ON THE TROTTER'S DISTANCE OF TWO WEIGHTED RANDOM SUMS OF d-DIMENSIONAL RANDOM VARIABLES

Tran Loc Hung

Hue University, Hue, Vietnam
E-mail: tlhung@hueuni.edu.vn

The main purpose of this paper is to establish some estimations for the Trotter's distance concerning two weighted random sums of d-dimensional random variables

Keywords: Trotter's distance, Random sum, d-dimensional random variables.

INTRODUCTION

Let $R^d = \{x \mid (x^{(1)},...,x^{(d)})\}$ be a d-dimensional Euclidean space $(d \geq 1)$ with norm $\| x \| = (\sum_{i=1}^{d} x^{(i)2})^{1/2}$. Suppose that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are two sequences of d-dimensional independent random vectors (r.vs.) defined on probability space $(\Omega, \Im, P)$. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer- valued r.vs. . We assume that, for every $n \geq 1$, the r.vs. N_n and the all d-dimensional r.vs. from two sequences $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent.

Since the appearance of the [9], various limit theorems concerning the asymptotic behavior for randomly indexed sums of independent r.vs. and rates of convergence either in central limit theorem or in the weak law of large numbers have been studied systematically, (cf.[1], [2], [8], , [10] and [12]).

The main aim of this note is to establish some estimations related to Trotter's distance between two weighted random sums of d-dimensional r. vs. $S_{N_n}^X$ and $S_{N_n}^Y$ (see definition of Trotter's distance for more details in [6], [3], [4] and [5])

$$d_T(S_{N_n}^X, S_{N_n}^Y, f) = \sup_{y \in R^d} \| Ef(S_{N_n}^X + y) - Ef(S_{N_n}^Y + y) \|, \tag{1}$$

where $f \in C(R^d)$ –the class of all bounded uniformly continuous functions on R^d with the norm $\| f \| = \sup_{x \in R^d} \| f(x) \|$; $S_{N_n}^X := \varphi(N_n) \sum_{i=1}^{N_n} X_i$; $S_{N_n}^Y := \varphi(N_n) \sum_{i=1}^{N_n} Y_i$, and φ is a positive function with $\varphi(N_n) \to 0$ as $n \to \infty$.

The received results are extensions of that given in [3], [4] and [6] for one-dimensional case, [7], [11] and [5] for d-dimensional case. The method used in this paper is the same as in [6], [3], [4], [5], [7] and [11].

PRELIMINARY

We need in the sequel the inequality related to Trotter's distance from (1) (cf. [6], [3] and [4] for the one-dimension case, [5] for the d-dimension case). If N is a positive integer-valued r.v. independent on $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$, then

$$d_T(\sum_{j=1}^{N} X_j, \sum_{j=1}^{N} Y_j; f) \le \sum_{j=1}^{\infty} P(N=n) \sum_{j=1}^{n} d_T(X_j, Y_j; f). \qquad (2)$$

The modulus of continuity of a function $f \in C(R^d)$ plays very important role in estimations related to Trotter's distance. The detailed discussions of the definition and properties of the modulus of continuity can be found in [7] and [11]).

Let $\{X_n, n \ge 1\}$ be a sequence of d-dimensional r.vs. having finite moments of order r, $0 < r < +\infty$. Then $\{X_n, n \ge 1\}$ is said to satisfy the generalized random Lindeberg's condition of order r, $r \ge 1$. if for every $\delta > 0$

$$\lim_{n\to+\infty} E\left[\sum_{i=1}^{N_n} \int_{\|x\| > \delta/\varphi(N_n)} \| x \|^r \, dF_{X_i}(x) / \sum_{i=1}^{N_n} E \| X_i \|^r \right] = 0. \qquad (3)$$

We note that if the r.vs. N_n takes the value n with probability one, and if $\varphi(n) = \| B_n \|$ (see definition of B_n in [7]), then (3) reduces to the classical Lindeberg's condition for d-dimensional r.vs. introduced by Rao in [7].

