INCOME FORECASTING IN THE HM-NETWORK WITH TIME-DEPENDENT PARAMETERS

E.V. Kosareva
Grodno State University
Grodno, BELARUS
e-mail: koluzaeva@gmail.com

Abstract

The HM-network with time-dependent parameters is investigated in present paper. The system of different-differential equations (DDE) for expected incomes of this network is obtained. In case of the closed HM-network the solution of the system of DDE is illustrated in the example.

1 Introduction

Queueing networks (QN) are widely used in practice as stochastic models of functioning of different real objects in computers, industry, insurance and other fields. Sometimes it is necessary not only to find their time-probability characteristics (an average number of customers, an average service time) but to estimate productivity or operating cost of such systems. For example, it is important to assess income which banking network receives from functioning of its subsystems; in the transport logistics the problem is to calculate the transportation coasts, etc. This problem has led to emergence of a new class of QN – HM(Howard-Matalytski)-networks with incomes. The HM-network needs to consider incomes from the message transitions between the systems. In this paper we describe the method of finding of the income of the Markov HM-network in case, when the number of service lines and the probabilities of message transition between systems depend from time. This particularity is due to the application of such models. For example, in the transport logistics the number of lorries moving between different destinations and the number of loading crews are not constants and change over time.

2 The system of DDE for expected incomes of the HM-network

Let's consider an open exponential QN of the arbitrary topology with one-type messages, which consists of n queueing systems (QS) $S_1, S_2, \dots, S_n$. The network state at the moment t is described by vector $(k,t) = (k_1, k_2, ..., k_n, t)$, where k_i – a count of messages in the system S_i at the moment t, $i = \overline{1,n}$. Let's define $m_i(t)$ – a number of service lines in the system S_i at the moment t, $i = \overline{1,n}$; $p_{ij}(t)$ – a probability of message transition from the system S_i (after its service) to the system S_j at the moment t, i, $j = \overline{0,n}$; the system S_0 corresponds to the external environment. The messages arrive to the network with probability $\lambda\Delta t + o(\Delta t)$ during the time interval $[t, t + \Delta t)$. The service of a message in the system line ends with probability $\mu_i(k_i, t)\Delta t + o(\Delta t)$ during the time interval $[t, t + \Delta t)$, $\mu_i(k_i, t) = \mu_i \min(m_i(t), k_i)$, μ_i – a service rate of each line in the i-th QS, $i = \overline{1, n}$. When a message moves from one system to another it brings to the last one some income and the income of the first system reduces accordingly. It is necessary to find expected (average) incomes of the network systems during time t provided the state of the network in the initial time is known.

Let's consider that the incomes from the message transitions between the systems are determinate functions depend on the network state and time. Let's define I_i – a vector of dimension n which consists of zeros except the i-th component, which equals 1, $i = \overline{1, n}$; $v_i(k, t)$ – an expected income of the system S_i during the time t, if the initial state of the network was k (we consider that $v_i(k, t)$ is differentiable of t); $r_i(k)$ – an income of the QS S_i per time unit if the network is in state k; $r_{0i}(k + I_i, t)$ – an income of the QS S_i if the network has changed its state from (k, t) to $(k + I_i, t + \Delta t)$ during the time Δt; $-R_{i0}\,(k - I_i, t)$ – an income of the QS S_i if the network has changed its state from (k, t) to $(k - I_i, t + \Delta t)$ during the time Δt; $r_{ij}\,(k + I_i - I_j, t)$ – an income of the QS S_i (a loss of the QS S_j) if the network has changed its state from (k, t) to $(k + I_i - I_j, t + \Delta t)$ during the time Δt, $i, j = \overline{1, n}$.

