PARAMETERS ESTIMATION OF GARCH(1,1) PROCESS WITH REGULARLY VARYING TAILED ERRORS

Le Hong Son

Department of probability theory and mathematical statistic, BSU
Minsk, Belarus
E-mail: lhsondhv@gmail.com

In this paper, we established the consistency and asymptotic distribution of estimation of parameters for GARCH(1,1) process with the errors, whose squares have regularly varying tail probabilities with the index α, $\alpha > 0$. Using a modification of Gaussian quasi-maximum likelihood estimation, we showed that, the estimation of our method is consistent, the asymptotic distribution can be normality, or stable random variable with other values of α.

Keywords: GARCH process, Heavy tails, regular variation, quasi-maximum likelihood, infinite variance, M-estimation.

INTRODUCTION

In the analysis of financial data, the best-known and most often used processes are Autoregressive Conditionally Heteroskedastic (ARCH) and its extensions generalizations – Generalized Autoregressive Conditionally Heteroskedastic (GARCH) processes, see F. Engle [6] and T. Bollerslev [3]. The first order GARCH processes (GARCH(1,1)) are given by

$$y_t = \sigma_t \varepsilon_t, \ and\ \sigma_t^2 = \omega_0 + \alpha_0 y_{t-1}^2 + \beta_0 \sigma_{t-1}^2, \ t = 0, \pm 1, \pm 2, \ldots, \qquad (1)$$

where $\omega_0 > 0$, $\alpha_0 > 0$, $\beta_0 > 0$. $\{\varepsilon_t\}$ is an independent and identically distributed sequence of random variables (r.v.s). The necessary and sufficient conditions for the existence of a unique strictly stationary and ergodic solution of (1) was studied by D. Nelson in [11], who showed that, the process (1) have an unique strictly stationary and ergodic solution if and only if $E\ln(\beta_0 + \alpha_0 \varepsilon_0^2) < 0$.

It is known that, provided the error distribution has finite fourth moment, quasi-maximum likelihood estimators for ARCH/GARCH processes are asymptotically normally distributed with the standard rate $\sqrt{n}$, see [1, 7, 9]. When the errors have heavy tail probability, parameters estimation of process GARCH has investigated by T. Mikosch, and D. Straumann in [10, 12]. They showed the consistency and asymptotic distribution of quasi-maximum likelihood estimation. In this paper, we consider the parameters estimation of process (1) with the errors, whose squares have regularly varying tail with index α, $\alpha > 1$, and use a modification of Gaussian quasi-maximum likelihood for estimation of parameters. At first, we survey the definition and some properties of multivariate regularly variation of a random vector. We say that, the distribution of random vector X is multivariate regularly varying with index $\alpha > 0$, if there exists a sequence of constants $\{x_n\}$ and a random vector $\Theta \in S^{m-1}$, where S^{m-1} is the unit sphere in R^m with respect to the norm $|\cdot|$, such that

$$nP(|X| > tx_n, X/|X| \in A) \xrightarrow{v} t^{-\alpha} P(\Theta \in A), \ t > 0, \text{ as } n \to \infty, \qquad (2)$$

where A is a Borel set, $A \subset S^{m-1}$. This is equivalent to the condition that for all $t > 0$

$$\frac{P(|X| > tx,\ X/|X| \in A)}{P(|X| > x)} \xrightarrow{v} t^{-\alpha} P(\Theta \in A), \text{ as } x \to \infty, \tag{3}$$

where "$\xrightarrow{v}$" denotes vague convergence on S^{m-1}. The distribution of Θ in this formula is called the spectral measure of X.

Remark 1. If X is α-stable random vector with stable index $\alpha \in (0,2)$, then it is multivariate regularly varying with the same index α, and the multivariate regular variation of X implies regular variation of $|X|$: $P(|X| > x) \to x^{-\alpha} L(x)$, as $x \to \infty$, where $L(x)$ is slowly varying function, i.e., $L(tx)/L(x) \to 1$ as $x \to \infty$ for every $t > 0$. See [8] for details.

We recall that a stationary sequence $\{X_t\}_{t \in Z}$ is strongly mixing if

$$\sup_{A \in \sigma\{X_t,\ t \le 0\},\ A \in \sigma\{X_t,\ t > k\}} |P(A \cap B - P(A)P(B))| =: a_k \to 0, \text{ as } k \to \infty. \tag{4}$$

If sequence $\{a_k\}_{k \in N}$ decays to zero at an exponential rate, then $\{X_t\}$ is said to be strongly mixing with geometric rate. An overview of properties of regular variation and the mixing properties may be found in [2, 8].