MAIN RESULTS

Before stating the main results, we first denote for $1 \le j \le r-1$; $j = 1, 2, ..., n$

$$\upsilon_i(j) := \sum_{\substack{1 \le i_1, \dots, i_j \le d \\ r_1 + \dots + r_j = j}} \left| \int_{R^d} x_{i_1}^{r_1} \dots x_{i_j}^{r_j} \, d(F_{X_i}(x) - F_{Y_i}(x)) \right|, \qquad (4)$$

$$\upsilon_{i,r} := \int_{R^d} \| x \|^r \, d(F_{X_i(x)} - F_{Y_i(x)}). \qquad (5)$$

Theorem 1. Assume that $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ are two sequences of d-dimensional independent r. vs. for which

$$\upsilon_i(j) = 0, \qquad (6)$$

and

$$\upsilon_{i,r} < +\infty, \qquad (7)$$

where $r \ge 1$ is a fixed integer. Let $\{N_n, n \ge 1\}$ be a sequence of positive integer-valued r.vs. independent of each X_i and Y_i, $i = 1, 2, ..., n$. Then, for any $f \in C^{r-1}(R^d)$

$$d_T(S_{N_n}^X, S_{N_n}^Y, f) \le 2E\left\{ [\varphi(N_n)]^{r-1} \omega(f^{(r-1)}; \varphi(N_n)) \sum_{i=1}^{N_n} \max(\upsilon_{i,r}^{1-1/r}; \upsilon_{i,r}) \right\} / (r-1)!.$$

If, in addition, $f^{(r-1)} \in Lip(\alpha, M)$, $0 < \alpha \le 1$, then

$$d_T(S_{N_n}^X, S_{N_n}^Y, f) \le 2ME\left\{ [\varphi(N_n)]^{r-1+\alpha} \sum_{i=1}^{N_n} \max(\upsilon_{i,r}^{1-1/r}; \upsilon_{i,r}) \right\} / (r-1)!.$$

Proof. Since $f \in C^{r-1}(R^d)$, the Taylor's formula can be written as, (see [11] and [5])

$$f(x+y) = \sum_{j=0}^{r-1} \frac{f^{(j)}(x)^j}{j} + ((r-1)!)^{-1} \left[f^{(r-1)}(\eta) - f^{(r-1)}(y) \right] (x)^{(r-1)},$$

where

$$\eta = y + \theta x, \ 0 < \theta < 1, \ (x)^j = \underbrace{(x, x, ..., x)}_{j} \in \underbrace{R^d \times R^d \times ... R^d}_{j}.$$

Hence

$$E\left[f(\varphi(n)X_i+y)\right]=\sum_{j=1}^{r-1}\frac{|\varphi(n)|^j}{j!}\int_{R^d} f^{(j)}(y)(x)^j\,dF_{X_i}(x)+$$
$$+\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\int_{R^d}\left[f^{(r-1)}(\eta_1)-f^{(r-1)}(y)\right](x)^{r-1}\,dF_{X_i}(x), \qquad (8)$$

where $\eta_1=y+\theta_1\varphi(n)x,\ 0<\theta_1<1$. On the other hand, similar arguments give us

$$E\left[f(\varphi(n)Y_i+y)\right]=\sum_{j=1}^{r-1}\frac{|\varphi(n)|^j}{j!}\int_{R^d} f^{(j)}(y)(x)^j\,dF_{Y_i}(x)+$$
$$+\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\int_{R^d}\left[f^{(r-1)}(\eta_2)-f^{(r-1)}(y)\right](x)^{r-1}\,dF_{Y_i}(x), \qquad (9)$$

where $\eta_2=y+\theta_2\varphi(n)x,\ 0<\theta_2<1$. Combining (8) and (9) and using the properties of $\omega(f^{(r-1)};\delta)$, (see in [7] and [5]), we have

