THEORY OF SHELLS AND THEORY OF CURVILINEAR RODS: A COMPARATIVE ANALYSIS

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Preliminary remarks. Now conferences on the theory of plates and shells are held regularly, articles in periodicals are constantly published, monographs and tutorial books are written. It is not the case with the theory of rods. The question arises: what is the reason for the lack of interest in the theory of rods? It would seem that the theory rods and the theory of shells have much in common, because they are theories of thin-walled structures. They are both moment theories, the basic equations are derived by the same methods, and when solving problems by numerical methods the same difficulties associated with the presence of small parameters arise. One can assume that a reason for the lack of active research in the area of rod theory is that the theory of rods is mathematically much simpler the theory of shells and all problems in theory of rods have long been solved? This is not the case. Further we give a few examples.

Curvilinear rods and shells. There is a difference between the curvilinear and rectilinear rods, exactly the same as that between the shells and plates. The mass centers of cross sections of rectilinear rods are on the supporting curve, as well as in the case of plate the mass centers are on the middle surface. In the case of curvilinear rods and shells the mass centers are offset towards the convexity. Therefore, the expressions for internal energy contain cross terms that are responsible for relationship of translational and angular deformities [1]. For example, in the case of physically linear theory of rods the internal energy density has the form:

\[
\rho_0 U = \frac{1}{2} \varepsilon \cdot P \cdot A \cdot P^T \cdot \varepsilon + \varepsilon \cdot P \cdot B \cdot P^T \cdot \varphi + \frac{1}{2} \varphi \cdot P \cdot C \cdot P^T \cdot \varphi
\]

where \( \varepsilon \) and \( \varphi \) are the strain vectors, \( P \) is the rotation tensor, \( A, B, C \) are the stiffness tensors in the reference configuration. According to theory of symmetry, the stiffness tensors of a rod without a natural twist, which is made of a homogeneous isotropic material, have the following structure:

\[
A = A_1 d_1 d_1 + A_2 d_2 d_2 + A_3 t t, \quad C = C_1 d_1 d_1 + C_2 d_2 d_2 + C_3 t t,
\]

\[
B = \frac{1}{R_c} (B_1 d_1 d_1 + B_2 d_2 d_2 + B_3 t t) + \frac{1}{R_t} \left[ (B_{23} d_2 t + B_{32} d_2 t) \cos \alpha + (B_{13} d_1 t + B_{31} d_1 t) \sin \alpha \right].
\]

Here \( t \) is the unit vector directed along the tangent to supporting curve, \( d_1, d_2 \) are the unit vectors directed along the principal axes of inertia of a cross section of the rod, \( R_c \) and \( R_t \) are radii of curvature and torsion, respectively, the angle \( \alpha \) determines the rotation of vectors \( d_1, d_2 \) with respect to vectors of the natural basis. It is easy to see that in the case of a rectilinear rod tensor \( B \) vanishes. In the theory of
shells the situation is similar. In the shell theory the values of all elastic moduli are known not only for shells made of a homogeneous isotropic material, but also for laminated shells. In the rod theory only the values of elastic moduli $A_i$ and $C_i$ are known. The values of coefficients of tensor $B$ are not known, and their determination is an important problem, because in many cases ignoring the tensor $B$ leads to paradoxical results. Now we give two problems to illustrate what has been said. As the first example, consider a rod that is a half circle in the non-deformed state. One end of the rod is hinged. Is it possible to bring the rod into linear state by applying a force on the other end of the rod? The solution of the problem without taking into account the tensor $B$ leads to a negative answer, but for practical reasons it is obvious that the answer should be positive. As the second example, consider a rod that is a cylindrical helical spring in the undeformed state. One end of the rod is fixed. Is it possible to straighten the rod by putting force on the other end? The solution of the problem without taking into account the tensor $B$ compels us to conclude that it is impossible. It is hard to believe in the correctness of this conclusion.

**Naturally twisted rods.** Another by-way of the rod theory is rectilinear rods with a natural twist, an example of which is the drill. From a physical point of view in the theory of shells there is nothing similar to a natural twist of the rods. However, from a mathematical viewpoint a natural twist leads to the same effects as the curvature because the tensor $B$ of naturally twisted rod is not equal to zero. In accordance with theory of symmetry in the case of rectilinear naturally twisted rod tensor $B$ has the form:

$$B = \varphi'(B_{01}d_1d_1 + B_{02}d_2d_2 + B_{03}tt),$$

where $\varphi$ is the twist angle of the rod. The characteristic property of naturally twisted rods is the mutual influence of torsional deformation and stretching. The problem of deformation of a naturally twisted rod by the distributed force which creates only tangential force and the moment applied on the end of the rod is the illustration of the aforesaid fact. If the natural twist is ignored then the force affects only tension and the moment affects only bending and torsion. If the natural twist is taken into account then the solution is characterized in that both force and torque affect stretching and torsion of the rod. A major obstacle to development of a theory of natural twisted rods is the lack of reliable information about the coefficients of the tensor $B$. The formula $B_{03} = E(J_p - J_r)$, where $E$ is Young’s modulus, $J_p$ is the polar moment of inertia of rod, $J_r$ is the geometric stiffness in torsion, is offered in [2]. The values of $B_{01}$ and $B_{02}$ are unknown. In the case of curvilinear naturally twisted rods the tensor $B$ contains several modules of elasticity the values of which are also unknown.

