SOME REMARKS ON ROBUST ESTIMATION OF POWER SPECTRA

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Abstract

Various robust modifications of the classical methods of power spectra estimation, both nonparametric and parametric, are considered. Their performance evaluation is studied in autoregressive models with contamination. It is found out that prospective robust estimates of power spectra are based on robust highly efficient estimates of autocovariances and on robust filtering algorithms. Several open problems for future research are formulated.

1 Introduction

Robust methods ensure high stability of statistical inference under uncontrolled deviations from the assumed distribution model. Much less attention is devoted in the literature to robust estimation of data spectra as compared to robust estimation of location, scale, regression and covariance [3, 4, 8]. However, it is necessary to study these problems due to their both theoretical and practical importance (estimation of time series power spectra in various applications, such as communication, geophysics, medicine, etc.), and also because of the instability of classical methods of power spectra estimation in the presence of outliers in the data [5].

There are several classical approaches to estimation of the power spectra of time series, e.g., via the nonparametric periodogram and the Blackman-Tukey formula methods, as well as via the parametric Yule-Walker and filter-based methods [2]. Thereafter, we may consider their various robust versions: to the best of our knowledge, a first systematic study of them is made in the dissertation of Bernhard Spangl [14].

In what follows, we partially use the aforementioned study as a baseline, mostly follow the classification of robust methods of power spectra estimation given in [14], specify them and propose some new approaches with their comparative performance evaluation. Basically, to obtain good robust estimates of power spectra, we use highly efficient robust estimates of scale and correlation.

Our main goals are both to outline the existing approaches to robust estimation of power spectra and to indicate open problems, so our paper is partially a review and partially a program for future research.

The remainder of the paper is as follows. In Section 2, classical methods of power spectra estimation are briefly enlisted. In Section 3, robust modifications of classical approaches are formulated. In Section 4, a few preliminary results on the comparative study of the performance evaluation of various robust methods are represented. In Section 5, some conclusions and open problems for future research are drawn.

2 Classical Estimation of Power Spectra

2.1 Nonparametric Estimation of Power Spectra

The nonparametric approach to estimation of power spectra is based on smoothed periodograms [2].

Let $x_t, t = 1, \ldots, n$ be a second-order stationary time-series with zero mean. Assume that the time intervals between two consecutive observations are equally spaced with duration Δt. Then the periodogram is defined as follows:

$$\widehat{S}_P(f) = \Delta t/n \Big| \sum_{t=1}^{n} x_t \exp\{-i2\pi f t \Delta t\} \Big|^2 \tag{1}$$

over the interval $[-f_{(n)}, f_{(n)}]$, where $f_{(n)}$ is the Nyquist frequency: $f_{(n)} = 1/(2\Delta t)$.

The Blackman-Tukey formula gives the representation of formula (1) via the sample autocovariances $\hat{c}_{xx}$ of the time series x_t [1]:

$$\widehat{S}_P(f) = \widehat{S}_{BT}(f) = \Delta t \sum_{h=-(n-1)}^{n-1} \hat{c}_{xx}(h) \exp\{-i2\pi f h \Delta t\}\,. \tag{2}$$

It can be seen that the periodogram $\widehat{S}_P(f)$ (1) at the frequency $f = f_k = k/(n\Delta t)$, where k is an integer such that $k \leq \lfloor n/2 \rfloor$, is equal to the squared absolute value of the discrete Fourier transform $X(f_k)$ of the sequence $x_1, \ldots, x_n$ given by the following formula

$$X(f_k) = \Delta t \sum_{t=1}^{n} x_t \exp\{-i2\pi f_k t \Delta t\}\,. \tag{3}$$

To reduce the bias and variance of the periodogram $\widehat{S}_P(f)$, the conventional techniques based on tapering and averaging of periodograms is used [2].

2.2 Parametric Estimation of Power Spectra

The widely used form of a parametric power spectra estimation procedure exploits an autoregressive model of order p for the underlying power spectrum $S(f)$. A stationary $AR(p)$ process x_t with zero mean is described by the following equation

$$x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + \epsilon_t\,, \tag{4}$$

where ϵ_t are i.i.d. Gaussian white noises with zero mean and variance σ_ϵ^2 . The power spectrum estimate $\widehat{S}_{AR}(f)$ has the form [2]

$$\widehat{S}_{AR}(f) = \frac{\Delta t \hat{\sigma}_\epsilon^2}{\left|1 - \sum_{j=1}^{p} \hat{\phi}_j \exp\{-i2\pi f j \Delta t\}\right|^2}\,, \quad |f| \leq f_{(n)}, \tag{5}$$

where $\hat{\phi}_1, \ldots, \hat{\phi}_p$ and $\hat{\sigma}_\epsilon^2$ are the maximum likelihood estimates of the model parameters.

