PARAMETER ESTIMATION FOR DIFFUSION PROCESSES: UNKNOWN FACTS IN THE WELL-KNOWN THEORY

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Abstract

The probabilistic distribution of local time of a homogeneous transient diffusion process is found. To this end, a second order differential equation corresponding to the process generator is considered, and properties of its monotone solutions as functions of a parameter are established with the help of analytic tools. At the same time a probabilistic representation of monotone solutions are used. Combining the techniques of differential equations theory and stochastic processes theory allowed to identify the parameter of exponential distribution of the local time.

Consider a family of one-dimensional homogeneous diffusion processes $\{X_t^x, t \geq 0, x \in \mathbb{R}\}$ defined on a standard filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ by a stochastic differential equation

$$dX_t^x = b(X_t^x)dt + a(X_t^x)dW_t, \quad t \geq 0. \tag{1}$$

Here $X_0^x = x \in \mathbb{R}$ is the initial condition, $\{W_t, t \geq 0\}$ is a standard Wiener process. When considering objects for which the initial condition is irrelevant, we will denote the process in question by. Let the coefficients of equation (1) be continuous and satisfy any conditions for existence of a weak solution. Let also $a(x) \neq 0$, $x \in \mathbb{R}$. Define the following objects related to the family $\{X_t^x, t \geq 0, x \in \mathbb{R}\}$:

1) For $f \in C^2(\mathbb{R})$ define the generator of diffusion process X as

$$\mathcal{L}f(x) = \frac{a^2(x)}{2} f''(x) + b(x)f'(x).$$

2) Define functions

$$\varphi(x_0, x) = \exp\left\{-2\int_{x_0}^{z} \frac{b(u)}{a^2(u)} du\right\}, \quad \Phi(x_0, x) = \int_{x_0}^{x} \varphi(x_0, z)dz, \quad x_0, x \in \mathbb{R}\cup\{-\infty, +\infty\}.$$

Note that for a fixed $x_0 \in \mathbb{R}$ the function $\Phi(x_0, \cdot)$ solves a second order homogeneous differential equation $\mathcal{L}\Phi(x_0, \cdot) = 0$.

3) For $x, y \in \mathbb{R}$ let $\tau_y^x = \inf\{t \geq 0, X_t^x = y\}$ be the first moment of hitting point y, and for $x \in (a, b)$ let $\tau_{a,b}^x = \inf\{t \geq 0, X_t^x \notin (a, b)\} = \tau_a^x \wedge \tau_b^x$ be the first moment of exiting interval (a, b). (We use the convention $\inf \varnothing = +\infty$.)

4) Define a local time of process X^x at $y \in \mathbb{R}$ on interval $[0,t]$ (the factor $a^2(y)$ is included to agree with the general Meyer–Tanaka definition of local times for semimartingales [1]):

$$L_t^x(y) = a^2(y) \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}\{|X_s^x - y| \le \varepsilon\} ds.$$

The limit exists almost surely and defines a continuous non-decreasing process $\{L_t^x(y), t \ge 0\}$ for all $x, y \in \mathbb{R}$. The local time on the whole interval $[0, +\infty)$ will be denoted by $L_\infty^x(y)$.

The aim of the paper is to determine the probabilistic distribution of L_∞^x. We remark that this problem was considered in papers [1] and [2], but the distribution parameter were not determined explicitly, rather as limits of some functionals of solutions to stochastic differential equations, see formula (4).

According to [5, 6], in the case where $\Phi(x, +\infty) = -\Phi(x, -\infty) = +\infty$ for some (equivalently, for all) $x \in \mathbb{R}$, the diffusion process X is recurrent, i.e. $P\{\overline{\lim}_{t\to+\infty} X_t^x = +\infty, \underline{\lim}_{t\to+\infty} X_t^x = -\infty\} = 1$. Therefore, $L_\infty^x(y) = +\infty$ for all $x, y \in \mathbb{R}$ a.s. The behavior of inverse local time process was studied in the recurrent case in [1, 3, 4].

For this reason, we will consider only the case of a transient process, where at least one of the integrals $\Phi(x_0, +\infty)$ and $\Phi(-\infty, x_0)$ is finite.

Further, note that it is enough to consider the case $x = y$. Indeed, by the strong Markov property of the process X, for any $l \ge 0$

$$P(L_\infty^x(y) > l) = P(L_\infty^y(y) > l) P(\tau_y^x < +\infty).$$

The probability $P(\tau_y^x < +\infty) = 1 - P(\tau_y^x = +\infty)$ can be found with the help of well-known formula (see e.g. [8, Section VIII.6, (18)]): for $x \in (a, b)$

$$P(X_{\tau_{a,b}}^x = b) = \frac{\Phi(a,x)}{\Phi(a,b)}.$$

Then the value of probabilty in question depends on x, y, and integrals $\Phi(x, +\infty)$, $\Phi(x, -\infty)$. Specifically, if $x > y$, then

$$P(\tau_y^x = \infty) = \lim_{a\to+\infty} P(X_{\tau_{y,a}}^x = a) = \lim_{a\to+\infty} \frac{\Phi(y,x)}{\Phi(y,a)};$$

so for $x > y$

$$P(\tau_y^x = +\infty) = \begin{cases} \frac{\Phi(y,x)}{\Phi(y,+\infty)}, & \Phi(x,+\infty) < +\infty, \\ 0, & \Phi(x,+\infty) = +\infty. \end{cases} \tag{2}$$

