THE MUTUAL VARIOGRAM ESTIMATE OF THE SIGNAL PROCESSES

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Abstract. The paper deals with the problem of a statistical analysis of time series connected with the estimation of mutual variogram, a measure of spatial correlation. G. Matheron has coined the term variogram, although earlier appearances of this function can be found in the scientific literature. We present the limiting expressions of the first two moments and the higher order cumulants of the classical mutual variogram estimate of the second-order-stationary stochastic process with discrete time. These expressions are then used to prove the theorem concerning the asymptotic distribution of the mutual variogram estimate.

1. Introduction

Consider a random process $Y^r(s)=\{Y_a(s), a=\overline{1,r}\}$, $s \in Z=\{0, \pm 1, \pm 2, \dots\}$. Suppose further that $Y^r(s), s \in Z$, is a zero-mean second-order-stationary stochastic process with unknown mutual variogram is given in G. Matheron [2],

$$2\gamma_{ab}(s_1-s_2)=\operatorname{cov}[(Y_a(s_1)-Y_a(s_2)),(Y_b(s_1)-Y_b(s_2))],$$

$s_1, s_2 \in Z,\ a,b=\overline{1,r}$.

It is easy to see that $\gamma_{ab}(-s)=\gamma_{ab}(s)$, $\gamma_{ab}(s)=\gamma_{ba}(s)$, $\gamma_{ab}(0)=0$, $s \in Z$, $a,b=\overline{1,r}$, and

$$\sum_{a,b=1}^{r} c_a c_b \gamma_{ab}(s) \geq 0,$$

$s \in Z,\ c_a \in Z,\ a=\overline{1,r},\ r \geq 1$.

The mutual variogram estimate $2\widetilde{\gamma}_{ab}(h)$ in terms of sequence of observations, $Y_a(1), Y_a(2), \dots, Y_a(n)$, is defined as

$$2\widetilde{\gamma}_{ab}(h)=\frac{1}{n-h}\sum_{s=1}^{n-h}\big(Y_a(s+h)-Y_a(s)\big)\big(Y_b(s+h)-Y_b(s)\big), \tag{1}$$

with $\widetilde{\gamma}_{ab}(-h)=\widetilde{\gamma}_{ab}(h)$, $h=\overline{0,n-1}$ and $\widetilde{\gamma}_{ab}(h)=0$ for $|h| \geq n$, $a,b=\overline{1,r}$.

It is the purpose of this paper to derive the asymptotic distribution of the mutual variogram estimate $2\widetilde{\gamma}_{ab}(h)$, $h=\overline{0,n-1}$. The approach is similar to the approach taken in the time series literature, and the reader is referred to Brillinger [1], Troush [3], Tsekhovaya [4], for theorems regarding the asymptotic distribution of the spectral density estimate, covariance estimate and variogram estimate of a time series.

2. First-, second-order moments

In order to state and prove the theorem concerning the asymptotic distribution of the mutual variogram estimate $2\tilde{\gamma}_{ab}(h)$, $h=\overline{0,n-1}$, it is first necessary to calculate the first two moments of the examined estimate (1).

The classical mutual variogram estimate $2\tilde{\gamma}_{ab}(h)$ is unbiased for $2\gamma_{ab}(h)$.

Theorem 1 *Let the mutual spectral densities $f_{ab}(x)$, $x \in \Pi=[-\pi,\pi]$, and cumulants spectral densities $f_{a_1b_1a_2b_2}(x_1,x_2x_3)$, $x_i \in \Pi$, $i=\overline{1,3}$, of the stationary stochastic processes $Y^r(s)$ are continuous on Π, Π^3, respectively. Then the asymptotic expressions for the covariance of the mutual variogram estimate given by (1) are*

$$\lim_{n\to\infty}(n-h^-)\operatorname{cov}(2\tilde{\gamma}_{a_1b_1}(h_1),2\tilde{\gamma}_{a_2b_2}(h_2))=$$

$$=2\pi(G_1^{h_1-h_2}(0)+G_2^{h_1-h_2}(0)), \tag{2}$$

where

$$G_1^{h_1-h_2}(x)=\cos\frac{(h_1-h_2)x}{2}G_1(x), \qquad G_2^{h_1-h_2}(x)=\cos\frac{(h_1-h_2)x}{2}G_2(x),$$

