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The Canonical Order and Optimization Problems

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Abstract: Using the partial order technique, we describe a subclass of objective functions taking their optimum at the greedy point of a given feasible polyhedron in $\mathbf{R}^n$.

Key Words: Canonical order, greedy algorithm, linear programming.

1 Introduction

The relation $\prec$ of the **canonical** order is defined on $\mathbf{R}^n$ by

$$x \prec y \Leftrightarrow g_s(x) \leq g_s(y) \qquad s = 1 \dots n$$

where $g_s(x) = \sum_{i=1}^{s} x_i$. The **greedy** solution x^g of the problem

$$\max\{f(x) | x \in D\} \tag{1}$$

is defined as the lexicographical maximum of the feasible set D.

As it was shown in [1, 2, 3], optimality of the greedy solution is closely connected with existence in the feasible set a single maximum w.r.t. $\prec$. Namely, the greedy algorithm on a polymatroid always provides the optimal solution for any nonnegative linear objective function, whereas polymatroids are a unique class of polyhedra in $\mathbf{R}^n$ which have a single maximum w.r.t. $\prec$ for any ordering of variables. In [4] we've obtained conditions defining a polyhedron in $\mathbf{R}^n$ with

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a single maximum x^* for a *fixed* order of variables. It was also shown that a class $\mathscr{F}$ of all continuously differentiable functions (C^1-functions), isotone w.r.t. $\prec$, is defined by

$$0 \le \frac{\partial f}{\partial x_{i+1}}(x) \le \frac{\partial f}{\partial x_i}(x) \qquad \text{for all} \quad x \in \mathbf{R}^n \ , \qquad i = 1 \dots n-1 \ . \tag{2}$$

Thus, for any $f \in \mathscr{F}$ and for certain class of feasible polyhedra it holds $x^{opt} = x^* = x^g$.

Here we consider the following problem.

Problem 1: Given an arbitrary feasible polyhedron $D \in \mathbf{R}^n$, describe a class $\mathscr{F}(D)$ of objective functions which take their maximum at the greedy point.

Our goal is to describe a subclass in $\mathscr{F}(D)$ using the partial order technique. The main idea consists in introducing on $\mathbf{R}^n$ a relation $\prec^h$ of the generalized canonical order, and describing a family of all orders $\prec^h$ for which the poset $(D, \prec^h)$ has a single maximum. Since a single maximum w.r.t. $\prec^h$ always coincides with the greedy solution, corresponding classes of isotone functions turns to belong to $\mathscr{F}(D)$.

2 Generalized Canonical Order

Let $\mathscr{M}$ be a class of strictly increasing separable concave functions on $\mathbf{R}^n$. For $h(x) = \sum_{i=1}^{n} h_i(x_i) \in \mathscr{M}$ we denine a relation $\prec^h$ of the **generalized** canonical order by

$$x \prec^h y \Leftrightarrow g_s^h(x) \le g_s^h(y) \qquad s = 1 \dots n$$

where $g_s^h(x) = \sum_{i=1}^{s} h_i(x_i)$. Note that $\prec^h$ defines a *partial* order (the antisymmetric property follows from strict increasing of h).

We say that an order $<_1$ **includes** an order $<_2$ if $x <_2 y$ yields $x <_1 y$. It is easy to show that the lexicographical order includes any canonical order $\prec^h$ which, in its turn, includes the coordinate order. Moreover, the order $\prec^h$ approaches the lexicographical order if $h_i(t) = \varepsilon^i t$ and $\varepsilon \to 0$. Similarly, it approaches the coordinate order if $h_i(t) = \varepsilon^{-i} t$ and $\varepsilon \to 0$.

Conditions defining the class $\mathscr{F}^h \subset C^1$ of isotone functions w.r.t $\prec^h$ can be easily obtained from (2) by substitution $x_i = h_i(z_i)$:

$$0 \leq \frac{h_i'(x_i)}{h_{i+1}'(x_{i+1})} \frac{\partial f}{\partial x_{i+1}}(x) \leq \frac{\partial f}{\partial x_i}(x) \qquad \text{for all} \quad x \in \mathbf{R}^n \ , \qquad i = 1 \ldots n-1 \ . \tag{3}$$

The class $\mathscr{F}^h$ contains, in particlar, each function g_s^h and any composition $F(y_1(x), \ldots, y_m(x))$, nondecreasing on y, and with $y_i(x) \in \mathscr{F}^h$.

