We consider the well-known problem of finding the vertex covering number of a graph. It is known that this problem is NP-hard. In this paper, we give an efficient algorithm for finding the vertex covering number for connected graphs whose blocks are either complete graphs or complete bipartite graphs or wheels or powers of cycles or powers of paths.

**Keywords:** Vertex covering number, Block graph, Cactus, Block-Cactus graph, Powers of Paths, Powers of cycles, Wheel graphs, Efficient algorithm.

**Introduction**

We consider only finite undirected simple graphs $G = (V, E)$ with the vertex set $V = V(G)$ and the edge set $E = E(G)$. Let $G$ be a graph and $v$ be a vertex of $G$. The neighborhood of $v$ in $G$ is the set $N_G(v) = \{ w \in V(G) : wv \in E(G) \}$. The closed neighborhood of $v$ in $G$ is the set $N_G[v] = N_G(v) \cup \{v\}$.

A vertex subset $U \subseteq V$ is a vertex cover of $G$ if each edge of $G$ is incident with at least one vertex in $U$. The minimum size of a vertex cover of $G$ is the vertex covering number of $G$ and is denoted by $\beta(G)$. The set of all vertex covers of $G$ is denoted by $\Theta(G)$.

The distance $d_G(u, v)$ between vertices $u$ and $v$ in $G$ is the length a shortest path of $G$ connecting $u$ and $v$. If $G$ doesn’t contain such path, then we set $d_G(u, v) = \infty$. For a graph $G = (V, E)$ and positive integer $k$, the $k$-th power of $G$ is the graph $G^k$ with the same vertex set $V$ and two vertices $u$ and $v$ are adjacent in $G^k$ if and only if $d_G(u, v) \leq k$. A path with $n$ vertices is denoted by $P_n$ and $C_n$ is a cycle with $n$ vertices. Wheel $W_n$ is a graph with $n + 1$ vertices that can be constructed by connecting a single vertex to each vertex of a cycle $C_n$.

A cut-vertex of a graph $G$ is a vertex whose removal increases the number of connected components of the graph. A block of $G$ is a maximal 2-connected subgraph of $G$. A block of $G$ is an end-block if it contains at most one cut-vertex.

In this paper we give an efficient algorithm for finding the vertex covering number of a connected graph, each block of which is a complete graph or a complete bipartite graph or a wheel or a power of a cycle or a power of a path. Such graphs (see, for an example, fig. 1) are referred to as graphs with special blocks. Note that the class of graphs with special
blocks includes such well-studied graphs as block graphs, cactus graphs and block-cactus graphs.

**Fig. 1.** An example of a graph with special blocks

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**The vertex covering number of graphs with Cut-Vertices**

Let $\text{CutVertices}$ be the set of cut-vertices and $\text{Blocks}$ be the set of blocks of a graph $G$. The block-cut-vertex graph $BC(G)$ of $G$ is defined as that graph whose vertices are the blocks and cut-vertices of $G$ and such that two vertices are adjacent if and only if one vertex is a block $B \in \text{Blocks}$ and the other vertex is a cut-vertex $c \in \text{CutVertices}$ belonging to $B$. It is well-known that $BC(G)$ is a tree for every connected graph $G$. For the given connected graph, its block-cut-vertex graph can be constructed in linear time.

Let $G$ be a graph with special blocks and $B$ is its an end-block with a single cut-vertex $x$. Consider the subgraph $H = G - V(B) \setminus \{x\}$ of $G$. The problem for $G$ can be divided into two subproblems: a subproblem for $B$ and other for $H$. Let us divide the set of all vertex covers of $G$ into two subsets $\Theta(G) = \Theta^+(G,x) \cup \Theta^-(G,x)$, where $\Theta^+(G,x) = \{U \in \Theta(G) : x \in U\}$ and $\Theta^-(G,x) = \{U \in \Theta(G) : x \notin U\}$. Define the following two numbers:

$$
\beta^+_x(G) = \min\{|U| : U \in \Theta^+(G,x)\},
\beta^-_x(G) = \min\{|U| : U \in \Theta^-(G,x)\}.
$$

**Claim 1.** If $B, H$ are connected graphs with exactly one common vertex $x$, then

$$
\beta^+_x(G) = \min\{\beta^+_x(B) + \beta^-_x(H) - 1, \beta^+_x(B) + \beta^-_x(H), \beta^+_x(B) + \beta^-_x(H)\},
$$

$$
\beta^-_x(G) = \beta^-_x(B) + \beta^-_x(H),
\beta(G) = \min\{\beta^+_x(G), \beta^-_x(G)\}.
$$

During running the algorithm for finding the vertex covering number of the given graph, we assign each vertex of the graph a label from the set $\{0, 1\}$. At the beginning of the algorithm, all labels are 0. When the subproblem for an end-block of $G$ with a single cut-vertex $x$ is solved we can set according value of the label associated with $x$. Further, we state a problem, in which it is used labels.