3 Robust Estimation of Power Spectra

3.1 Preliminaries

A natural way to provide robustness of the classical estimates of power spectra is based on using highly robust and efficient estimates of location, scale and correlation in the classical estimates. Here we enlist several highly robust and efficient estimates of scale and correlation.

Robust Scale: The median absolute deviation $MAD_n(x) = med\,|x - med\,x|$ is a highly robust estimate of scale with the maximal value of the breakdown point 0.5, but its efficiency is only 0.37 at the normal distribution [3]. In [7], a highly efficient robust estimate of scale Q_n has been proposed: it is close to the lower quartile of the absolute pairwise differences $|x_i - x_j|$, and it has the maximal breakdown point 0.5 as for MAD_n but much higher efficiency 0.82. The drawback of this estimate is its low computation speed; the computation of Q_n requires an order of greater time than of MAD_n.

In [12], an M-estimate of scale denoted by FQ_n whose influence function is approximately equal to the influence function of the estimate Q_n is proposed

$$FQ_n(x) = 1.483\,MAD_n(x)\,\left(1 - (Z_0 - n/\sqrt{2})/Z_2\right)\,, \tag{6}$$

$$Z_k = \sum_{i=1}^{n} u_i^k e^{-u_i^2/2}\,, \quad u_i = (x_i - med\,x)/(1.483\,MAD_n),\ k = 0, 2; \quad i = 1, \ldots, n.$$

The efficiency and breakdown point of FQ_n are equal to 0.81 and to 0.5, respectively.

Robust Correlation: A remarkable robust minimax bias and variance MAD correlation coefficient with the breakdown point 0.5 and efficiency 0.37 is given by

$$r_{MAD}(x, y) = (MAD^2(u) - MAD^2(v))\big/(MAD^2(u) + MAD^2(v)), \tag{7}$$

where u and v are the robust principal variables [10]

$$u = \frac{x - med\,x}{\sqrt{2}\,MAD\,x} + \frac{y - med\,y}{\sqrt{2}\,MAD\,y}, \quad v = \frac{x - med\,x}{\sqrt{2}\,MAD\,x} - \frac{y - med\,y}{\sqrt{2}\,MAD\,y}.$$

Much higher efficiency 0.81 with the same breakdown point 0.5 can be provided by using the FQ correlation coefficient [11]

$$r_{FQ}(x, y) = (FQ^2(u) - FQ^2(v))\big/(FQ^2(u) + FQ^2(v))\,. \tag{8}$$

3.2 Robust Analogs of the Discrete Fourier Transform

Since computation of the discrete Fourier transform (DFT) (3) is the first step in periodogram estimation of power spectra, consider several robust analogs of the DFT.

Robust L_p-Norm DFT Analogs: As the classical DFT (3) $X(f)$ can be obtained via the L_2-norm approximation to the data $y_t(f) = x_t \exp\{-i2\pi f\,t\Delta t\},\ t = 1, \ldots, n$:

$$X(f) \propto \arg\min_Z \sum_{t=1}^{n} \left|y_t(f) - Z\right|^2,$$

the L_p-norm analog of $X(f)$ (up to the scale factor) is defined as follows :

$$X_{L_p}(f) \propto \arg\min_Z \left\{ \sum_{t=1}^{n} \left| y_t(f) - Z \right|^p \right\}^{1/p}, \quad 1 \le p < \infty. \tag{9}$$

The case of $1 \le p < 2$, and especially the L_1-norm or the median Fourier transform, are of our particular interest [9, 13, 14]. The other possibilities such as the component-wise, spatial medians, and trimmed mean analogs of the DFT are also considered in [9, 14].

