Journal of Engineering Physics and Thermophysics, Vol. 74, No. 2, 2001

SOLUTION OF NONLINEAR TWO-DIMENSIONAL DIFFERENTIAL EQUATIONS OF TRANSFER WITH DISCONTINUITY BOUNDARY CONDITIONS ON THE SURFACE OF AN ISOTROPIC SEMIINFINITE BODY IN ITS HEATING THROUGH A CIRCLE OF KNOWN RADIUS

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An analytical solution of the heat-conduction problem is obtained by the method of linearization of a nonlinear equation of transfer and combined application of integral transforms to the linearized problem for an isotropic half-space heated through a circular region $0 < r < R$ *on its surface* $z = 0$.

As is known, in high-intensity heat-transfer processes, e.g., in thermal explosion ("shock"), chemical reactions, etc., temperature undergoes substantial changes in small time intervals. In describing transport phenomena occurring within a wide range of temperature variation, one must take into account the dependence of the coefficients of transfer on temperature. Under these conditions, the flow of thermal energy becomes nonlinear and to determine the temperature field, one must solve the nonlinear differential equation of transfer

$$c\,(T)\,\gamma\,(T)\,\frac{\partial T}{\partial \tau} = \mathrm{div}\,(\lambda\,(T)\,\mathrm{grad}\,(T))\,, \qquad (1)$$

where $\lambda(T)$, $c(T)$, and $\gamma(T)$ are, respectively, the coefficients of thermal conductivity, specific heat, and density of the considered isotropic body that depend only on the temperature $T = T(r, z, \tau)$ and do not depend on the coordinates (r, z are the cylindrical coordinates; τ is the time).

We have a semiinfinite isotropic body whose initial temperature is $T(r, z, 0) = T_0 = \text{const}$. At the initial instant of time, a part of the surface $z = 0$ is heated by a variable heat flux $q(r, z)$ through a circle of known radius $0 < r < R$. The remaining part of the surface ($z = 0$, $R < r < \infty$) is heat-insulated, i.e., the equality $T_z(r, 0, \tau) = 0$ ($R < r < \infty$, $z = 0$, $\tau > 0$) holds. It is necessary to find the temperature field $T(r, z, \tau)$ for $z > 0$, $r \geq 0$, and $\tau > 0$.

Thus, using the function of excess temperature $\theta = \theta(r, z, \tau) = T(r, z, \tau) - T_0 = T - T_0$, we reduce the problem posed to the necessity of solving the nonlinear equation (1), which in the cylindrical coordinates r, z has the form

$$c\,(\theta + T_0)\,\gamma\,(\theta + T_0)\,\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial r}\left(\lambda\,(\theta + T_0)\,\frac{\partial \theta}{\partial r}\right) + \frac{1}{r}\,\lambda\,(\theta + T_0)\,\frac{\partial \theta}{\partial r} + \frac{\partial}{\partial z}\left(\lambda\,(\theta + T_0)\,\frac{\partial \theta}{\partial z}\right), \quad r, z, \tau > 0\,, \qquad (2)$$

with the homogeneous initial condition

$$\theta\,(r, z, 0) = 0\,, \qquad (3)$$

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1062-0125/01/7402-0471$25.00 ©2001 Plenum Publishing Corporation

the unmixed discontinuity boundary condition for $z = 0$

$$-\frac{\partial\theta(r,0,\tau)}{\partial z} = \begin{cases} \dfrac{q(r,\tau)}{\lambda(\theta+T_0)}, & 0 < r < R, \quad \tau > 0, \quad \lambda(\theta+T_0) > 0, \\ 0, & R < r < \infty, \quad \tau > 0, \end{cases} \tag{4}$$

and the boundary conditions for $z = \infty$, $r = \infty$, and $r = 0$

$$\frac{\partial\theta(r,\infty,\tau)}{\partial z} = \frac{\partial\theta(\infty,z,\tau)}{\partial r} = \frac{\partial\theta(0,z,\tau)}{\partial r} = 0 . \tag{5}$$

We linearize Eq. (2) by introducing a new integral function [1, 2]

$$U = U(T) = U(\theta) = \frac{1}{\lambda_0}\int_{T_0}^{\theta+T_0}\lambda(T')\,dT' = \frac{1}{\lambda_0}\int_0^{\theta}\lambda(\theta'+T_0)\,d\theta', \tag{6}$$

where λ_0 is the coefficient of thermal conductivity at $T = T_0$ ($\theta = 0$).