For $x < y$

$$P(\tau_y^x = \infty) = \lim_{a\to-\infty} \Big(1 - P(X_{\tau_{a,y}}^x = y)\Big) = \lim_{a\to-\infty} \frac{\Phi(a,y) - \Phi(a,x)}{\Phi(a,y)} =$$
$$= \lim_{a\to-\infty} \frac{\phi(a,x)\Phi(x,y)}{-\phi(a,y)\Phi(y,a)} = \lim_{a\to-\infty} \frac{-\phi(a,x)\phi(x,y)\Phi(y,x)}{-\phi(a,y)\Phi(y,a)} = \lim_{a\to-\infty} \frac{\Phi(y,x)}{\Phi(y,a)};$$

therefore

$$P(\tau_y^x = +\infty) = \begin{cases} \frac{\Phi(y,x)}{\Phi(y,-\infty)}, & -\Phi(x,-\infty) < +\infty, \\ 1, & -\Phi(x,-\infty) = +\infty. \end{cases} \tag{3}$$

Thus it is indeed sufficient to determine distributions of variables $L^x_\infty(x)$. To this end, we will use the following facts.

1. According to [1, Theorem 1], $P(L^x_\infty(x) > l) = \exp(-l\psi_x(0))$, where

$$\psi_x(0) = \psi^{x,+}(0) + \psi^{x,-}(0), \qquad \psi^{x,\pm}(0) = \pm\frac{1}{2}\lim_{\lambda\downarrow 0}\frac{y'_{\lambda,\pm}(x)}{y_{\lambda,\pm}(x)}; \tag{4}$$

the functions $y_{\lambda,+}$ $y_{\lambda,-}$ are, respectively, the increasing and decreasing solutions of equation ($\lambda > 0$ is fixed)

$$\mathcal{L}y = \lambda y. \tag{5}$$

2. According to [6], functions $y_{\lambda,+}$ and $y_{\lambda,-}$ have probabilistic representations

$$y_{\lambda,+}(x) = \begin{cases} Ee^{-\lambda\tau_0^x}, & x < 0, \\ (Ee^{-\lambda\tau_x^0})^{-1}, & x \geq 0 \end{cases} \qquad y_{\lambda,-}(x) = \begin{cases} Ee^{-\lambda\tau_0^x}, & x \geq 0, \\ (Ee^{-\lambda\tau_x^0})^{-1}, & x < 0. \end{cases} \tag{6}$$

(We set $e^{-\lambda t} = 0$ for $t = +\infty$, $\lambda > 0$.)

3. Any solution $y_\lambda(x)$ to equation (5) admits an integral representation

$$y_\lambda(x) = C_1(\lambda) + C_2(\lambda)\Phi(x_0, x) + 2\lambda\int_{x_0}^{x}\frac{\Phi(s,x)}{a^2(s)}y_\lambda(s)ds, \tag{7}$$

$$y'_\lambda(x) = C_2(\lambda)\varphi(x_0, x) + 2\lambda\int_{x_0}^{x}\frac{\varphi(s,x)}{a^2(s)}y_\lambda(s)ds. \tag{8}$$

Theorem 1. *The following formula holds true:*

$$\psi_x(0) = \frac{1}{2}\left(\frac{1}{\Phi(x,+\infty)} - \frac{1}{\Phi(x,-\infty)}\right), \tag{9}$$

where $\frac{1}{\infty} := 0$.

Corollary 1. *1. In each of the cases: $x = y$; $x < y$ and $-\Phi(0,-\infty) = +\infty$; $x > y$ and $\Phi(0,+\infty) = +\infty$, the local time $L^x_\infty(y)$ is exponentially distributed with parameter $\psi_y(0)$ given by* (9).

2. If $x < y$ and $-\Phi(0,-\infty) < +\infty$, then the local time $L^x_\infty(y)$ is distributed as $\kappa\xi$, where ξ is exponentially distributed with parameter $\psi_y(0)$, κ is an independent of ξ Bernoulli random variable with

$$P(\kappa = 0) = 1 - P(\kappa = 1) = \frac{\Phi(y,x)}{\Phi(y,-\infty)}.$$

3. If $x > y$ and $\Phi(0,+\infty) < +\infty$, then the local time $L^x_\infty(y)$ is distributed as $\kappa\xi$, where ξ is exponentially distributed with parameter $\psi_y(0)$, κ is an independent of ξ Bernoulli random variable with

$$P(\kappa = 0) = 1 - P(\kappa = 1) = \frac{\Phi(y,x)}{\Phi(y,+\infty)}.$$

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