Robust Cross-Product DFT Analogs: Here we exploit the well-known relation connecting the cross-product, covariance and means:

$$\sum x_t z_t = n\, cov(x, z) + n\, \overline{x}\, \overline{z}. \tag{10}$$

Since the DFT is decomposed into the real (cosine) and imaginary (sine) parts as $X(f) = X^c(f) + iX^s(f)$, we apply formula (10) to them using robust estimates of covariances and means. Denoting the results of application of formula (10) to cosine and sine parts as $X^c_{CP}(f)$ and $X^s_{CP}(f)$, respectively, we define robust cross-product analog of the classical DFT as follows:

$$X_{CP}(f) = X^c_{CP}(f) + iX^s_{CP}(f)\,. \tag{11}$$

In the case of the conventional estimate of the covariance in (10), namely, the sample covariance $n^{-1}\sum(x_t - \overline{x})(z_t - \overline{z})$, we get the classical definition of the DFT.

3.3 Robust Nonparametric Estimation via Periodograms and the Blackman-Tukey Formula

Now we apply the aforementioned robust analogs of the DFT as well as highly robust and efficient estimates of scale and correlation to the classical nonparametric estimation of power spectra.

Robust Nonparametric Estimation via Periodograms: Here we apply the robust L_p-norm and cross-product analogs of the DFT to the classical periodogram $\widehat{S}_P(f)$ (1):

$$\widehat{S}_{L_p}(f) \propto \left| X_{L_p}(f) \right|^2, \quad \widehat{S}_{CP}(f) \propto \left| X_{CP}(f) \right|^2. \tag{12}$$

In what follows, the L_1- or the median periodogram is of our particular interest.

Robust Nonparametric Estimation via the Blackman-Tukey Formula: In order to construct robust modifications of the Blackman-Tukey formula, we have to consider robust estimates of autocovariances $\widehat{c}_{xx}(h)$ instead of the conventional ones used in (2). These robust estimates are based on the highly robust MAD and FQ estimates of scale and correlation (6) - (8):

$$\widehat{c}_{MAD}(h) = r_{MAD}(x_t, x_{t-h}) MAD(x_t) MAD(x_{t-h}) = r_{MAD}(h) MAD^2(x)\,, \tag{13}$$
$$\widehat{c}_{FQ}(h) = r_{FQ}(x_t, x_{t-h}) FQ(x_t) FQ(x_{t-h}) = r_{FQ}(h) FQ^2(x)\,.$$

To provide the required Teplitz property (symmetry, semipositive definiteness, equal elements on sub-diagonals) of the autocovariance matrix $\widehat{C}_{xx}$ built of the element-wise robust autocovariances (13), a new effective transform is used [6]. Thus, the Teplitz transformed estimates are substituted into formula (2), and the corresponding robust estimates of power spectra are denoted as $\widehat{S}_{MAD}(f)$ and $\widehat{S}_{FQ}(f)$, respectively.

3.4 Robust Parametric Estimation of Power Spectra via the Yule-Walker Equations

A classical approach to estimation of autoregressive parameters $\phi_1, \ldots, \phi_p$ in (4) is based on the solution of the linear system of the Yule-Walker equations [2]:

$$\begin{cases} \widehat{c}(1) = \widehat{c}(0)\widehat{\phi}_1 + \widehat{c}(1)\widehat{\phi}_2 + \cdots + \widehat{c}(p-1)\widehat{\phi}_p \\ \widehat{c}(2) = \widehat{c}(1)\widehat{\phi}_1 + \widehat{c}(2)\widehat{\phi}_2 + \cdots + \widehat{c}(p-2)\widehat{\phi}_p \\ \cdots \\ \widehat{c}(p) = \widehat{c}(p-1)\widehat{\phi}_1 + \widehat{c}(p-2)\widehat{\phi}_2 + \cdots + \widehat{c}(0)\widehat{\phi}_p\,. \end{cases} \tag{14}$$

The estimate of the innovation noise variance is defined by the following equation

$$\widehat{c}(0) = \widehat{c}(1)\widehat{\phi}_1 + \widehat{c}(2)\widehat{\phi}_2 + \cdots + \widehat{c}(p)\widehat{\phi}_p + \widehat{\sigma}^2_\epsilon\,. \tag{15}$$

Substituting robust estimates of autocovariances (13) into (14) and (15), we get the robust analogs of the Yule-Walker equations. Solving these equations, we arrive at the robust estimate of power spectra in the form (5).