Let's consider that the network state is (k, t). The network can stay in this state or pass into the states $(k - I_i)$, $(k + I_i)$, $(k + I_i - I_j)$ during the time Δt, $i, j = \overline{1, n}$. We has analyzed all possible network's transitions during the time Δt, their probabilities and the incomes of the QS S_i from these transitions. So the system of DDE for expected income of the system S_i was obtained by analogy with [1]:

$$\frac{dv_i(k,t)}{dt} = r_i(k) + \lambda p_{0i}(t) r_{0i}(k + I_i, t) - \left[\lambda + \sum_{j=1}^{n} \mu_j(k_j, t) u(k_j)\right] v_i(k, t) +$$

$$+ \sum_{j=1}^{n} \left[\lambda p_{0j}(t) v_i(k + I_j, t) + \mu_j(k_j, t) u(k_j) p_{j0}(t) v_i(k - I_j, t)\right] +$$

$$+ \sum_{j=1, j\neq i}^{n} \left[\mu_j(k_j, t) u(k_j) p_{ji}(t) v_i(k + I_i - I_j, t) + \mu_i(k_i, t) u(k_i) p_{ij}(t) v_i(k - I_i + I_j, t)\right] +$$

$$+ \sum_{j=1, j\neq i}^{n} \left[\mu_j(k_j, t) u(k_j) p_{ji}(t) r_{ij}(k + I_i - I_j, t) - \mu_i(k_i, t) u(k_i) p_{ij}(t) r_{ji}(k - I_i + I_j, t)\right] +$$

$$+ \sum_{c,s=1, c,s\neq i}^{n} \mu_s(k_s, t) u(k_s) p_{sc}(t) v_i(k + I_c - I_s, t) - \mu_i(k_i, t) u(k_i) p_{i0}(t) R_{i0}(k - I_i, t). \quad (1)$$

The number of the equations in this system equals to number of the network states.

In case of the closed networks the system (1) can be reduced to the system of a finite number of linear nonhomogeneous ODE with variable coefficients. This system can be written in matrix form

$$\frac{dV_i(t)}{dt} = Q_i(t) + A_i(t)V_i(t), \tag{2}$$

where $V_i^T(t) = (v_i(1,t),\, v_i(2,t),\, ...,\, v_i(l,t))$ – an unknown vector of the incomes of the system S_i, l – a number of the network states and it equals C^K_{n+K-1}, K – a number of messages functioning in the network. A general solution of the system (2) is sum of $V_i^0(t)$ (general solution of the system of homogeneous ODE that corresponds to the system (2)) and $V_i^1(t)$ (particular solution of the system (2)). The solution $V_i^0(t)$ is expressed in terms of the fundamental matrix

$$V_i^0(t) = \Phi(t)C, \tag{3}$$

where C – l-dimension vector of arbitrary numbers. If product of matrix $A_i(t)$ and its integral is commutative, i.e. $A_i(t)\int A_i(t)dt = \int A_i(t)dtA_i(t)$, then matrix $\Phi(t)$ can be found as $\Phi(t) = e^{\int A_i(t)dt}$. This property holds in case of symmetric and diagonal matrices.

The most common method of solution of the systems of nonhomogeneous ODE is the method of variation of Lagrange. The vector $C(t)$, which components are continuously differentiable functions of the independent variable t, is substituted in (3) instead of constant vector C:

$$V_i(t) = \Phi(t)C(t). \tag{4}$$

Considering expressions (4) and (2) we find unknown vector $C(t)$: $\dfrac{d\left(\Phi(t)C(t)\right)}{dt} = Q_i(t) + A_i(t)\Phi(t)C(t) \Rightarrow \Phi'(t)C(t) + \Phi(t)C'(t) = Q_i(t) + A_i(t)\Phi(t)C(t)$. Since (3) is solution of homogeneous system that corresponds to the system (2) then $\Phi'(t)C(t) = A_i(t)\Phi(t)C(t)$ and it follows that $\Phi(t)C'(t) = Q_i(t)$. Taking into account that matrix $\Phi(t)$ is nonsingular we multiply the last equation by $\Phi^{-1}(t)$:

$$\Phi^{-1}(t)\Phi(t)C'(t) = \Phi^{-1}(t)Q_i(t), \text{ i.e. } C'(t) = \Phi^{-1}(t)Q_i(t). \tag{5}$$

After integration we'll obtain vector $C(t)$ and solution of the system (4) with the initial conditions $V_i = v_{i0}$.