Our aim is to describe a family of all orders $\prec^h$ such that $\mathscr{F}^h \subset \mathscr{F}(D)$. Note that if $(D, \prec^h)$ has a single maximum x^* then $x^* = x^{opt}$ for any $f \in \mathscr{F}^h$. On the other hand, if x^* exists, then $x^* = x^g$, since the lexicographical order includes $\prec^h$. Thus, the problem is reduced to the next one.

Problem 2: Given an arbitrary polyhedron in $\mathbf{R}^n$, describe the set $\mathscr{L}(D)$ of all $h \in \mathscr{M}$ for which $(D, \prec^h)$ has a single maximum.

Remark: Suppose we want to solve (1) with incomplete knowledge about the objective function f. All information we have is that f is isotone w.r.t. some order $\prec^h$. Then the functions $w_i(x) = h_i'(x_i)/h_{i+1}'(x_{i+1})$ report *how much* information we have. For example, if $w_i(x) \equiv \varepsilon \to 0$, then condition (3) reduces to inequalities $\frac{\partial f}{\partial x_i}(x) \geq 0$ reporting only that f is a nondecreasing function. In this sense, problem 2 may have another interpretation: how much information about the objective function is enough to solve problem (1) if the objective function is unknown?

We'll need the following lemma.

Lemma 1: The greedy solution x^g is a single maximum of $(D, \prec^h)$ iff it is the optimal solution for n problems

$$\max\{g_s^h(x) | x \in D\} \qquad s = 1 \ldots n \ . \tag{4}$$

Indeed, the optimal solution for (4) satisfies $x \prec^h x^g$ for all $x \in D$.

For $h \in \mathscr{M}$ define an n-vector $\nabla g_s^h(x) = (h_1'(x_1), \ldots, h_s'(x_s), 0, \ldots, 0)$ and a matrix

$$\nabla G^h(x) = \begin{bmatrix} \nabla g_1^h(x) \\ \vdots \\ \nabla g_n^h(x) \end{bmatrix} \ .$$

Let $D = \{x \in \mathbf{R}^n\colon Ax \leq b\}$ be a nondegenerate polyhedra and A_B be a basic submatrix corresponding to x^g (x^g is always a basic solution by definition).

Theorem 1: The greedy solution x^g is a single maximum of $(D, \prec^h)$ iff

$$\nabla G^h(x^g) A_B^{-1} \geq 0 \; . \tag{5}$$

Proof: Problems (4) are problems with concave objective functions and linear restrictions. According to the Kuhn–Tucker conditions for problems of this kind, x^g is the optimal solution for $\max\{g_s^h(x) | x \in D\}$ iff there exists a vector $\lambda \geq 0$ such that

$$\nabla g_s^h(x^g) - \lambda A_B = 0 \; . \tag{6}$$

Writing (6) for every $s = 1 \ldots n$, we obtain (5). ■

Corollary 1: The greedy solution is optimal for any $f \in \mathscr{F}^h$ iff the feasible polyhedron satisfies (5).

Note that condition (5) is defined by values of h_i' only at the greedy point. Denote $\alpha_i = h_i'(x_i^g)$ and consider (5) as a system of inequalities w.r.t. α under the condition $h_i'(x^g) > 0$:

$$\sum_{i=1}^{s} \alpha_i \bar{a}_{ij} \geq 0 \qquad s = 1 \ldots n \; , \qquad j = 1 \ldots n \; , \tag{7}$$

$$\alpha_i > 0 \qquad i = 1 \ldots n \; ,$$

where $\bar{a}_{ij}$ is an element of A_B^{-1}.

Let $\mathscr{A}(D)$ be the set of feasible solutions of (7). The following theorem gives the solution of problem 2.