3.5 Robust Parametric Estimation via Filtering

A wide collection of robust methods of power spectra estimation is given by various robust filters (Kalman, Masreliez, ACM-type, robust least squares, filter-cleaners, etc.) providing preliminary cleaning the data with the subsequent power spectra estimation. An extended comparative experimental study of robust filters is made in [14]; below we compare some of those results with ours.

4 Performance Evaluation

In Monte Carlo experiment, the autoregressive model AR(2): $x_t = x_{t-1} - 0.9\,x_{t-2} + \epsilon_t$ and AR(4): $x_t = x_{t-1} - 0.9\,x_{t-2} + 0.5\,x_{t-3} - 0.1\,x_{t-4} + \epsilon_t$ are used, where the innovation noise $\epsilon \sim (1-\alpha)\,N(x;0,1) + \alpha\,N(x;0,10)$ with different levels of contamination α. Nonparametric and parametric estimates are tested on different sample sizes n and numbers of trials M. Some results are exhibited in Figs. 1-4.

5 Concluding Remarks

- From Fig. 1 it follows that the classical periodogram is catastrophically bad under contamination. From Figs. 1-2 it can be also seen that the robust Yule-Walker method considerably outperforms robust filter methods.
- From Fig. 3-4 it follows that the bias of estimation by the robust Yule-Walker method increases with growing dimension and contamination. Under heavy contamination, the nonparametric median periodogram and the robust Blackman-Tukey method outperform the Yule-Walker method in estimating the frequency location, although they have a considerable bias in amplitude.

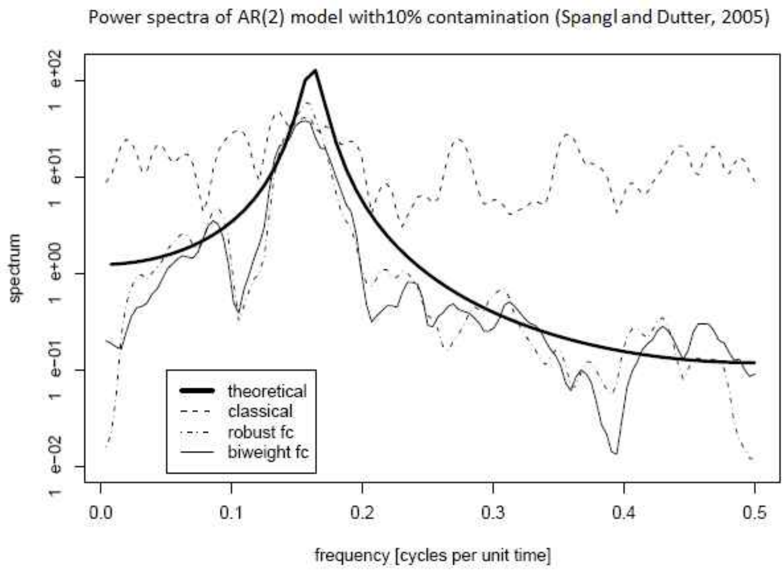

Figure 1: Power spectra estimation with robust filter-cleaners: $n = 100, M = 400$.

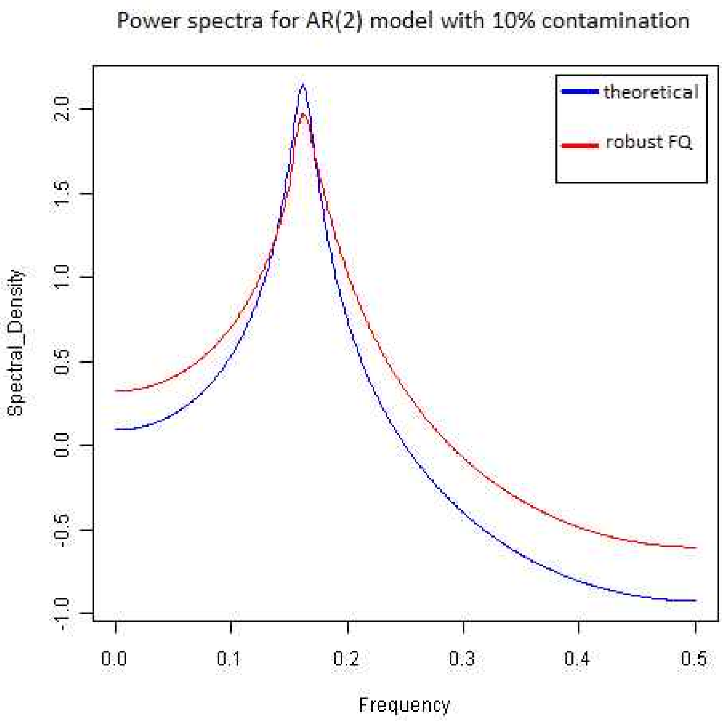

Figure 2: Power spectra estimation by the Yule-Walker method: $n = 128, M = 2000$.

