Algorithms of Calculate Wavelets Spectral Estimate with Doubeshies Scaling Function

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Abstract: *The main task of this research work is applying techniques of wavelet analysis in spectral analysis of stationary random processes. Algorithm of calculation obtained for the spectral estimate of stationary random processes with discrete time via Doubeshies scaling function is studied.*

Keywords**:** Spectral estimate, wavelets methods, scaling function Doubeshies.

1. INTRODUCTION

One of the main problems in spectral analysis of time series is consistent estimate formation of the second order spectral density via finite realization of stationary process. In numerous researches dealing with mentioned task periodogram methods based on inverse Fourier transform are used.

It's essential to mention that various research questions for statistics of consistent estimate obtained by periodogram smoothing via spectral windows are published, for example, in monographs [1-3] and publications [4-10].

Recently application of wavelet-analysis methods in time series study is quite relevant, as obtained by this method results are frequently more informative and can directly deal with such input data peculiarities which are hard to handle with the traditional approach.

Instead of using a deterministic approach applied scientists usually use a stochastic approach to model the data and to estimate the energy distribution (e.g. in electrical engineering, geophysics, economics or neurophysiology). One reason is that in a stochastic setup certain fluctuations of the Fourier-transform of the data can be interpreted more naturally.

1. SCALING FUNCTION

Function $\varphi(x) \in L_2(R)$ creating the set of spaces:

$$V_j = \left\{ \sum_{k\in\mathbf{Z}} c_k \varphi_{j,k}(x), \sum_{k\in\mathbf{Z}} |c_k|^2 < \infty, x \in \mathbf{R} \right\}, \ j \in \mathbf{Z},$$

for which

1. $V_j \subset V_{j+1}$ for each $j \in \mathbf{Z}$,
2. $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L_2(\mathbf{R})$,
3. $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$

is called scaling function.
The system of function

$$\varphi_{j,k}(x) = 2^{\frac{j}{2}} \varphi(2^j x - k),$$

$j \in \mathbf{Z}$, $x \in \mathbf{R}$, $k \in \mathbf{Z}$ is orthonormal in $L_2(\mathbf{R})$ and function (1) is formed orthonormal basis of spase V_j, $j \in \mathbf{Z}$.

Consider scaling function Doubeshies $\varphi(x) \in L_2(\mathbf{R})$ of order $L \in \mathbf{N}$. Scaling function $\varphi(x)$ is continuous and have the next properties (see [4]):

$$\int_{\mathbf{R}} \varphi(x)\,dx = 1,$$

$$\varphi(x) = \sqrt{2}\sum_{k\in Z} h_k \varphi(2x-k), \quad x \in \mathbf{R},$$

$$\operatorname{supp}\varphi(x) = [0; 2L-1],$$

where h_k, $k \in \mathbf{Z}$ - filter of scaling function, $\operatorname{supp}\varphi(x)$ – support of scaling function.

Table 1. Example of Doubeshies scaling function (order $L = 2, L = 3$ filter's

$L = 2$	
h_0	0.48296291314453410
h_1	0.836516337378077
h_2	0.2241438680420134
h_3	-0.1294095225512603
$L = 3$	
h_0	0.3326705529500825
h_1	0.8068915093110924
h_2	0.4598775021184914
h_3	- 0.1350110200102546
h_4	- 0.0854412738820267
h_5	0.0352262918857095

On fig 1-2 is illustrated some examples of scaling function Doubeshies with compact support.

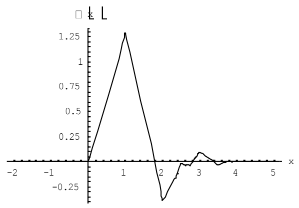

Fig.1 –Scaling function Doubeshies. $L = 3$,
$\max_{x\in\mathbf{R}} |\varphi(x)| = 1.28634$
$\operatorname{supp}\varphi(x) \subset [0,5]$

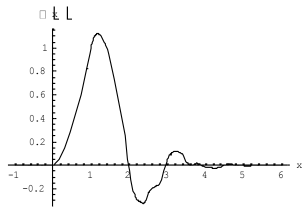

Fig.2 –Scaling function Doubeshies. $L = 4$,
$\max_{x\in\mathbf{R}} |\varphi(x)| = 1.12165$
$\operatorname{supp}\varphi(x) \subset [0,7]$