**Thin-walled rods and shells.** The problems of deformation of thin-walled rods are closest to the theory of shells. In general, the theory of rods is mathematically much simpler than the theory of shells. However, every rule has exceptions. For example, consider the problem of the free vibrations of a cylindrical spiral shell. If solving this problem by means of the shell theory we neglect the tension and shear deformation, that in this case is justified, we obtain a fairly simple differential equation in terms of displacement normal to the surface of the shell. The possibility
of such a simplified formulation of problem in the framework of rod theory is far from obvious. This raises question of the possibility to construct a general theory of thin-walled rods that is based rather on the comparison with the shell theory than on the comparison with the three-dimensional theory.

**Conclusion.** At present the theory of rods is not only applied engineering sciences. There are many unsolved theoretical problems in the rod theory. To solve these problems we can use the methods and approaches that are well developed in the shell theory.

**References**

**FREE INTERFACIAL VIBRATIONS OF THIN ELASTIC INFINITE CYLINDRICAL SHELLS**

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**Abstract.** Free interfacial vibrations in closed and non-closed infinite circular cylindrical shells composed of two semi-infinite orthotropic shells with different elastic properties are studied. Conditions of full contact are imposed at the interface of separation of materials. In the case of non-closed cylindrical shells hinge-mounted conditions at the edge generators are set. The choice of the coordinate system and the form of shell are shown in Figures 1 and 2. Here all the values, corresponding to the right shell are noted by superscript (1) and to the left shell by superscript (2).

**Figure 1**

**Figure 2**

Dispersion equations for finding the natural frequencies of interfacial vibrations of composed cylindrical shells are obtained using the system of equations corresponding to the classical theory of orthotropic cylindrical shells [1].

Asymptotic links are established between the dispersion equations of problems in hand and analogous problems for infinite composed plate and plate-strip

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\[ \text{Det} \left\| p_{ij} \right\|_{i,j=1}^{8} = \frac{B_{16}^{(1)}}{B_{11}^{(1)}} \frac{N^{(1)}(\eta_m^{(1)}) N^{(2)}(\eta_m^{(2)})}{N^{(r)}(\eta_m^{(r)})} \times \]
\[ \times \left( K_3^{(1)}(\eta_m^{(1)}) K_3^{(2)}(\eta_m^{(2)}) G(\eta_m^{(1)}, \eta_m^{(2)}) L(\eta_m^{(1)}, \eta_m^{(2)}) + O\left(r_0/m^2\right) \right) = 0, \ m = 1, \infty. \]