Let $G = (V, E)$ be a graph and each vertex $v$ of $G$ has the label $f(v) \in \{0, 1\}$. In the next text the following notation is used. The $f$-vertex covering number of $G$ is

$$
\beta(G, f) = \min\{|\{y \in U : f(y) \neq 1\} : U \in \Theta(G)\}.
$$

For $x \in V(G)$, define the following two numbers
\[ \beta^*_i(G,f) = \min\{ \{\ y \in U : f(y) \neq 1 \} : U \in \Theta^*(G,x) \}, \]
\[ \beta^*_i(G,f) = \min\{ \{\ y \in U : f(y) \neq 1 \} : U \in \Theta^*(G,x) \}. \]

**Claim 2.** If \( f(v) = 0 \) for each vertex \( v \in V(G) \) of a graph \( G \), then \( \beta(G) = \beta(G,f) \).

**Lemma.** Let \( G = (V,E) \) be a graph and \( x \in V \). For any function \( f : V \to \{0,1\} \), the following equalities are correct:
\[ \beta_i^+(G,f) = |\{ v \in N_G(x) : f(v) = 0 \}| + \beta(G - N_G[x], f), \]
\[ \beta(G,f) = \min \{ \beta^+_i(G,f), \beta^+_i(G,f) \}, \]
\[ \beta^+_i(G) = 1 + \beta(G - x), \]
\[ \beta^+_i(G) = |N_G(x)| + \beta(G - N_G(x)), \]
\[ \beta^+_i(G,f) = \beta(G - x,f) + (1 - f(x)). \]

Let \( F = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\} \) be a family of graph classes containing only 2-connected graphs. For \( i = 1, 2, \ldots, p \), let \( A_i \) be algorithm for finding the pair of numbers \( (\beta^+_i(B,f), \beta^+_i(B,f)) \), where \( B \) belongs to \( \Gamma_i \), \( x \in V(B) \) and \( f : V(B) \to \{0,1\} \).

An algorithm for finding the vertex covering number of a graph \( G \), in which each block belongs at least one of classes in \( F \).

1. Construct the block-cut-vertex graph \( BC(G) \); // Here \( BC(G) \) is a tree.
2. \( \text{cost} := 0; \)
3. For each vertex \( v \in V(G) \) set \( f(v) := 0; \)
4. While the set \( \text{Blocks} \) is non-empty, do the following steps:
   a. If \( \text{Blocks} \) contains only one a block \( B \), then take \( B \) and an arbitrary vertex \( x \in V(B) \), otherwise take an arbitrary end-block \( B \in \text{Blocks} \) and let \( x \) be the cut-vertex of \( G \) that is adjacent to \( B \) in \( BC(G) \);
   b. Find a graph class \( \Gamma_i \) in \( F \) such that \( B \in \Gamma_i \);
   c. Compute the pair of numbers \( (\beta^-_i(B,f), \beta^+_i(B,f)) \) by the algorithm according to the graph class \( \Gamma_i \);
   d. \( \text{cost} := \text{cost} + \min \{ \beta^-_i(B,f), \beta^+_i(B,f) \} \);
   e. If \( \beta^-_i(B,f) \geq \beta^+_i(B,f) \) then set the label \( f(x) := 1; \)
   f. Remove the vertex \( B \) from \( BC(G) \), set \( \text{Blocks} := \text{Blocks} \setminus \{B\} \) and if \( x \) is a leaf of \( BC(G) \), then remove \( x \) from \( BC(G) \).
5. \( \beta(G) := \text{cost} \).

Note that the algorithm solves the problem in polynomial time for such graphs, in which each block \( B \) belongs to the class graphs for which the problem of finding the pair of numbers \( (\beta^-_i(B,f), \beta^+_i(B,f)) \) is polynomially solvable. The next, we give subroutines for finding this pair of numbers when \( B \) is a complete graph, a complete bipartite graph, a power of a path, a power of a cycle and a wheel.

**Algorithms for some types of 2-connected graphs**

**Algorithm:** VCCG (Finding \( (\beta^-_i(B,f), \beta^+_i(B,f)) \) when \( B \) is a complete graph)

**Input:** a complete graph \( B \), labels \( f(v) \) for each vertex \( v \) of \( B \) and \( x \in V(B) \).
\[ p := \{ v \in V(B) \setminus \{ x \} : f(v) = 0 \}; \]

if \( p = 0 \) then return \((0, 1 - f(x))\); else return \((p, p - f(x))\).

**Algorithm: VCCB** (Finding \((\beta^*_B(B, f), \beta^*_B(B, f))\) when \(B\) is a complete bipartite graph)

**Input:** a complete bipartite graph \(B\) with bipartition \(V(B) = X \cup Y\), labels \(f(v)\) for each vertex \(v\) of \(B\) and \(x \in V(B)\).

let \(x \in X; // otherwise, rename parts of \(B\) such that \(x \in X\)
\(a := \{ v \in X \setminus \{ x \} : f(v) = 0 \}; \)
\(b := \{ v \in Y : f(v) = 0 \}; \)
\(\beta^*_B(B, f) := b; \)
\(\beta^*_B(B, f) := \min\{a, b\} + 1 - f(x); \)
return \((\beta^*_B(B, f), \beta^*_B(B, f))\).