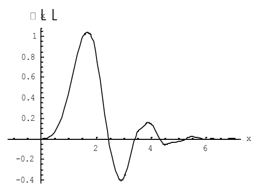

Fig.3 –Scaling function Doubeshies. $L = 6$.
$\max_{x\in\mathbf{R}} |\varphi(x)| = 1.0378$
$\operatorname{supp}\varphi(x) \subset [0,11]$

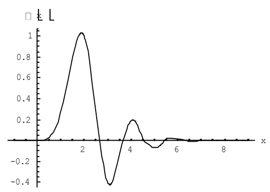

Fig.4 –Scaling function Doubeshies. $L = 7$.
$\max_{x\in\mathbf{R}} |\varphi(x)| = 1.0301$
$\operatorname{supp}\varphi(x) \subset [0,13]$

We can constructed $2\pi-$periodical scaling, using scaling function Doubeshies via formula:

$$\tilde{\varphi}_{j,k}(\lambda)=\frac{2^{\frac{J}{2}}}{\sqrt{2\pi}}\sum_{n=-1}^{(2L-1)2^{-j}+1}\sum_{n=-1}\varphi\left(\frac{2^j\lambda}{2\pi}+2^j n-k\right),\ \lambda\in\Pi .$$

Function (1) is formed orthonormal basis of spase $L_2(\Pi)$.

On Fig.5 illustrated some examples of $2\pi-$periodical scaling function.

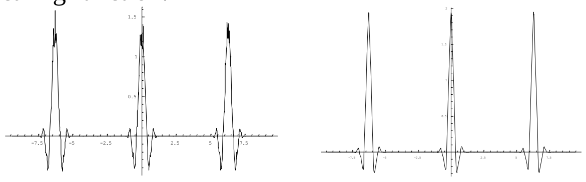

Fig. 5 – $2\pi-$periodical scaling function.

2. SPECTRAL ESTIMATE

Let $X(t)$, $t\in\mathbf{Z}$, be a wide-sense stationary stochastic process with $EX(t)=0$, $t\in\mathbf{Z}$, belonging to a set of random processes $\chi(\lambda, f,\alpha, L, C_2)$.

The set $\chi(\lambda, f,\alpha, L, C_2)$ is defined as the set of wide-sense stationary processes X(*t*), , $t\in\mathbf{Z}$, whose spectral density is $f(\lambda)$, $\lambda\in\Pi=[-\pi,\pi]$, having a fourth-order semiinvariant spectral density $f_4(\lambda_1,\lambda_2,\lambda_3)$, $\lambda_j\in\Pi$, $j=\overline{1,3}$, and such that for fixed $\lambda\in\Pi$ the spectral density f satisfies $f\in Lip_\alpha(L)$ and the fourth-order semi-invariant spectral density is bounded by a constant $C_2>0$.

The definition of the class $\chi(\lambda, f,\alpha, L, C_2)$ can be found in Zhurbenko [9]. I t contains processes with spectral densities whose peaks and troughs increase with T, for example $AR-$process with peaks.

For a process $X\in\chi(\lambda, f,\alpha, L, C_2)$, the rate of convergence of the mean-square deviation of a linear wavelet estimate of the spectral density is studied in [6].

The coefficients of the asymptotically dominant term, which depend on the smoothness of the spectral density, are calculated for some scaling functions and data tapering windows you can find in [5].

Thus the information on value α according to the aprioristic information on spectral density for investigated stochastic process. Such information, as a rule, undertakes on the basis of supervision over several realizations for the concrete phenomenon.

Our theoretical results in this paper are also used for developing computational algorithms for wavelet estimates of the spectral density. These algorithms enable us

1) to select a data tapering windows;
2) to choose a scaling function;
3) to compute the level of decomposition.
4) in order to construct the estimate minimizing the mean square deviation, depending on the sample length and the smoothness of the spectral density.

As spectral estimate $f(\lambda)$ let's consider statistics:

$$\hat{f}(\lambda)=\sum_{k=1}^{2^J}\hat{\alpha}_{J,k}\tilde{\varphi}_{J,k}(\lambda) \tag{1}$$

where

$$\hat{\alpha}_{J,k}=\int_\Pi I_T(\alpha)\tilde{\varphi}_{J,k}(\alpha)d\alpha, \tag{2}$$

– are wavelet-coefficients estimates in (1), and $I_T(\lambda)$ – modified periodogram:

$$I_T(\lambda)=\frac{1}{2\pi H_2^{(T)}(0)}d_T(\lambda)d_T(-\lambda) \tag{3}$$

$$d_T(\lambda)=\sum_{t=0}^{T-1}h_T(t)X(t)e^{-i\lambda t} \tag{4}$$

$$H_k^{(T)}(\lambda)=\sum_{t=0}^{T-1}\left(h_T(t)\right)^k e^{-i\lambda t} \tag{5}$$