ASYMPTOTIC DISTRIBUTION OF THE ESTIMATION

Let consider process (1) with the errors, whose squares have regularly varying tail probability with the index α, $\alpha > 0$. The modification of likelihood function is defined as following

$$L_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} l_t(\theta),\ l_t(\theta) = -\left[\ln \sigma_t^2(\theta) + \frac{1}{p} \left(\frac{y_t^2}{\sigma_t^2(\theta)} \right)^p \right],\ t = 1, 2, \ldots, n,$$

where $\sigma_t^2(\theta)$ is parameterized conditional variance of the process, $\sigma_t^2(\theta) = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2(\theta)$, with the parameters θ, $\theta = (\omega, \alpha, \beta), \omega > 0, \alpha > 0, \beta \ge 0, E(\beta + \alpha \varepsilon_0^2)^p < 1$, $p > 0$. At first, consider the first and second derivatives of the modification of the underlying log-likelihood $L_n(\theta)$. Easy to see that

$$L_n'(\theta) = \frac{1}{n} \sum_{t=1}^{n} \left[\frac{(y_t^2)^p (\sigma_t^2(\theta))' (\sigma_t^2(\theta))^{p-1}}{(\sigma_t^2(\theta))^{2p}} - \frac{(\sigma_t^2(\theta))'}{\sigma_t^2(\theta)} \right] = \frac{1}{n} \sum_{t=1}^{n} \left[\left(\frac{y_t^2}{\sigma_t^2(\theta)} \right)^p - 1 \right] \frac{(\sigma_t^2(\theta))'}{\sigma_t^2(\theta)}.$$

Therefore, since $\sigma_t^2(\theta_0) = \sigma_t^2$, we have

$$L_n'(\theta_0) = \frac{1}{n} \sum_{t=1}^{n} [(\varepsilon_t^2)^p - 1] \frac{1}{\sigma_t^2} (\sigma_t^2(\theta_0))' = \frac{1}{n} \sum_{t=1}^{n} [(\varepsilon_t^2)^p - 1] A_t, \tag{5}$$

where

$$A_t = \frac{1}{\sigma_t^2} (\sigma_t^2(\theta_0))' = \left(\frac{1}{1-\beta_0} \frac{1}{\sigma_t^2}, \frac{1}{\sigma_t^2} \sum_{j=1}^{\infty} \beta_0^{\,j-1} \varepsilon_{t-j}^2, \frac{1}{\sigma_t^2} \sum_{j=1}^{\infty} \beta_0^{\,j-1} \frac{\sigma_{t-j}^2}{\sigma_t^2} \right). \tag{6}$$

By the same way

$$L_n''(\theta) = \frac{1}{n} \sum_{t=1}^{n} \left\{ \left[\left(\frac{y_t^2}{\sigma_t^2(\theta)} \right)^p - 1 \right] \frac{(\sigma_t^2(\theta))''}{\sigma_t^2(\theta)} - \left[1 + (p-1) \left(\frac{y_t^2}{\sigma_t^2(\theta)} \right)^p \right] \left(\frac{(\sigma_t^2(\theta))'}{\sigma_t^2(\theta)} \right)^T \frac{(\sigma_t^2(\theta))'}{\sigma_t^2(\theta)} \right\}. \tag{7}$$

$$L_n''(\theta_0)=\frac{1}{n}\sum_{t=1}^{n}\left\{\left[(\varepsilon_t^2)^p-1\right]\frac{(\sigma_t^2(\theta_0))''}{\sigma_t^2(\theta_0)}-\left[1+(p-1)(\varepsilon_t^2)^p\right]\left(\frac{(\sigma_t^2(\theta_0))'}{\sigma_t^2(\theta_0)}\right)^T\frac{(\sigma_t^2(\theta_0))'}{\sigma_t^2(\theta_0)}\right\}. \qquad (8)$$

Remark 2. From (5) and (7) we see that, $L_n'(\theta_0)$ and $L_n''(\theta_0)$ are processes of measurable functions of strictly stationary and ergodic process $\{\varepsilon_t\}$. Therefore they are also strictly stationary and ergodic processes.