Here we have defined
\[ N^{(r)}(\eta_m^{(r)}) = \left(y_1^{(r)} + y_3^{(r)}\right) \left(y_1^{(r)} + y_4^{(r)}\right) \left(y_2^{(r)} + y_3^{(r)}\right) \left(y_2^{(r)} + y_4^{(r)}\right), \]
\[ G(\eta_m^{(1)}, \eta_m^{(2)}) = K_1^{(1)}(\eta_m^{(1)}) Q^{(2)}(\eta_m^{(2)}) + \left( \frac{B_{11}^{(2)}}{B_{11}^{(1)}} \right)^2 K_1^{(2)}(\eta_m^{(2)}) + \]
\[ \frac{B_{11}^{(2)}}{B_{11}^{(1)}} \left\{ 2 \left( y_3^{(1)} y_4^{(1)} + \frac{B_{12}^{(1)}}{B_{11}^{(1)}} y_3^{(2)} y_3^{(1)} + \frac{B_{12}^{(1)}}{B_{11}^{(1)}} y_3^{(2)} y_4^{(1)} \right) \right\} + \left( y_3^{(1)} y_4^{(1)} + \frac{B_{12}^{(1)}}{B_{11}^{(1)}} y_3^{(2)} y_4^{(1)} \right) \left( y_3^{(1)} y_4^{(1)} + \frac{B_{12}^{(1)}}{B_{11}^{(1)}} y_3^{(2)} y_4^{(1)} \right) , \]
\[ K_1^{(r)}(\eta_m^{(r)}) = \left( y_3^{(r)} \right)^2 \left( y_4^{(r)} \right)^2 + 4 \frac{B_{66}^{(r)}}{B_{11}^{(r)}} (y_3^{(r)} y_4^{(r)}) - \left( \frac{B_{12}^{(r)}}{B_{11}^{(r)}} \right)^2, \]
\[ L(\eta_m^{(1)}, \eta_m^{(2)}) = K_2^{(1)}(\eta_m^{(1)}) Q^{(2)}(\eta_m^{(2)}) + \left( \frac{B_{66}^{(2)}}{B_{66}^{(1)}} \right)^2 K_2^{(2)}(\eta_m^{(2)}) Q^{(1)}(\eta_m^{(1)}) + \]
\[ + \frac{B_{66}^{(2)}}{B_{66}^{(1)}} \left\{ 2 \left( y_1^{(1)} y_2^{(1)} - \frac{B_{12}^{(1)}}{B_{11}^{(1)}} \right) \left( 1 - \left( \eta_m^{(1)} \right)^2 \right) \right\} + \left( y_1^{(1)} y_2^{(1)} - \frac{B_{12}^{(1)}}{B_{11}^{(1)}} \right) \left( 1 - \left( \eta_m^{(1)} \right)^2 \right) , \]
\[ K_2^{(r)}(\eta_m^{(r)}) = \left( 1 - \left( \eta_m^{(r)} \right)^2 \right) \left( \frac{B_{12}^{(r)} B_{22}^{(r)} - B_{12}^{(r)} B_{66}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} - \left( \eta_m^{(r)} \right)^2 \right) - \left( \eta_m^{(r)} \right)^2 \left( y_1^{(r)} y_2^{(r)} \right), \]
\[ Q^{(r)}(\eta_m^{(r)}) = y_1^{(r)} y_2^{(r)} + \frac{B_{66}^{(r)}}{B_{11}^{(r)}} \left( 1 - \left( \eta_m^{(r)} \right)^2 \right) . \]

Also, \( K_3^{(r)}(\eta_m^{(r)}) \) is defined in [2]. Here \( y_1^{(r)}, y_2^{(r)} \) and \( y_3^{(r)}, y_4^{(r)} \) are different roots of equations
\[ \left( y_1^{(r)} \right)^4 - \frac{B_{12}^{(r)} B_{22}^{(r)} - B_{12}^{(r)} B_{66}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} \left( y_1^{(r)} \right)^2 + \frac{B_{11}^{(r)} + B_{66}^{(r)}}{B_{11}^{(r)}} \left( \eta_m^{(r)} \right)^2 \left( y_1^{(r)} \right)^2 + \]
\[ + \left( 1 - \left( \eta_m^{(r)} \right)^2 \right) \left( \frac{B_{22}^{(r)}}{B_{11}^{(r)}} - \frac{B_{66}^{(r)}}{B_{11}^{(r)}} \left( \eta_m^{(r)} \right)^2 \right) = 0, \ r = 1, 2, \]

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with positive real parts. $B_{ij}^{(r)}$, $r = 1, 2$, are coefficients of elasticity of composed shells

$$a^2 = \frac{k^2 h^2}{12} \left( \eta_m^{(r)} \right)^2 = \frac{\rho^{(r)} \omega^2}{m^2 k^2 B_{66}^{(r)}}, \quad R^{-1} = kr_0/2, \quad m = 1, \infty.$$  

\( R \) is the radius of the directing circle, \( \omega \) is the angular frequency, \( \rho^{(r)}, \ r = 1, 2 \), are the densities of the materials, \( k = 2\pi/2 \) for closed cylindrical shells and \( k = \pi/s \) for non-closed shells, \( s \) is the length of directing circle, \( m \) is the wave number. From equation (1) it follows, that at \( r_0^2/m^2 \to 0 \) the dispersion equation of the considered problems split into equations

$$G \left( \eta_m^{(1)}, \eta_m^{(2)} \right) = 0, \quad L \left( \eta_m^{(1)}, \eta_m^{(2)} \right) = 0, \quad K_3^{(1)} \left( \eta_m^{(1)} \right) = 0, \quad K_3^{(2)} \left( \eta_m^{(2)} \right) = 0. \quad (2)$$

The first two equations from (2) are analogous to Stoneley dispersion equation for bending and plane vibrations composed of infinite plate and plate-strip [3]. Plane interfacial vibrations of composed cylindrical shell correspond to the roots of the third and forth equations of (2). This manifests the fact of using the equation of the corresponding classical theory of orthotropic cylindrical shells [1]. Numerical analysis shows the efficiency of the asymptotic formulas (1) and (2).

References

ON THE APPLICATION OF THE I. VEKUA’S METHOD FOR THE GEOMETRICALLY NONLINEAR THEORY OF NON-SHALLOW SPHERICAL SHELLS

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I. Vekua constructed several versions of the refined linear theory of thin and