TESTING FOR STRUCTURAL HETEROGENEITY IN VECTOR AUTOREGRESSIVE MODEL BY MEANS OF STATISTICAL CLASSIFICATION METHODS

V. Malugin
Belarusian State University
Minsk, BELARUS
e-mail: Malugin@bsu.by

Abstract

The problems of analysis of complex systems, described by VARX model with structural heterogeneity, are studied in this article. It is supposed that structural heterogeneity may be conditioned as a result of two reasons: 1) presence of different system state classes; 2) presence of structural changes. It is supposed that every state of the system is characterized by the stationary model, i.e. the general model is a segmented stationary VARX. The methods of discriminant analysis for VARX models are suggested for the classification of the complex system states. The results of the classification are afterwards used for the estimation of the moments of single or multiple structural changes by means of statistical testing. The simulation results show the acceptable accuracy of the proposed methods.

1 Segmented stationary VARX and problems of their analysis

The problem of structural heterogeneity in multivariate econometric models has attracted considerable attention of researchers during the last decade. A number of well-known results concerning testing and estimating the moments of the structural change have been obtained for the so called segmented stationary models [5]. The theoretical research mainly concern studying of limiting distributions for the test statistics. The proposed algorithms are quite complex and their investigation on the small samples has only been approached with the Monte Carlo experiments.

In this paper, we propose quite simple algorithms for estimation of switching regime (structural changes) moments. These algorithms are based on statistical methods for classification of the multivariate observations [2]. The results of the asymptotic risk analysis for the considered type of decision rules with respect to the multivariate regression observations classification are presented in [4].

Suppose that for a complex system with $L$ ($L \geq 2$) classes of states (regime of functioning) the class of state at time $t$ is described by a random variable $d_t \in S(L)$. The relationship between endogenous $x = (x_1, ..., x_N)' \in X \subseteq \mathbb{R}^N$ and exogenous $z = (z_1, ..., z_M)' \in Z \subseteq \mathbb{R}^M$ variables is described by a vector autoregressive model with a $p$-th order $VARX(p)$ [3] that has the following heterogeneous structure:
\[ x_t = A'_d(t)X_{t-1} + B'_d(t)z_t + \eta_{d(t),t}, \quad t = 1, \ldots, T, \quad \eta_{t,a} \sim \text{NI}(0, \Sigma_a), \]

where \( X'_{t-1} = (x'_{t-1}, \ldots, x'_{t-p}) \in \mathbb{R}^{Np} \), \( X_0 = (x'_0, \ldots, x'_{1-p}) \in \mathbb{R}^{Np} \) - given initial value, \( z_t \in \mathbb{Z} \subseteq \mathbb{R}^M \) - stationary vector time series; \( A'_d(t) = (A'_d(t),1, \ldots, A'_d(t),p) \) is a block matrix with dimensions \( N \times pN \), where matrices \( \{A_{a,l}\} \ (l = 1, \ldots, p) \) satisfy the stationarity conditions for VAR \( \forall \alpha \in S(L) \) [3].

It is assumed that the model (1) is valid for the structural heterogeneity assumption:

\[ P\{B_{a}\zeta = B_{\gamma}z\} = 0, \quad z \in \mathbb{Z} \subseteq \mathbb{R}^M, \quad B_{\alpha} \neq B_{\gamma}, \quad \alpha \neq \gamma, \quad \alpha, \gamma \in S(L). \]

The model (1) is a segmented stationary model within the range \( \mathfrak{N} = [1, \ldots, T] \). The range \( \mathfrak{N} \) includes \( s \geq 1 \) switching of regimes (structural changes) at unknown moments \( \{\tau_l\} (l = 1, \ldots, s) \) \( 1 < \tau_1 \ldots < \ldots < \tau_s < T \).