- The obtained results are preliminary indicating many open problems, most theoretical: analysis of the statistical asymptotic properties of the proposed methods, reducing their bias and variance on finite samples, study of the properties of the

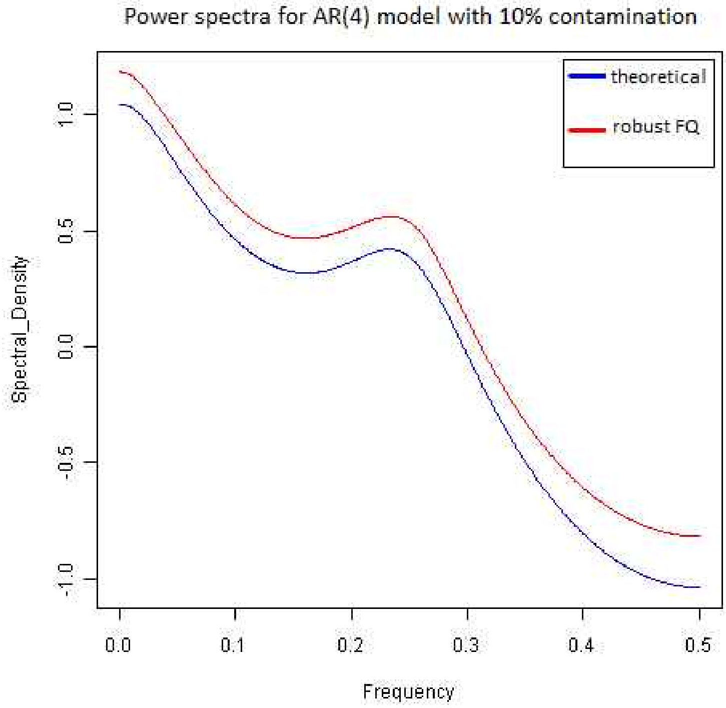

Figure 3: Power spectra estimation by the Yule-Walker method: $n = 128, M = 2000$.

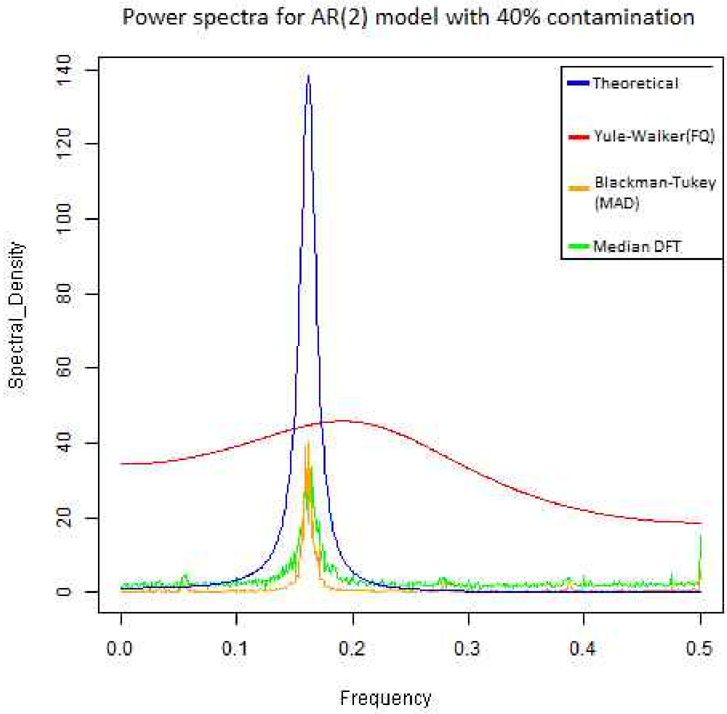

Figure 4: Power spectra estimation by nonparametric methods: $n = 1024, M = 50$.

direct and inverse L_p-norm analogs of the Fourier transform ($1 < p < \infty$) as well as of the direct L_1-norm analog, testing cross-product analogs of the DFT.

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