$k\in\mathbf{N}$, $T\in\mathbf{N}$, and function $h_T(t)=h\left(\frac{t}{T}\right)$, $h:[0,1]\to\mathbf{R}$ - data tapers, it's behavior is studied sufficiently in [2, 6].

$$\tilde{\varphi}_{J,k}(\lambda)=\sum_{n\in Z}(2\pi)^{-1/2}\varphi_{J,k}\left((2\pi)^{-1}\lambda+n\right) \tag{6}$$

$2\pi-$periodical scaling function;

$$\varphi_{J,k}(x)=2^{J/2}\varphi\left(2^J x-k\right),$$

$J\in\mathbf{N_0}=\{0,1,2,..\}$, $\lambda\in\Pi$, $\varphi(x)$ - scaling function $x\in\mathbf{R}$; We have to mention, that the results cited in this article are obtained with condition of data tapers variation restriction.

In the spectral analysis we can use the next data taper:

Function $h(x)$, $x\in[-1,1]$, bounded variation, $h(x)=1$ in point $x=0$, and $h(x)=0$ for $|x|\geq 1$, and $h(x)=0$ for $x<0$.

Function $h(x)$, bounded variation, with $h(x)=1$ in point $x=\frac{1}{2}$, and $h(x)=0$ for $x\leq 0$ и $x\geq 1$.

Examples of data taper functions are more low resulted:

1. Hemming's window:

$$h(x)=0.54+0.46\cos(\pi x),\ x\in[-1,1].$$

2. A triangular window:

$$h(x)=1-|x|,\ x\in[-1,1].$$

2. Rice's, Bohner's, Parzen's window:

$$h(x)=1-x^2,\ x\in[-1,1].$$

A heuristic explanation is the following. Straightforward calculation gives for the expectation of the wavelet estimation:

$$E\hat{f}(\lambda)=\iint_{\Pi^2} f(\lambda+x+\gamma)\Phi_2^{(T)}(\gamma)K_J(\lambda,\lambda+x)dxd\gamma$$

where

$$K_J(\lambda,\alpha)=\sum_{k=1}^{2^J}\tilde{\varphi}_{J,k}(\lambda)\tilde{\varphi}_{J,k}(\alpha)$$

$$\Phi_2^{(T)}(x)=\frac{H_1^{(T)}(x)H_1^{(T)}(-x)}{2\pi H_2^{(T)}(0)}\ -$$

two kernel function. One is depended from scaling function, second from data taper.

In paper [6] is proved, that for first moment's estimate (1) we have:

$$E\hat{f}(\lambda)=f(\lambda)+O\left(\frac{1}{2^{J\alpha}}\right)+R_T,$$

where

$$R_T = \begin{cases} O\left(\frac{1}{T^\alpha}\right), \text{if } 0<\alpha<1, \\ O\left(\frac{\ln(\pi T)}{T}\right), \text{if } \alpha = 1. \end{cases}$$

In fact, one can prove, that

$$E\hat{f}(\lambda) = f(\lambda) + O(T^{-2})$$

for a sufficiently smooth data taper and for a scaling function higher order.

And for dispersion is valid:

$$D(\hat{f}(\lambda)) = \frac{2\pi H_4^{(T)}(0)}{\left(H_2^{(T)}(0)\right)^2} \sum_{k_1=1}^{2^J} \sum_{k_2=1}^{2^J} \tilde{\varphi}_{J,k_1}(\lambda)\tilde{\varphi}_{J,k_2}(\lambda) \times$$

$$\int_\Pi \tilde{\varphi}_{J,k_1}(\alpha)\left(\tilde{\varphi}_{J,k_2}(-\alpha) + \tilde{\varphi}_{J,k_2}(\alpha)\right) f^2(\alpha)d\alpha$$

$$+ O\left(\frac{1}{T}\right) + O\left(\frac{2^J \ln^5(\pi T)}{T^{1+\alpha}}\right) +$$

$$O\left(\frac{2^{J(\mu+1)}\ln^5(\pi T)}{T^{1+\mu}}\right) + O\left(\frac{2^{2J}\ln^3(\pi T)}{T^2}\right).$$

Wavelet estimate $\hat{f}(\lambda)$, define by formula (1), is consistent in mean-square sense estimate of spectral density $f(\lambda)$, $\lambda \in \Pi$ for all $2^J \le RT^{1-\rho}$, where $0<\rho<1$, $0<R<\infty$, is some fixed constant.