The matrix $\Phi(t)$ is found as $\Phi(t) = H^{-1}\exp(J)H, J = H^{-1}\left(\int A_i(t)dt\right)H$, where J – Jordan form of the matrix $A_i(t)$, H – a matrix of eigenvectors of $A_i(t)$, H^{-1}– an inverse matrix.

3 Example

Let's consider the closed QN consisting of $n = 3$ QS. The number of messages functioning in the network is $K = 2$. The number of network states is $C^2_{3+2-1} = 6$; the network states are ((2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)) (we numbered

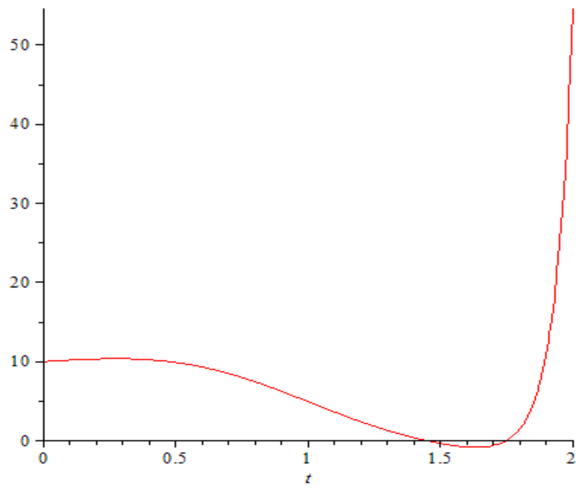

Figure 1: Income $v_1(1,t)$

them from 1 to 6). The probabilities of message transitions between the systems are constants $p_{ij} = 0.5$, $i,j = \overline{1,\ 3}$, $i \neq j$. The number of service lines in systems is $m_i(t) = \max(0, k_i(l))t$, therefore $\mu_i(k_i,t)u(k_i) = \mu_i \max(0, k_i(l))t$, $i = \overline{1,3}$, $l = \overline{1,6}$. Let's suppose $\mu_i = 1$, $i = \overline{1,3}$. The incomes from message transitions are $r_{ij}(k,t) = k_i t + k_j$, $r_i(k) = k_i$, $i,j = \overline{1,\ 3}$. We'll find the expected income of the first QS supposing that we know $v_0 = (10, 15, 5, 0, 5, 5)$ – the income of the first QS at the initial time. Substituting the parameters of the considering network to the system (1) we obtain matrix $A_1(t)$ and vector $Q_1(t)$

$$A_1(t) = \begin{pmatrix} -2t^2 & 0 & 0 & t^2 & t^2 & 0 \\ 0 & -2t^2 & 0 & t^2 & 0 & t^2 \\ 0 & 0 & -2t^2 & 0 & t^2 & t^2 \\ \frac{1}{2}t^2 & \frac{1}{2}t^2 & 0 & -2t^2 & \frac{1}{2}t^2 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & 0 & \frac{1}{2}t^2 & \frac{1}{2}t^2 & -2t^2 & \frac{1}{2}t^2 \\ 0 & \frac{1}{2}t^2 & \frac{1}{2}t^2 & \frac{1}{2}t^2 & \frac{1}{2}t^2 & -2t^2 \end{pmatrix}, \qquad Q_1(t) = \begin{pmatrix} 2 - 2t^3 - 2t^2 \\ t^2(t+1) \\ t^2(t+1) \\ t^3 + 1 - 2t^2 \\ t^3 + 1 - 2t^2 \\ t^3 \end{pmatrix}.$$

Matrices H, H^{-1}, J, $\Phi(t)$ and $\Phi^{-1}(t)$ can be found, for example, by means of the computer algebra system Maple. Note that the exact solution of the system (5) can not be found in some cases. In our example the system (5) is solved approximately by means of the expansion of the integrand in Taylor series. The plot of the income of the first system, if the network was at the first state in the initial time, is presented on pic.1.

References

[1] Matlytski M.A., Koluzaeva E.V. (2009). Markov queueing networks of arbitrary topology with incomes. *Doklady of the National Academy of Sciences of Belarus.* Vol. **53**.N. 3, pp. 10-17.