**Algorithm: VCP** (Finding \((\beta^*_B(B, f), \beta^*_B(B, f))\) when \(B\) is a power of a path)

**Input:** \(B = P_n^k \ (k < n - 1)\), where \(V(P_n) = \{v_1, v_2, \ldots, v_n\}\) and \(E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}\), labels \(f(v)\) for each vertex \(v\) of \(B\) and \(x \in V(B)\).

\(H := B - N_B[x]; \)
\(\beta^*_B(B, f) := \{ v \in N_B(x) : f(v) = 0 \}; \)
for each the connected component \(S\) of \(H\) do
let \(w \in V(S); \)
\((a, b) := VCP(S, f, w); \)
\(\beta^*_B(B, f) := \beta^*_B(B, f) + \min\{a, b\}; \)
end for
let \(x = v_q; \)
if \(q \leq n/2\) then \(C := \{ x = v_q, v_{q+1}, \ldots, v_{q+k} \}; \)
else \(C := \{ x = v_q, v_{q+1}, \ldots, v_{k+1} \}; \)
\(H := B - C\; \quad \text{cost} := |\{ u \in C : f(u) = 0 \}|; \)
for each the connected component \(S\) of \(H\) do
let \(w \in V(S); \)
\((a, b) := VCP(S, f, w); \)
\(\text{cost} := \text{cost} + \min\{a, b\}; \)
end for
for each \(v \in C \setminus \{x\} \) do
\(\text{cost} := |\{ u \in N_B(v) : f(u) = 0 \}|; \)
\(D := B - N_B[v]; \)
for each the connected component \(S\) of \(D\) do
let \(w \in V(S); \)
\((a, b) := VCP(S, f, w); \)
\(\text{cost} := \text{cost} + \min\{a, b\}; \)
end for
if \(\text{cost} < \text{cost}\) then \(\text{cost} := \text{cost}; \)
end for
\(\beta^*_B(B, f) := \text{cost}; \)
Algorithm: VCC (Finding \((\beta_x^v(B, f), \beta_x^v(B, f))\) when \(B\) is a power of a cycle)

**Input:** \(B = C_n^k\) (\(k < \lfloor n/2 \rfloor\)), where \(V(C_n) = \{v_1, v_2, ..., v_n\}\), \(E(C_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}\), labels \(f(v)\) for each vertex \(v\) of \(B\) and \(x \in V(B)\).

Let \(w \in V(B) \setminus N_B[x]\);
\((a, b) := VCP(B - N_B[x], f, w)\);
\(\beta_x^v(B, f) := |\{v \in N_B(x) : f(v) = 0\}| + \min\{a, b\}\);
\(C := \{x = v_x, v_{x+1 \mod(n+1)}, ..., v_{x+k \mod(n+1)}\}\);

Let \(w \in V(B) \setminus C\);
\((a, b) := VCP(B - C, f, w)\);
\(cost := b + |\{u \in C : f(u) = 0\}|\);

For each \(v \in C \setminus \{x\}\) do
\(cost1 := |\{u \in N_B(v) : f(u) = 0\}|\);
\(D := B - N_B[v]\);
Let \(w \in V(D)\);
\((a, b) := VCP(D, f, w)\);
\(cost1 := cost1 + \min(a, b)\);
If \(cost1 < cost\) then \(cost := cost1\);
End for
\(\beta_x^v(B, f) := cost\);

Return \((\beta_x^v(B, f), \beta_x^v(B, f))\).

Algorithm: VCW (Finding \((\beta_x^v(B, f), \beta_x^v(B, f))\) when \(B\) is a wheel)

**Input:** a wheel \(B = W_n\), labels \(f(v)\) for each vertex \(v\) of \(B\) and the vertex \(x\) of \(B\).

Let \(v\) is a center of the wheel \(B\);
If \(v = x\) then
\((a, b) := VCC(B - v, f, x)\);
\(\beta_x^v(B, f) := (1 - f(v)) + b\);
\(\beta_x^v(B, f) := |\{u \in V(B - v) : f(u) = 0\}|\);
Else
Let \(B' := B - v\);
\((a, b) := VCC(B', f, x)\);
\(\beta_x^v(B, f) := \min\{|1 - f(v)| + b, |\{u \in V(B') : f(u) = 0\}|\}\);
Let \(w \in \{u \in V(B - N_B[x]) : \deg(u) = 1\}\);
\((a, b) := VCP(B - N_B[x], f, w)\);
\(\beta_x^v(B, f) := |u \in N_B(x) : f(u) = 0| + \min\{a, b\}\);
End if
Return \((\beta_x^v(B, f), \beta_x^v(B, f))\).

**Theorem.** The vertex covering number of a graph with special blocks with \(n\) vertices can be found in time \(O(n^2)\).

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