Both state vector \( d_t \in S(L) \) and parameters \( \{A_{d(t)}, B_{d(t)}, \Sigma_{d(t)}\} \) are unknown. A learning realization of time series \( (X_\alpha^L, Z_\alpha^L) \) is given, provided that \( X_\alpha^L = (x_{11}, \ldots, x_{TTL}) \in \mathbb{X}^{TL}, \quad Z_\alpha^L = (z_{11}, \ldots, z_{TTL}) \in \mathbb{Z}^{TL}, \quad T \rightarrow \infty \) \( \forall \alpha \in \{0, 1, \ldots, s\} \), \( T \rightarrow \infty \) \( \forall \alpha \in \{0, 1, \ldots, s\} \).

We consider the following problems:

- Problem 1. Estimation of state vector \( d = (d_t) \in S^T(L) \);
- Problem 2. Estimation of the switching regime (structural changes) moments \( \{\tau_l\} (l = 1, \ldots, s) \);
- Problem 3. Estimation of the parameters \( \{A_{d(t)}, B_{d(t)}, \Sigma_{d(t)}\} \) for the given moments \( \{\tau_l\} \) and testing of the statistical significance of the structural changes.

The estimator of state vector \( \hat{d} = (\hat{d}_t) \in S^T(L) \) is obtained by means of statistical classification rules applied to the tested time series \( \{x_t, z_t\} (t = 1, \ldots, T) \) (Problem 1). The solution of Problem 2 is based on the elimination of false signals for the structural changes in the sequence \( \hat{d}_1, \ldots, \hat{d}_T \in S(L) \). The proportion of false signals for different classes of states is determined by conditional probabilities of errors \( \{r(l)\} \ (l \in S(L)) \) for the used classification decision rules. These probabilities are estimated on the learning sample. To recognize a real signal of structural changes we use a statistical test. Problem 3 is solved by means of traditional estimation and hypothesis testing methods with the use of dummy variables for the given values \( \tau_1, \ldots, \tau_s \).

Peculiarities of application for the described approach depend on the version of the model (1) and the conditions for solving of problems 1, 2 and 3, which may be as follows:

- C.1. Equation (1) describes a model with \( L \) different classes of states, provided that \( L \) is less than the total number of regimes, i.e. \( 2 \leq L < s + 1 \) \( s \geq 2 \). The learning realizations of time series \( (X^L_\alpha, Z^L_\alpha) \forall \alpha \in S(L) \) are given.

- C.2. Equation (1) describes a model with multiple structural changes where \( L = s + 1 > 2 \). In this case \( T_{l1}^L = |J_{l-1,l}| = \tau_l^- - \tau_{l-1}^+ + 1 \) \( l = 1, \ldots, s + 1 \) where \( J_{l-1,l} = (\tau_{l-1}^+, \ldots, \tau_l^-) \) and \( J_l = [\tau_l^-, \ldots, \tau_l^+] \) is an interval including \( l \)-th structural change.
2 Statistical Classification Rules

For solving of Problems 1 and 2 under conditions C.1, C.2 we propose a "plug-in-rule" Bayesian classifier for the “point-wise classification” of observations described by the model (1) provided that $P \{d_t = \alpha\} = \pi_\alpha > 0 (\alpha \in S(L))$. The problem of the “point-wise classification” is represented in the estimation of vector $d_t \in S(L)$ for the given values $(x_t, z_t), \ldots, x_{t-1}, z_{t-1}, t = 1, \ldots, T$.

Theorem 1. Let model (1) satisfy the mentioned above assumptions and $\{\hat{A}_{\alpha,l}\}, \{\hat{B}_{\alpha}\}, \{\hat{\pi}_\alpha\} (l = 1, \ldots, p)$ be LS (MLE)-estimates of its parameters on the learning sample $(X^L_\alpha, Z^L_\alpha)$, then the consistent “point-wise classification” plug-in-rule for the given values $(x_t, z_t), \ldots, x_{t-1}$ is described by the following relations:

$$\hat{d}_t \equiv \hat{d}(x_t; z_t) = \arg \min_{\alpha \in S(L)} \left\{ \text{tr}(\hat{\Sigma}^{-1}_\alpha S_\alpha) + \ln |\hat{\Sigma}_\alpha| - \frac{2\ln \hat{\pi}_\alpha}{T} \right\},$$

