1 - \frac{1}{z}\right)R_0(z). \tag{6}$$

Put (6) into (4), we obtain

$$D(z) = \frac{(1-\rho)\tilde{B}(\lambda-\lambda z)(1-z)}{(\bar{c}+cz)\tilde{B}(\lambda-\lambda z)-z} \times \frac{(\bar{c}+cz)R_0(z)}{1-\rho}. \quad (7)$$

The second factor on the right part of (7) is the generating function for the number of customers in the retrial queueing system with feedback given that the server is idle when the retrial times follow a general distribution. Note that if $\bar{c} = 1$, one can obtain the same result as in [6] for the M/G/1 retrial queue without feedback.

REFERENCES

1. *Choi B. D., Kim Y. C., Lee Y. W.* The M/M/c retrial queue with geometric loss and feedback // Computers and Mathematics with Applications. 1998. V. 36. P. 41–52.
2. *Falin G. I., Templeto J. G. C.* Retrial Queues. Chapman & Hall, 1997.
3. *Farahmand K.* Single line queue with recurrent repeat demands // Queueing Systems. 1996. V. 22. P. 425–435.
4. *Krishna Kumar B. et al.* The M/G/1 retrial queue with feedback and starting failures // Applied Mathematical Modelling. 2002. V. 26. P. 1057–1075.
5. *Takacs I.* A single server queue with feedback // Bell System Technical J. 1963. V. 42. P. 505–519.
6. *Yang T. et al.* Approximation method for the M/G/1 retrial queue with general retrial times // EJOR. 1994. V. 76. P. 552–562.