Theorem 2: For an arbitrary polyhedron D in $\mathbf{R}^n$ the set $\mathscr{L}(D)$ is always nonempty and is defined as follows

$$\mathscr{L}(D) = \bigcup_{\alpha \in \mathscr{A}(D)} \{h \in \mathscr{M}\colon \nabla g_n^h(x^g) = \alpha\} \; .$$

Proof: It suffices to show that $\mathscr{L}(D)$ is nonempty, i.e. to find $h \in \mathscr{M}$ such that $(D, \prec^h)$ has a single maximum.

Let $\varepsilon > 0$ satisfies the following: if x and y are basic solutions and $x \neq y$, then

$$\varepsilon < |x_i - y_i| < \frac{1}{\varepsilon}.$$

Define $h(x) := \alpha x$ with $\alpha := \left(1, \frac{\varepsilon^2}{2}, \left(\frac{\varepsilon^2}{2}\right)^2, \ldots, \left(\frac{\varepsilon^2}{2}\right)^{n-1}\right)$. Then x^g is a single maximum w.r.t. $\prec^h$ by lemma 1. Indeed, let x be any basic solution and let $i = \min\{1 \leq j \leq n: x_j \neq x_j^g\}$. Then for any linear $f(x) = cx \in \mathscr{F}^h$ (including $g_s^h(x)$) we have

$$\begin{aligned} c(x^g - x) &= c_i(x_i^g - x_i) + c_{i+1}(x_{i+1}^g - x_{i+1}) + \cdots + c_n(x_n^g - x_n) \\ &\geq c_i\varepsilon - c_{i+1}\frac{1}{\varepsilon} - \cdots - c_n\frac{1}{\varepsilon} \\ &\geq c_i\left(\varepsilon - \frac{\varepsilon^2}{2}\frac{1}{\varepsilon} - \left(\frac{\varepsilon^2}{2}\right)^2\frac{1}{\varepsilon} - \cdots - \left(\frac{\varepsilon^2}{2}\right)^{n-1}\frac{1}{\varepsilon}\right) > 0 . \quad \blacksquare \end{aligned}$$

3 Some Examples

Consider a problem

$$\max\{f(x): ax \leq b, 0 \leq x \leq d\} \tag{8}$$

where $a, b > 0$. It is easy to see that the greedy solution for (8) is of the form

$$x^g = \left(d_1, \ldots, d_{k-1}, \frac{1}{a_k}\left(b - \sum_{i=1}^{k-1} a_i d_i\right), 0, \ldots, 0\right). \tag{9}$$

where $1 \leq k \leq n$. The problem is to describe $\mathscr{L}(D)$, i.e. to find all $h \in \mathscr{M}$ such that x^g is optimal for any $f \in \mathscr{F}^h$.

Corollary 2: A function $h \in \mathcal{M}$ belongs to $\mathcal{L}(D)$ iff for any i and j such that $1 \leq i < k < j \leq n$, it holds

$$\frac{h_i'(x_i^g)}{a_i} \geq \frac{h_k'(x_k^g)}{a_k} \geq \frac{h_j'(x_j^g)}{a_j} \ . \tag{10}$$

Proof: The basic matrix and its inverse corresponding to x^g are the following

$$A_B = \begin{bmatrix} & I & & 0 & \\ a_1 & \cdots & a_k & \cdots & a_n \\ & 0 & & -I & \end{bmatrix} \quad \text{and}$$

$$A_B^{-1} = \begin{bmatrix} & I & & 0 & \\ -\dfrac{a_1}{a_k} & \cdots & \dfrac{1}{a_k} & \cdots & \dfrac{a_n}{a_k} \\ & 0 & & -I & \end{bmatrix} .$$

Hence, as theorem 2 implies, $h \in \mathcal{M}$ belongs to $\mathcal{L}(D)$ iff

$$h_i'(x_i^g) - \frac{a_i}{a_k} h_k'(x_k^g) \geq 0 \qquad \text{for } i < k \qquad \text{and}$$

$$-h_j'(x_i^g) + \frac{a_j}{a_k} h_k'(x_k^g) \geq 0 \qquad \text{for } k < j \ . \quad \blacksquare$$

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