(2)

$$S_\alpha = \tilde{\eta}_{\alpha,t} \tilde{\eta}_{\alpha,t}^\tau, \tilde{\eta}_{\alpha,t} = x_t - \hat{A}_{\alpha}^\tau x_{t-1} - \hat{B}_{\alpha} z_t, t = 1, \ldots, T.$$  

(3)

3 Testing for the Structural Changes

The proposed method for estimation of the switching regime (structural changes) moments includes three following steps:

Step 0. Development and probability error evaluation of the plug-in-rule. At this step we use the learning sample $X^L_\alpha \in X^T(L), Z^L_\alpha \in Z^T(L)$ in order to estimate the conditional probabilities of errors $\{\tilde{r}^{(l)}\}(l \in S(L))$ and confidence intervals $[\delta_l^- (q, \tilde{T}_l), \delta_l^+ (q, \tilde{T}_l)]$ with the given confidence level $q$ for the plug-in-rule (2), (3).

Step 1. Estimation of the state vector. Plug-in-rule (2)–(3) is used to classify $\{x_t, z_t\}(t = 1, \ldots, T)$. The results of this step is the estimate $\hat{d} = (\hat{d}_1, \ldots, \hat{d}_T) \in S^T(L)$.

Step 2. Estimation of the switching regime (structural changes) moments. The following steps are carried out at this step:

1. Random sequence $\hat{d}_1, \ldots, \hat{d}_T \in S(L)$ is divided into $\gamma = T/m$ fragments with a fixed length $m$ ($m \ll T$). For each fragment $\hat{d}_j = (\hat{d}_{(j-1)m+1}, \ldots, \hat{d}_{jm}) \in \mathbb{R}^m (j = 1, \ldots, \gamma)$ the following two statistics are calculated:

$$k_j^{(l)} = \sum_{i=(j-1)m+1}^{jm} \delta_{d_i,1}, \quad k_j^{(l)} = \frac{k_j}{m} (j = 1, \ldots, \gamma), \quad \{\tau_l\} (l = 1, \ldots, s);$$

(4)

where $k_j^{(l)} \in \{0, 1, \ldots, m\}$ – frequency errors, provided that the complex system is in a state $\Omega_l$, while the alternative state is $\Omega_{l+1}$, and the conditional probabilities of errors are equal to $r_0^{(l)} (l = 0, 1, \ldots, L)$. It is known that statistics $k_j^{(l)}$ has a binomial distribution with parameters $m, r_0^{(l)} [1]$.

2. For each fragment $j$ ($j = 1, \ldots, \gamma$) the following hypotheses are tested with respect to the given values $0 < \alpha_0, \beta_0 \ll 1$ for the probabilities of the first and the second error type respectively:
\[ H_0 : r = r_0^{(l)} , \]
\[ H_1 : r = r_1^{(l)} > r_0^{(l)} , \]

where \( r_0^{(l)} \) and \( r_1^{(l)} = 1 - r_0^{(l+1)} \) are acceptable and unacceptable levels of errors for class \( \Omega_t \) correspondingly, provided that there is no strong evidence of the structural change \((H_0 \text{ is not rejected}), \ r_0^{(l)} \text{ and } r_1^{(l)} \) are determined by the formulas:

\[
\begin{align*}
 r_0^{(l)} &= \begin{cases} 
 \tilde{r}^{(l)}(t), & \text{if } \tilde{T}_t \rightarrow \infty, \\
 \delta_t^+(q, \tilde{T}_t), & \text{if } \tilde{T}_t < \infty,
\end{cases} \\
 r_1^{(l)} &= \begin{cases} 
 1 - \tilde{r}_1^{(l+1)}, & \text{if } \tilde{T}_{t+1} \rightarrow \infty, \\
 1 - \delta_{t+1}^+(q, \tilde{T}_{t+1}), & \text{if } \tilde{T}_{t+1} < \infty,
\end{cases}
\end{align*}
\]

\[
\delta^+(q, T^E) = \frac{1}{T^E + \phi^2_q} \left( k + \frac{\phi^2_q}{2} \mp \phi_q \sqrt{\frac{k(T^E - k)}{T^E} + \frac{\phi^2_q}{4}} \right),
\]

where \( [1]: k \) – the number of errors in the point-wise classification of \( T^E \) observations; \( \phi_q \equiv \Phi^{-1}(q) \) – quantile of the standard normal distribution \( N_1(0, 1) \) with the confidence level \( q \).