Lemma 1. *Assume that, A_t is defined at (6), $\{\varepsilon_t^2\}_{t\in Z}$ are regularly varying with index α, $\alpha>0$, then $\{[(\varepsilon_t^2)^p-1]A_t\}_{t\in Z}$, $p>0$, are multivariate regularly varying with the index α/p.*

Proof. Since the regularly varying with index α of $\{\varepsilon_t^2\}_{t\in Z}$, we have that, $\{(\varepsilon_t^2)^p\}_{t\in Z}$ are regularly varying with index α/p. Since [4], it suffices to show that, for all $k>0$, the absolute moments order k^{th} of components of A_t are finite. Actually, since $\sigma_t^2\ge\omega$, we have

$$E\left(\frac{1}{1-\beta_0}\frac{1}{\sigma_t^2}\right)^k\le\frac{1}{[(1-\beta_0)\omega_0]^k}<\infty. \qquad (9)$$

Next, for every $j=1,2,\ldots$,

$$\frac{1}{\sigma_t^2}=\frac{1}{\omega_0+(\beta_0+\alpha_0\varepsilon_{t-1}^2)\sigma_{t-1}^2}\le\frac{1}{(\beta_0+\alpha_0\varepsilon_{t-1}^2)\sigma_{t-1}^2}\le\ldots\le\prod_{i=1}^{j}\frac{1}{(\beta_0+\alpha_0\varepsilon_{t-1}^2)\sigma_{t-j}^2}.$$

Therefore, since $\sigma_t^2\ge\omega_0$, for all $t\in Z$,

$$\frac{1}{\sigma_t^2}\sum_{j=1}^{\infty}\beta_0^{\,j-1}\frac{\sigma_{t-j}^2}{\sigma_t^2}\le\frac{1}{\omega_0}\sum_{j=1}^{\infty}\beta_0^{\,j-1}\prod_{i=1}^{j}\frac{1}{(\beta_0+\alpha_0\varepsilon_{t-1}^2)}=\frac{1}{\omega_0}\sum_{j=1}^{\infty}\beta_0^{-1}\prod_{i=1}^{j}\frac{\beta_0}{(\beta_0+\alpha_0\varepsilon_{t-1}^2)}. \qquad (10)$$

Put $q=E[\beta_0/(\beta_0+\alpha_0\varepsilon_{t-1}^2)]^k<1$, from (10), using Minkowski's inequality

$$\left[E\left(\frac{1}{\sigma_t^2}\sum_{j=1}^{\infty}\beta_0^{\,j-1}\frac{\sigma_{t-j}^2}{\sigma_t^2}\right)^k\right]^{1/k}\le\frac{1}{\omega_0}\sum_{j=1}^{\infty}\beta_0^{-1}\left[E\left(\prod_{i=1}^{j}\frac{\beta_0}{(\beta_0+\alpha_0\varepsilon_{t-1}^2)}\right)^k\right]^{1/k}\le\frac{1}{\omega_0}\sum_{j=1}^{\infty}\beta_0^{-1}q^{j/k}<\infty. \qquad (11)$$

At the end, since (5), we have

$$\frac{1}{\sigma_t^2}\sum_{j=1}^{\infty}\beta_0^{\,j-1}\varepsilon_{t-j}^2\le\frac{1}{\omega_0}\sum_{j=1}^{\infty}\beta_0^{\,j-1}\varepsilon_{t-j}^2\prod_{i=1}^{j}\frac{1}{(\beta_0+\alpha_0\varepsilon_{t-i}^2)}\le\frac{1}{\omega_0\alpha_0}\sum_{j=1}^{\infty}\frac{\alpha_0\varepsilon_{t-j}^2}{(\beta_0+\alpha_0\varepsilon_{t-i}^2)}\prod_{i=1}^{j}\frac{\beta_0}{(\beta_0+\alpha_0\varepsilon_{t-i}^2)}.$$

By the same way, using Minkowski's inequality, for all $k>0$, we obtain

$$\left[E\left(\frac{1}{\sigma_t^2}\sum_{i=1}^{\infty}\beta_0^{\,j-1}\varepsilon_{t-j}^2\right)^k\right]^{1/k}\le\frac{1}{\omega_0\alpha_0}\sum_{j=1}^{\infty}\left[E\left(\frac{\alpha_0\varepsilon_{t-j}^2}{(\beta_0+\alpha_0\varepsilon_{t-i}^2)}\right)^k\right]^{1/k}\left[E\left(\prod_{i=1}^{j}\frac{\beta_0}{(\beta_0+\alpha_0\varepsilon_{t-i}^2)}\right)^k\right]^{1/k}$$

$$\le\frac{1}{\omega_0\alpha_0}\sum_{j=1}^{\infty}q^{j/k}<\infty. \qquad (12)$$

Combining (9), (11) and (12) we get the proof of the lemma.

Theorem 2. Assume process (1) with the errors, whose squares have regularly varying tail probabilities with index α, $\alpha>0$. $\alpha/p>1$, $E(\varepsilon_0^2)^{\alpha/p}=1$, then there exist $\hat{\theta}_n$ be local minimum of $L_n(\theta)$ in some neighborhood of θ_0, and

$$\hat{\theta}_n\xrightarrow{a.s.}\theta_0,\text{ as } n\to\infty.$$

Proof. The existence of local minimum $\hat{\theta}_n$ of $L_n(\theta)$ in some neighborhood of θ_0 can be proved as Lee at al. in [9]. Since remark 2, the consistency of $\hat{\theta}_n$ in this theorem can be proved as the same argument in [1, 7] with noting that, if $(\varepsilon_0^2)^p$ have regularly varying tail probabilities with index α / p, $\alpha / p > 1$, then $E(\varepsilon_0^2)^p < \infty$.