$$G_1(x)=\iint_{\Pi^2} f_{a_1b_1a_2b_2}(x-y,y,z)q(x,y,z)dydz,$$

$$G_2(x)=\int_{\Pi}(f_{a_1a_2}(x-y)f_{b_1b_2}(y)+f_{a_1b_2}(x-y)f_{b_1a_2}(y))q(x,y)dy,$$

$$\begin{aligned}q(x,y,z)=e^{i\frac{x(h_2-h_1)}{2}}\Big(&1+e^{ix(h_1-h_2)}-e^{i(xh_1+zh_2)}-e^{ix(h_1-h_2)-izh_2}+e^{ixh_1}-\\&-e^{ix(h_1-h_2)-iyh_1}+e^{i(x-y)h_1+ih_2z}+e^{i(x-y)(h_1-h_2)-iyh_2-ih_2z}-e^{i(x-y)h_1}-\\&-e^{iyh_1-ixh_2}+e^{iyh_1+ih_2z}+e^{iyh_1-i(x+z)h_2}-e^{iyh_1}+e^{-ixh_2}-e^{ih_2z}-e^{-ih_2(x+z)}\Big),\end{aligned}$$

$$\begin{aligned}q(x,y)=e^{i\frac{x(h_2-h_1)}{2}}\Big(&1+e^{ix(h_1-h_2)}-e^{ixh_1-(x-y)h_2}-e^{ixh_1-yh_2}+e^{ixh_1}-\\&-e^{-ixh_2+i(x-y)h_1}+e^{i(x-y)(h_1-h_2)}+e^{i(x-y)h_1-yh_2}-e^{i(x-y)h_1}-e^{iyh_1-ixh_2}+\\&+e^{iyh_1-i(x-y)h_2}+e^{iy(h_1-h_2)}-e^{iyh_1}+e^{-ixh_2}-e^{-i(x-y)h_2}-e^{-ih_2y}\Big),\end{aligned}$$

$h^-=\min\{h_1,h_2\}, h_1,h_2=\overline{0,n-1}$, $a_1,b_1,a_2,b_2=\overline{1,r}$.

Corollary *Under the assumptions of Theorem 1.* $\lim_{n\to\infty}\operatorname{cov}[2\tilde{\gamma}_{a_1b_1}(h_1),2\tilde{\gamma}_{a_2b_2}(h_2)]=0$ *and* $\lim_{n\to\infty}D[2\tilde{\gamma}_{ab}(h)]=0$, $h,h_1,h_2=\overline{0,n-1}$, $a,b,a_1,b_1,a_2,b_2=\overline{1,r}$.

3. Higher order cumulants

In order to found the asymptotic distribution of the mutual variogram estimate $2\tilde{\gamma}_{ab}(h)$, $h=\overline{0,n-1}$, it is necessary to investigate an asymptotic behavior of the cumulant $cum\{2\tilde{\gamma}_{a_1b_1}(h_1),...,2\tilde{\gamma}_{a_pb_p}(h_p)\}$, $h_j=\overline{0,n-1}$, $a_j,b_j=\overline{1,r}$, $j=\overline{1,p}$.

Theorem 2 *Let processes* $Y^r(s)$, $s\in Z$, *be a stochastic process with continuous cumulants spectral densities up to order* s, $s=\overline{2,p}$, *for any* p, $p>2$. *Further, let the mutual variogram estimate* $2\tilde{\gamma}_{ab}(h)$, *given by* (1). *Then*

$$\lim_{n\to\infty} cum\{2\tilde{\gamma}_{a_1b_1}(h_1),...,2\tilde{\gamma}_{a_pb_p}(h_p)\}=0,$$

$h_j=\overline{0,n-1}$, $a_j,b_j=\overline{1,r}$, $j=\overline{1,p}$, $h=\overline{0,n-1}$, $n=1,2,3,...$

4. Asymptotic distribution

Theorem 3 *Let all the assumptions of Theorem 2 be satisfied. Then the mutual variogram estimate* $2\tilde{\gamma}_{ab}(h)$, $h=\overline{0,n-1}$, *is asymptotically normally distributed with mean* $2\gamma(h)$, $h\in Z$, *and asymptotic covariance structure* (2).

References

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