$$d_T(\varphi(n)X_i,\varphi(n)Y_i,f)\le$$
$$\le\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\sup\int_{R^d}\| f^{(r-1)}(\eta)-f^{(r-1)}(y)\|\,\| x\|^{r-1}|\,d(F_{Y_i}(x)-dF_{Y_i}(x))|$$
$$\le\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\int_{R^d}\| x\|^{r-1}\,\omega(f^{(r-1)},\varphi(n)\| x\|)\,|\,d(F_{Y_i}(x)-dF_{Y_i}(x))|$$
$$\le\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\omega(f^{(r-1)},\varphi(n))\int_{R^d}\| x\|^{r-1}\,(1+\| x\|)\,|\,d(F_{Y_i}(x)-dF_{Y_i}(x))|$$
$$\le\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\omega(f^{(r-1)},\varphi(n))(\upsilon_{i,r};\upsilon_{i,r}^{1-1/r})$$
$$\le 2\left[(r-1)!\right]^{-1}\left[\varphi(n)\right]^{r-1}\omega(f^{(r-1)},\varphi(n))\max(\upsilon_{i,};\upsilon_{i,r}^{1-1/r}).$$

This, together with (2), completes the proof of Theorem.

Theorem 2. Assume that $\{X_n,n\ge 1\}$ and $\{Y_n,n\ge 1\}$ are two sequences of independent and identically distributed (i.i.d.) d-dimensional r.vs satisfying condition (6) for some $r\ge 1$ and assume that $\{N_n,n\ge 1\}$ is a sequence of r.vs. defined as in Theorem 1. Furthermore, suppose that

$$\upsilon_{i,r-1+\delta}<+\infty,\ 0<\delta<1,\ i=1,2,\dots,n. \qquad (10)$$

Then, for any $f\in C^{r-1}(R^d)$

$$d_T(S_{N_n}^X,S_{N_n}^Y,f)=0\left\{E\left([\varphi(N_n)]^{r-1+\delta}\sum_{i=1}^{N_n}\upsilon_{i,r-1+\delta}\right)\right\}.$$

Proof. By same way that used in the proof of Theorem 1.

Theorem 3. Suppose that $\{X_n,n\ge 1\}$ and $\{Y_n,n\ge 1\}$ are two sequences of d-dimensional independent r.vs. with zero means and satisfying condition (4), and moreover assume that

$$\upsilon_{r,i}^X=E\| X_i\|<+\infty,\ \upsilon_{r,i}^Y=E\| Y_i\|<+\infty,\ E[\varphi(N_n)]^r<+\infty. \qquad (11)$$

Assume further that

$$\lim_{n\to+\infty}\varphi(N_n)=0\,, \qquad (12)$$

and as $n\to+\infty$

$$[\varphi(N_n)]^r\left(\sum_{i=1}^{N_n}\upsilon_{r,i}^X+\sum_{i=1}^{N_n}\upsilon_{r,i}^Y\right)=0\left\{E\left([\varphi(N_n)]^r\left(\sum_{i=1}^{N_n}\upsilon_{r,i}^X+\sum_{i=1}^{N_n}\upsilon_{r,i}^Y\right)\right)\right\}\ \text{a.s.} \qquad (13)$$

Then, for each $f\in C^r(R^d)$,

$$d_T(S_{N_n}^X,S_{N_n}^Y,f)=o\left\{E\left([\varphi(N_n)]^r\left(\sum_{i=1}^{N_n}\upsilon_{r,i}^X+\sum_{i=1}^{N_n}\upsilon_{r,i}^Y\right)/r!\right)\right\}.$$

Proof. By an argument analogous to that used in the proof of Theorem 1, we have for each $f \in C^r(R^d)$

$$E[f(\varphi(n)X_i + y)] = \sum_{j=1}^{r} \frac{|\varphi(n)|^j}{j!} \int_{R^d} f^{(j)}(y)(x)^j dF_{X_i}(x) +$$
$$+[(r)!]^{-1}[\varphi(n)]^r \int_{R^d} f^{(r)}(\eta_3) - f^{(r)}(y)(x)^r dF_{X_i}(x),$$

where $\eta_3 = y + \theta_3\varphi(n)x$, $0 < \theta_3 < 1$. Since $f \in C^r(R^d)$, for every $\varepsilon > 0$, there exists $\delta > 0$, such that $\| \eta_3 - y \| < \delta$ implies $\| f^{(r)}(\eta_3) - f^{(r)}(y) \| < \varepsilon$. For this δ we split up the above integral into

$$\int_{\|x\|<\delta/\varphi(n)} [f^{(r)}(\eta_3) - f^{(r)}(y)](x)^r dF_{X_i}(x) + \int_{\|x\|\ge\delta/\varphi(n)} [f^{(r)}(\eta_3) - f^{(r)}(y)](x)^r dF_{X_i}(x) =$$
$$= I_1^X + I_2^X.$$

For I_1^X one has $\| \eta_3 - y \| < \varphi(n) \| x \| < \delta$, implying $I_1^X \le \varepsilon\upsilon_{r,i}^X$.