The introduced function $U(\theta)$ is conceptually a potential whose gradient is proportional to the heat flux [2].

From relation (6) it follows that

$$\frac{\partial U}{\partial\tau} = \frac{\lambda(\theta+T_0)}{\lambda_0}\frac{\partial\theta}{\partial\tau}; \quad \frac{\partial U}{\partial r} = \frac{\lambda(\theta+T_0)}{\lambda_0}\frac{\partial\theta}{\partial r}; \quad \frac{\partial U}{\partial z} = \frac{\lambda(\theta+T_0)}{\lambda_0}\frac{\partial\theta}{\partial z}, \tag{7}$$

and Eq. (2) takes the form

$$\frac{1}{a(T)}\frac{\partial U}{\partial\tau} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}, \tag{8}$$

where

$$a(T) = \frac{\lambda(T)}{c(T)\gamma(T)}; \quad c(T)\gamma(T) \neq 0 . \tag{9}$$

Thus, when the new variable U (relation (6)) is used the form of the heat-conduction equation is retained for the linear case, but the coefficient of thermal diffusivity a depends on temperature. Here, in most cases, the variation in the coefficient of thermal diffusivity a as a function of the temperature variation is less important than the similar variation in $\lambda(T)$. Consequently, it may approximately be assumed that $a(T) = a_0$ = const ($a_0 > 0$ is the thermal diffusivity at $T = T_0$), since, for example, for metals at temperatures close to absolute zero $\lambda(T)$ and $c(T)$ are proportional to an absolute temperature [2].

After linearization of the nonlinear equation (2) in the particular case $a(T) = a_0$ the mathematical formulation of the problem is written as

$$\frac{1}{a_0}\frac{\partial U}{\partial\tau} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}, \quad r, z, \tau > 0, \tag{10}$$

with the initial condition ($\tau = 0$)

$$U = 0 \tag{11}$$

and the boundary conditions

$$-\frac{\partial U}{\partial z} = \begin{cases} \dfrac{q(r,\tau)}{\lambda_0}, & 0 < r < R, \\ 0, & R < r < \infty, \end{cases} \quad \text{for} \quad z = 0; \tag{12}$$

$$\frac{\partial U}{\partial z} = 0 \quad \text{for} \quad z = \infty; \quad \frac{\partial U}{\partial r} = 0 \quad \text{for} \quad r = \infty; \quad \frac{\partial U}{\partial r} = 0 \quad \text{for} \quad r = 0. \tag{13}$$

We use the Laplace transform

$$\overline{U} = L[U] = \int_0^\infty \exp(-s\tau)\, U\, d\tau = \int_0^\infty \exp(-s\tau)\, d\tau \int_{T_0}^{T(r,z,\tau)} \frac{\lambda(T')}{\lambda_0} dT', \quad \operatorname{Re} s > 0, \tag{14}$$

with account for which Eq. (10) takes the form

$$\frac{\partial^2 \overline{U}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{U}}{\partial r} + \frac{\partial^2 \overline{U}}{\partial z^2} - \frac{s}{a_0}\overline{U} = 0, \tag{15}$$

and the boundary conditions are

$$-\frac{\partial \overline{U}}{\partial z} = \begin{cases} \dfrac{\overline{q}(r,s)}{\lambda_0}, & 0 < r < R, \quad \operatorname{Re} s > 0, \\ 0, & R < r < \infty, \end{cases} \quad \text{for} \quad z = 0; \tag{16}$$

$$\frac{\partial \overline{U}}{\partial z} = 0 \quad \text{for} \quad z = \infty; \quad \frac{\partial \overline{U}}{\partial r} = 0 \quad \text{for} \quad r = \infty; \quad \frac{\partial \overline{U}}{\partial r} = 0 \quad \text{for} \quad r = 0, \tag{17}$$

where

$$\overline{q}(r,s) = L[q(r,\tau)] = \int_0^\infty \exp(-s\tau)\, q(r,\tau)\, d\tau, \quad \operatorname{Re} s > 0. \tag{18}$$

We note that the conditions of the existence of integral (18) can be found in [1] and we then assume that these conditions are satisfied.