As argument in [1], we can prove that, there exists the following non-singular matrix

$$B = E\left[\left(\frac{(\sigma_0^2(\theta_0))'}{\sigma_0^2}\right)^T \frac{(\sigma_0^2(\theta_0))'}{\sigma_0^2}\right].$$

Theorem 3. Assume process (1) and the function $L_n(\theta)$, which satisfy conditions in theorem 1, $\hat{\theta}_n$ is defined in theorem 2, we have

i) if $\alpha / p \geq 2$, then $n^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \Sigma G^{-1})$ as $n \to \infty$,

where $\Sigma = \tau C$, $\tau = E(\varepsilon_0^2)^{2p} - 1$, $C = E(A_0 A_0^T)$, $G = -E[1 + (p-1)|\varepsilon_0^2|^p]B$.

ii) if $1 < \alpha / p < 2$, $\{[(\varepsilon_t^2)^p - 1]A_t\}$ is strongly mixing with geometric rate, then

$$na_n^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow{d} S_{\alpha/p} \text{ as } n \to \infty,$$

where $S_{\alpha/p}$ is α / p-stable r.v., a_n is defined by $a_n = \inf\{x \in R^+ : nP((\varepsilon_0^2)^p > x) < 1\}$.

Proof. Since remark 2, (8), the assumption that $E(\varepsilon_0^2)^{\alpha/p} = 1$ and by ergodic theorem in [5], we have

$$L_n''(\theta_0) \xrightarrow{a.s} G, \text{ as } n \to \infty. \tag{13}$$

Thus, since theorem 2, there exists $n_0 \in N$, such that

$$L_n'(\hat{\theta}_n) = 0, \text{ for all } n \geq n_0.$$

By the mean value theorem, with some point η in some sufficiently small neighborhood of θ_0, we obtain

$$(\hat{\theta}_n - \theta_0)L_n''(\eta) = L_n'(\hat{\theta}_n) - L_n'(\theta_0) = -L_n'(\theta_0). \tag{14}$$

In the case $\alpha / p \geq 2$, since (5), (13) and (14), we have

$$n^{1/2}(\hat{\theta}_n - \theta_0) \sim n^{-1/2}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]A_t G^{-1}, \text{ as } n \to \infty. \tag{15}$$

Put D_t is arbitrary linear combination of components of A_t, $t = 1, 2, ..., n$, by Cramer-Wold theorem (see [2], Theorem 7.7), it is sufficiently to show the asymptotic distribution of sequence $n^{-1/2}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]D_t$. From lemma 1, D_t has finite covariance. Hence, $\{[(\varepsilon_t^2)^p - 1]D_t\}$ is a stationary ergodic martingale difference sequence. Appling the central limit theorem for martingale difference sequence (see [2], Theorem 23.1), where $D = Var[[(\varepsilon_t^2)^p - 1]D_1] = \tau ED_1^2$,

$$n^{-1/2}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]A_t \xrightarrow{d} .N(0, D_1).$$

Therefore, where $\Sigma = \tau C$ is the covariance matrix $(\varepsilon_0^2 - 1)A_0$, we obtain

$$n^{-1/2}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]A_t \xrightarrow{d} N(0, \Sigma).$$

Combining (13), (14) and (15) we get the proof of the first part of the theorem.

In the case $1 < \alpha / p < 2$, since (5), (13) and (14), we have

$$na_n^{-1}(\hat{\theta}_n - \theta_0) \sim a_n^{-1}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]A_t G^{-1}, \text{ as } n \to \infty. \quad (16)$$

Since (14), lemma 1, and the strongly mixing with geometric rate property of sequence $\{[(\varepsilon_t^2)^p - 1]A_t\}$, we derive that, the sequence $\{[(\varepsilon_t^2)^p - 1]A_t\}$ satisfies conditions A1, A2 and A3 of theorem 7.4.1 in [12]. Therefore, since this theorem, we have

$$a_n^{-1}\sum_{t=1}^{n}[(\varepsilon_t^2)^p - 1]A_t \xrightarrow{d} S_{\alpha/p}, \text{ as } n \to \infty, \quad (17)$$

where $S_{\alpha/p}$ is a α / p-stable r.v.

Combining (16), (17) and Cramer-Wold theorem, we get the proof of the theorem.

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