For I_2^X one has $\| f^{(r)}(\eta_3) - f^{(r)}(y) \| \le 2 \| f^{(r)} \|$, giving

$$| I_2^X | \le 2 \| f^{(r)} \| \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{X_i}(x).$$

By an argument analogous to the previous one, we get for the sequence $\{Y_n, n \ge 1\}$

$$| I_1^Y | \le \varepsilon\upsilon_{r,i}^Y; \ | I_2^Y | \le 2 \| f^{(r)} \| \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{Y_i}(x).$$

Thus

$$d_T(\varphi(n)X_i, \varphi(n)Y_i, f) \le \varepsilon[\varphi(n)]^r(\upsilon_{r,i}^X + \upsilon_{r,i}^Y) +$$
$$2 \| f^{(r)} \| \left\{ \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{X_i}(x) + \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{Y_i}(x) \right\} / r!.$$

Therefore

$$(r!) \left\{ [\varphi(n)]^r \left(\sum_{i=1}^{N_n} \upsilon_{r,i}^X + \sum_{i=1}^{N_n} \upsilon_{r,i}^Y \right) \right\}^{-1} = \sum_{i=1}^{n} d_T(\varphi(n)X_i, \varphi(n)Y_i; f) \le$$
$$\varepsilon + 2 \| f^{(r)} \| \left\{ \sum_{i=1}^{n} \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{X_i}(x) / \sum_{i=1}^{n} \upsilon_{r,i}^X \right\} +$$
$$+ 2 \| f^{(r)} \| \left\{ \sum_{i=1}^{n} \int_{\|x\|\ge\delta/\varphi(n)} \| x \|^r dF_{Y_i}(x) / \sum_{i=1}^{n} \upsilon_{r,i}^Y \right\}.$$

Taking expectations of both sides and using (2), we have

$$d_t(S_{N_n}^X, S_{N_n}^Y, f)(r!)E\left\{ [\varphi(N_n)]^r \left(\sum_{i=1}^{N_n} \upsilon_{r,i}^X + \sum_{i=1}^{N_n} \upsilon_{r,i}^Y \right) \right\}^{-1} \le$$
$$\le \varepsilon + 2 \| f^{(r)} \| E\left\{ \sum_{i=1}^{N_n} \int_{\|x\|\ge\delta/\varphi(N_n)} \| x \|^r dF_{X_i}(x) / \sum_{i=1}^{N_n} \upsilon_{r,i}^X \right\} +$$
$$+ 2 \| f^{(r)} \| \left\{ \sum_{i=1}^{N_n} \int_{\|x\|\ge\delta/\varphi(N_n)} \| x \|^r dF_{Y_i}(x) / \sum_{i=1}^{N_n} \upsilon_{r,i}^Y \right\}. \quad (19)$$

According to the conditions (12), (13) and (2), the right side of foregoing inequality can be made arbitrarily small for $n \to +\infty$. Now on account of (13) there exist $C_1 > 0$, $n_0 > 0$ such that

$$C_1\left\{[\varphi(N_n)]^r\left(\sum_{i=1}^{N_n}\upsilon_{r,i}^X+\sum_{i=1}^{N_n}\upsilon_{r,i}^Y\right)\right\}^{-1}\geq\left\{E\left([\varphi(N_n)]^r\left(\sum_{i=1}^{N_n}\upsilon_{r,i}^X+\sum_{i=1}^{N_n}\upsilon_{r,i}^Y\right)\right)\right\}^{-1}, \quad (20)$$

for each $n > n_0$. Since the left side of the (20) vanishes for $n \to +\infty$. It completes the proof.

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