Next, applying the integral Hankel transform to (15) [1]

$$\overline{U}_{\mathrm{H}} = \int_0^\infty \overline{U}\, J_0(pr)\, r dr, \tag{19}$$

where $J_0(pr)$ is the first-kind Bessel function of zero order, and allowing for the boundary conditions (17), we obtain the equation

$$\frac{d^2\overline{U}_{\text{H}}}{dz^2} - \left(p^2 + \frac{s}{a_0}\right)\overline{U}_{\text{H}} = 0\,, \quad \text{Re}\, s > 0\,. \tag{20}$$

In the case $\partial\overline{U}_{\text{H}}/\partial z = 0$ for $z = \infty$ the solution of Eq. (20) has the form

$$\overline{U}_{\text{H}} = \overline{A}_1\,(p, s)\,\exp\left(-z\sqrt{p^2 + \frac{s}{a_0}}\right), \quad \text{Re}\, s > 0\,. \tag{21}$$

Applying the inversion formula to the Hankel transform, we obtain the solution for the function $\overline{U} = L[U]$:

$$\overline{U} = \int_0^{\infty} \overline{A}_1\,(p, s)\,\exp\left(-z\sqrt{p^2 + \frac{s}{a_0}}\right) J_0\,(pr)\,pdp\,, \quad \text{Re}\, s > 0\,. \tag{22}$$

We find the value of $\overline{A}_1(p, s)$ from condition (16) for $z = 0$:

$$\overline{A}_1\,(p, s) = \frac{1}{\lambda_0\sqrt{p^2 + \frac{s}{a_0}}}\int_0^R J_0\,(px)\;\overline{q}\,(x, s)\,xdx\,, \;\; \text{Re}\, s > 0\,, \tag{23}$$

i.e., the solution has the form

$$\overline{U} = \int_0^{\infty} \exp\left(-z\sqrt{p^2 + \frac{s}{a_0}}\right)\frac{J_0\,(pr)\,pdp}{\sqrt{p^2 + \frac{s}{a_0}}}\int_0^R \overline{q}\,(x, s)\,J_0\,(px)\,xdx\,, \quad \text{Re}\, s > 0\,. \tag{24}$$

Using the inversion formula of the Laplace integral [1], we represent the inverse transform $U = U(T(r, z, \tau))$ as

$$U = \frac{\sqrt{a_0}}{\lambda_0\sqrt{\pi}}\int_0^{\infty} J_0\,(pr)\,pdp\int_0^R J_0\,(px)\,xdx\int_0^{\tau}\frac{q\,(x, \tau - \xi)}{\sqrt{\xi}}\exp\left(-a_0p^2\xi - \frac{z^2}{4a_0\xi}\right)d\xi\,. \tag{25}$$

Allowing for relation (6), we obtain the integral equation for determining $\lambda(T)$ within the temperature range (T_0, T) for the known values of $U(T(r, z, \tau))$ or the equation for determining the temperature field $T(r, z, \tau) - T_0 = \theta(r, z, \tau)$ for the known dependence $\lambda(T)$ within the same temperature range (T_0, T):

$$\int_{T_0}^{T(r,z,\tau)}\frac{\lambda\,(T')}{\lambda_0}\,dT' = \frac{\sqrt{a_0}}{\lambda_0\sqrt{\pi}}\int_0^{\infty} J_0\,(pr)\,pdp\int_0^R J_0\,(px)\,xdx\int_0^{\tau}\frac{q\,(x, \tau - \xi)}{\sqrt{\xi}}\exp\left(-a_0p^2\xi - \frac{z^2}{4a_0\xi}\right)d\xi\,. \tag{26}$$

Relying on formula (26), we consider, for example, the primal problem of determination of the temperature field $\theta(r, z, \tau) = T(r, z, \tau) - T_0$ in the particular case of variation in $\lambda(T)$ according to the law

$$\frac{\lambda(T)}{\lambda_0} = 1 + \beta T \,, \tag{27}$$

where $\beta\lambda_0 = \tan\varphi$ is the angular coefficient of the straight line $\lambda(T) = \lambda_0 + \beta\lambda_0 T$, $0 < \varphi < 2\pi$.

Substituting (27) into the left-hand side of Eq. (26), we obtain the quadratic equation

$$\frac{\beta}{2} T^2 + T - \left(T_0 + \frac{\beta}{2} T_0^2 + U \right) = 0 \,, \quad \beta \neq 0 \,, \tag{28}$$

where U is determined by formula (25).