The test for hypotheses (5) is used for the approximation of binomial distribution with normal distribution and has the form

\[ \text{hypothesis } H_0 \begin{cases} 
 \text{is not rejected if } \kappa_j^{(l)} < \kappa^{(l)}, \\
 \text{rejected if } \kappa_j^{(l)} \geq \kappa^{(l)},
\end{cases} \]

where

\[
\kappa^{(l)} = r_0^{(l)} + \frac{0.5}{m} + \Phi^{-1}(1 - \alpha_0) \frac{\sigma_0}{\sqrt{m}} \quad \text{and} \quad \sigma_0 = \sqrt{r_0^{(l)}(1 - r_0^{(l)})},
\]

\[
m = \left( \frac{\sigma_0 \Phi^{-1}(1 - \alpha_0) + \sigma_i \Phi^{-1}(1 - \beta_0)}{\omega} \right)^2 , \quad \sigma_i = \sqrt{r_i^{(l)}(1 - r_i^{(l)})(i = 0, 1),} \quad \omega = r_1^{(l)} - r_0^{(l)}. \]

4 Performance Evaluation

We propose several options for the basic algorithm that implements the method described above. These options can be used in cases C.1 or C.2 with single and multiple structural changes models. Let us present Monte Carlo experiment results for one of these algorithms for complex system with different classes of system states. As a test model we use the VARX with partial structural changes in the matrix of regression coefficients for which: \( N = 2, \ M = 3, \ p = 1, \ T = 1000 \). Vector of exogenous variables \( z = (z_i) \in Z \) has a uniform distribution in \( Z = Z^M \in \mathbb{R}^M, \ Z = [1, 10] \). We consider three versions of the model VARX, which have differences in the value of the Mahalanobis distance between the classes \( \Delta(\hat{z}) = \sqrt{\hat{z}'(B_1 - B_0)'\Sigma^{-1}(B_1 - B_0)}\hat{z} \) at point \( \hat{z} = (\hat{z}_i) \in Z, \ \hat{z}_i = 5.5 \in Z \ (i = 1, 2, 3) \). Testing time series in all cases includes three structural changes at times \( \tau_1 = 251, \ \tau_2 = 501, \ \tau_3 = 751 \). The results of Monte Carlo experiment are given in Tables 1, where \( r_0(\hat{z}) \) is a probability error of the Bayesian classifier for the Mahalanobis distance, that is equal to \( \Delta(\hat{z}) \).

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Table 1: Estimates of the switching regime moments

<table>
<thead>
<tr>
<th>( \Delta(\tilde{z}) )</th>
<th>r_0(\tilde{z})</th>
<th>( \tau_1 (\tau_1 = 251) )</th>
<th>( \tau_2 (\tau_2 = 501) )</th>
<th>( \tau_3 (\tau_3 = 751) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.23</td>
<td>0.2693</td>
<td>286</td>
<td>507</td>
<td>767</td>
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<tr>
<td>2.46</td>
<td>0.1093</td>
<td>250</td>
<td>505</td>
<td>755</td>
</tr>
<tr>
<td>4.92</td>
<td>0.0069</td>
<td>252</td>
<td>500</td>
<td>752</td>
</tr>
</tbody>
</table>

### 5 Conclusions

Under certain conditions regarding the possibility of forming the learning sample the proposed algorithm for estimation of the switching regimes (or structural changes) moments in VARX has the following benefits: it is less dependent on both the particular specifications of econometric models and the a priori assumptions about the model of structural changes; it is relatively simple in terms of practical implementation and is better applicable for estimating moments of single structural changes, multiple structural changes and structural changes in real time.

### References