3. CALCULATION

Step 1. Chooze data taper $h_T(t)$. Data taper $h_T(t) = h\left(\frac{t}{T}\right)$ we can find from condition of minimization of value:

$$\Delta = \int_{-\pi}^{\pi} |x|^\alpha \left|\Phi_2^{(T)}(x)\right| dx + \frac{2}{T}\sum_{k=0}^{\left[\frac{T}{2}\right]-1}\left(\left|\frac{H_2^{(T)}\left(\frac{2\pi k}{T}\right)}{H_2^{(T)}(0)}\right|\right)^2 \to \min_H$$

where H - some bounded set of data taper $h(x), x \in \mathbf{R}$.

The integral

$$\int_{-\pi}^{\pi} |x|^\alpha \left|\Phi_2^{(T)}(x)\right| dx$$

and the sum

$$\frac{2}{T}\sum_{k=0}^{\left[\frac{T}{2}\right]-1}\left(\left|\frac{H_2^{(T)}\left(\frac{2\pi k}{T}\right)}{H_2^{(T)}(0)}\right|\right)^2$$

we can calculate, using standard numerical methods;

Data taper, choosing via this method is optimal in sense of minimum mean-square deviation of modified periodogram.

Thus, the advantages of data tapers could also be established theoretically. An important problem is the choice of data taper. No rigorous results exist for this problem. No rigorous results exists for this problem. It is obvious, that the choice depends on the true (unknown) spectral density, in particular on the relation of the peaks and troughs to each other.

Step 2. Calculate modified periodogram $I_T(\lambda)$, via formulas (3) and (4), using data taper $h(x)$, $x \in [-1,1]$, which we find on step 1.

Step 3. Choose scaling function:

$$\left(\frac{(2\pi)^\alpha 2L \max_\varphi |\varphi(x)|}{T^\alpha}\left[\int_{\mathbf{R}} |z|^\alpha |\varphi(z)| dz\right]\right)^2 \to \min_\Phi$$

where $\Phi = \{\varphi_L(x), x \in \mathbf{R}, L \in \mathbf{N}\}$ - the set of Doubeshies scaling function.

On choose of scaling function two characteristics is influence: maximum of scaling function and it's support.

Step 4. Calculate level J

$$J = [\log_2(T)],$$

where $[\bullet]-$ the whole part of number.

Step 5. For calculation of initial factors we will put $J_0 = T$.

Step 6. Coefficients $\hat{\alpha}_{J_0,k}$ calculate, using formula of left rectangle (see step 6.1), or using quadrature formula (see step 6.2).

Step 6.1. Calculate coefficients via formula of left rectangle:

$$\hat{\alpha}_{J_0,k} = \sqrt{2^{-J_0+1}\pi}\,\frac{(2L-1)}{N_1}\times$$

$$\times \sum_{l=0}^{N_1-1} I_T\left(2^{-J_0+1}\pi\left(k + \frac{2L-1}{N_1}l\right)\right)\varphi\left(\frac{2L-1}{N_1}l\right),$$

where $N_1 -$ quantity of pats for integration interval ($N_1 \le T$), $k = \overline{0, 2^{J_0}-1}$.

Step 6.2. Calculate coefficients via quadrature formulas

$$\hat{\alpha}_{J_0,k} \approx 2^{\frac{-J_0+1}{2}}\sqrt{\pi}\sum_{k=1}^{r}\omega_k I_T\left(\frac{(x_k+k)}{2^{J_0-1}}\right),$$

$k = \overline{0, 2^{J_0}-1}$; where method of calculate of abscissas x_k $k = \overline{1,r}$ and weight ω_k, $k = \overline{1,r}$ is considered in paper [7].