Solution of Eq. (28) is trivial and has the form

$$T_{1,2}(r, z, \tau) = \frac{1}{\beta}\left(-1 \pm \sqrt{1 + 2\beta\left(T_0 + \frac{\beta}{2} T_0^2 + U \right)} \right). \tag{29}$$

The value of the roots of $T_{1,2}(r, z, \tau)$ is selected from physical considerations and depends on the initial temperature $T_0 > 0$ or $T_0 < 0$ and also on positive or negative values of β. In the case $\beta = 0$, $\lambda(T') = \lambda_0$ holds and the solution for $U = \theta(r, z, \tau) = T(r, z, \tau) - T_0$ is determined by formula (25).

We now consider the case of assigning mixed boundary conditions of the form

$$-\frac{\partial\theta(r, 0, \tau)}{\partial z} = \frac{q(r, \tau)}{\lambda(\theta + T_0)} \,, \quad 0 < r < R \,, \quad \lambda(\theta + T_0) > 0 \,,$$

$$\theta(r, 0, \tau) = 0 \,, \quad R < r < \infty \,. \tag{30}$$

In this case, solution of the nonlinear equation (2) is also connected with its initial linearization using relation (6); however, determination of the function U from Eq. (10) necessitates solution of the so-called paired integral equations with the L-parameter of the form (see, e.g., [3–6]):

$$\int_0^\infty \overline{C}^*(p, s) \sqrt{p^2 + \frac{s}{a_0}} \, J_0(pr)\, dp = \frac{\overline{q}(r, s)}{\lambda_0} \,, \quad 0 < r < R \,, \quad \mathrm{Re}\, s > 0 \,,$$

$$\int_0^\infty \overline{C}^*(p, s)\, J_0(pr)\, dp = 0 \,, \quad R < r < \infty \,, \quad \mathrm{Re}\, s > 0 \,. \tag{31}$$

The method of solution of the paired equations (31), i.e., the method of determination of the function $\overline{C}^*(p, s)$ is considered in [3–6] in sufficient detail. At the same time, the solution of Eq. (15) for the function $\overline{U}$ is determined by a formula similar to (22):

$$\overline{U} = \int_0^\infty \overline{C}^*(p, s) \exp\left(-z \sqrt{p^2 + \frac{s}{a_0}} \right) J_0(pr)\, dp \,, \quad \mathrm{Re}\, s > 0 \,, \tag{32}$$

with the value of $\overline{C}^*(p, s)$ being specified by the relation

$$\bar{C}^*(p, s) = \frac{p}{\sqrt{p^2 + \frac{s}{a_0}}} \int_0^R \bar{\varphi}^*(t, s) \sin\left(t\sqrt{p^2 + \frac{s}{a_0}}\right) dt\,, \quad \mathrm{Re}\, s > 0\,, \tag{33}$$

where the analytical function $\bar{\varphi}^*(t, s)$ satisfies the equation

$$\int_0^r \frac{t\bar{\varphi}^*(t, s)}{\sqrt{r^2 - t^2}} \exp\left(-\sqrt{\frac{s}{a_0}(r^2 - t^2)}\right) dt + \int_0^R \bar{\varphi}^*(t, s) \sin\left(t\sqrt{\frac{s}{a_0}}\right) dt -$$

$$- \int_r^R \frac{t\bar{\varphi}^*(t, s)}{\sqrt{t^2 - r^2}} \sin\left(\sqrt{\frac{s}{a_0}(t^2 - r^2)}\right) dt = \frac{1}{\lambda_0} \int_0^r \bar{q}(\rho, s)\, \rho d\rho\,, \quad 0 < r < R\,, \quad \mathrm{Re}\, s > 0\,. \tag{34}$$

We note that the methods of determination of the function $\bar{\varphi}^*(t, s)$ are described in [5].

Hereafter the determination of the temperature field $T(r, z, \tau)$ for this problem with mixed boundary conditions virtually does not differ from the case of the nonlinear Neumann problem considered above. Finding the inverse transform $U = L^{-1}[\bar{U}]$ is a rather labor-consuming task, while the inverse transform $T(r, z, \tau)$ with the mixed boundary conditions (30) can be determined using formula (6) for the known dependence $\lambda(T)$.

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