Table 2. Abscissas x_k, $k = \overline{1,r}$ and weight ω_k, $k = \overline{1,r}$, of quadrature formulas for Doubeshies system of order $L = 2,3$.

	k	x_k	ω_k
$L=2$ $r=3$	1	0.565179	0.899173
	2	1.565179	0.132858
	3	2.565179	-0.032031
$L=2$ $r=4$	1	0.247825	0.268749
	2	0.747825	0.561228
	3	1.247825	0.298997
	4	1.747825	-0.128975
$L=2$ $r=5$	1	0.253425	0.276273
	2	0.753425	0.557197
	3	1.253425	0.296560
	4	1.753425	-0.130903
	5	2.253425	0.000872

	k	x_k	ω_k
$L=3$ $r=3$	1	0.804695	0.990491
	2	2.804695	0.012666
	3	4.804695	-0.003156
$L=3$ $r=4$	1	0.701350	0.817228
	2	1.701350	0.264924
	3	2.701350	-0.097581
	4	3.701350	0.015430
$L=3$	1	0.661075	0.747720

$r=5$	2	1.661075	0.384507
	3	2.661075	-0.174764
	4	3.661075	0.048801
	5	4.661075	-0.006264

Step 7. Calculate coefficient $\hat{\alpha}_{J_0,k},\ldots,\hat{\alpha}_{J-1,k}$, using modified formulas

$$\hat{\alpha}_{j,k}=\frac{1}{L}\sum_{n=0}^{2L-1}h_n\hat{\alpha}_{j+1,(n+2k)\bmod 2^{j+1}},$$

where h_k, $k\in\mathbf{Z}$ - filters of Doubeshies scaling function.

Step 8. Build wavelet estimate $\hat{f}(\lambda)$, using formula

$$\hat{f}(\lambda)=\sum_{k=1}^{2^J}\hat{\alpha}_{J,k}\tilde{\varphi}_{J,k}(\lambda)$$

on level J.

Estimate $\hat{f}(\lambda)$, constructed via considered algorithm is optimal in minimum biases square.

We remark that exist other spectral estimates that have similar resolution properties as the tapered periodogram. Those statistics are usually non linear / non quadratic and therefore very difficult to investigate theoretically.

4. USING WAVELET ESTIMATE IN PARAMETRIC MODEL

An alternative to nonparametric spectral density estimation is the fitting of a parametric spectral density As an example we now discus the fitting of an $AR-$ model [5].

Let's consider an autoregressive process of *p*-th order $p\in\mathrm{N}=\{1,2,\ldots\}$:

$$\sum_{j=0}^{p}a_jX_{t-j}=\varepsilon_t,\ (a_0=1),$$

where ε_t is a sequence of independent, identically distributed random variates, $t\in\mathbf{Z}=\{0,\pm1,\pm2,\ldots\}$. Further on we'll assume that $\varepsilon_t\sim N(0,\sigma^2)$.

Theoretical spectral density of the process AR(p) looks like:

$$f(\lambda)=\frac{\sigma^2}{2\pi}\left|1-\sum_{j=1}^{p}a_je^{-i\cdot j\lambda}\right|^{-2} \qquad (7)$$

Taking in consideration [3] we can rewrite:

$$\sum_{j=0}^{p}a_jz^j=\prod_{j=1}^{p}(1-u_jz),\ u_j\in\mathbf{C}.$$

If $u_j=\theta_je^{-i\mu_j}$, then at $\theta_j\to1$ spectral density $f(\lambda)$, $\lambda\in\Pi$ of the random process has peaks on a frequency $\mu_j\in\Pi$, $j=\overline{1,p}$, $p\in\mathrm{N}$ and is defined by the following ratio:

$$f(\lambda)=\frac{\sigma^2}{2\pi}\prod_{j=1}^{p}\left|1-\theta_je^{i(\lambda-\mu_j)}\right|^{-2}.$$

One possible approach is to minimize a distance between the theoretical parametric density and the periodogram with respect to the parameters. A natural distance function comes from considering the asymptotic Kullback-Leibler information divergence. It is possible to prove that the asymptotic information divergence of a process with true spectral density $f(\lambda)$ and a Gaussian model with spectral density $f_\theta(\lambda)$ is

$$L(\theta)=\frac{1}{4\pi}\int_\Pi\left(\ln f_\theta(\lambda)+\frac{f(\lambda)}{f_\theta(\lambda)}\right)d\lambda+const.$$

Since $f(\lambda)$ is unknown we use the wavelet estimation instead and an empirical distance

$$L_T(\theta)=\frac{1}{4\pi}\int_\Pi\left(\ln f_\theta(\lambda)+\frac{\hat{f}_T(\lambda)}{f_\theta(\lambda)}\right)d\lambda.$$

Minimizing $L_T(\theta)$ with respect to θ gives the estimate $\hat{\theta}_T$. For AR process $f_\theta(\lambda)$ is given by (7), $\hat{f}_T(\lambda)$ is given by (1) and calculate via algorithm from part 3 from this paper.

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