MINIMIZING VALUE-AT-RISK IN A PORTFOLIO OPTIMIZATION PROBLEM USING A MULTIOBJECTIVE GENETIC ALGORITHM*

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ABSTRACT:
In this paper we develop a general framework for market risk optimization. The model is valid for any given risk measure (e.g. deviation, Value-at-Risk, Conditional Value-at-Risk...). Our empirical procedure is focused on VaR. The reason for choosing this particular risk measure is the complexity of the risk-return optimization problems that it generates (non-convex and non-differential). We solve the problem using a multiobjective genetic algorithm (GA). The algorithm is very efficient and it can handle hundreds of assets in reasonable computer time. One of the advantages of this approach is that it is easily extendable. We could simultaneously introduce cardinality constrains, non-linear, non-differentiable transaction cost structures, buy-in thresholds or round lots, all of them constraints that lead to non-convex, non-differential models or consider another risk measure without needing to modify the GA.

KEY WORDS: Artificial intelligence; Investment criteria; Portfolio selection

1. INTRODUCTION

In Markowitz (1952) risk is defined in terms of the possible variation of expected portfolio returns. Traditionally it has been assumed that the portfolio return is normally-distributed. Under this assumption, two statistical measures, mean and standard deviation, can be used to balance return and risk. The optimal portfolio is selected from the efficient frontier that is the set of Pareto optimal portfolios with two conflicting criteria: mean and variance. However, it is known that the distributions of many financial return series are non-normal, with skewness and kurtosis pervasive (Tsay, 2002). Moreover, the use of standard deviation as an appropriate measure for risk implies that investors weigh the probability of negative returns equally against positive returns. This has motivated that many authors use alternative risk measures or include additional higher-order moments to solve the portfolio selection problem (Konno *et al*., 1993).

The fact that investors have loss aversion has guided the design of new risk measures. Value-at-Risk (VaR) and Conditional VaR (CVaR) are two of these measures. VaR of a portfolio is the lowest amount such that the loss will not exceed it with probability 1-α (usually 95% or 99%). CVaR is the conditional expectation of losses above the VaR. Currently it is common practice for investment analysts to define portfolio selection risk by means of these two measures.

The minimization of the variance or the CVaR for a given return is a convex, differential problem that can be solved using standard methods. For instance, Rockafellar (2000) show that CVaR can be efficiently minimized using linear programming and nonsmooth optimization techniques. Andersson *et al*. (2001) present an approach for minimizing CVaR. In this case, CVaR is used for quantifying credit risk of emerging market bonds portfolios.

On the other hand, when VaR is considered as the risk measure to minimize, it leads to a non-convex and non-differential risk-return optimization problem. This problem is tackled in the literature in various ways. Using Arzac (1977) framework, Jansen *et al*. (2000) and Campbell *et al* (2001) use a safety-first theory approach to maximize expected return subject to a VaR constraint. Jansen *et al* (2000) compute VaR using a semi-parametric approach that only models the tail of the distribution parametrically. Gaivoronski (2005) present a method for calculating the minimum VaR of a portfolio subject to a specific minimum return by using a smoothing algorithm which smooths out the local noisy component of the VaR function. At present, only heuristic methods are available to find VaR-optimal portfolios Gilli *et al* (2006).

* Financial support under project ECO 2008-02846 is acknowledged

In order to avoid the use of smoothing techniques, in this paper we propose a genetic algorithm (GA) approach to deal with the problem of minimizing VaR or any other measure that leads to non-convex and non-differential risk-return optimization problems.

The application of GAs to the portfolio selection problem is not new in the literature. For instance, Yang (2006) introduces a GA into a state dependent dynamic portfolio optimization system in order to improve the portfolio efficiency over the classical mean-variance method by reducing the estimation risk. Lin (2008) uses GA for portfolio selection problems with minimum transaction lots. Ong *et al*. (2005) provide an application of multiobjective genetic algorithm to obtain efficient frontiers to improve the accuracy of the mean-variance approach when a small sample is available. Subbu *et al.* (2005) present an optimization approach in which a multiobjective genetic algorithm is combined with linear programming to identify efficient frontiers under multiple risk measures. While the majority of the works cited previously solve the mean-variance problem, in this paper the risk measure under consideration is VaR computed with historical simulation. We evaluate differences between the subset of efficient portfolios when VaR or variance is the risk measure minimized. Finally, a comparison of in-sample and out-sample results and periods under different conditions, that is, high and small volatility and bullish and bearish markets is shown.

Also, we present a multiobjective evolutionary approach that optimizes simultaneously the return and the level of risk and evaluates the differences between mean-variance and mean-VaR efficient portfolios. Coello (2006) presents a very complete overview of what it is called multiobjective evolutionary optimization, i.e. the use of evolutionary algorithms (of which GAs are a sub-class) to solve multiobjective problems. A multiobjective optimization problem is a problem which has two or more objectives to optimize simultaneously. Usually these objectives are in conflict, that is, improving one of them leads to worsening the others. Therefore, most multiobjective optimization problems do not have a single solution but a set of solutions equally optimal. These solutions are "trade-offs" among the objectives and form what it is called the Pareto optimal set. By definition, a solution is Pareto optimal (or nondominated) "if there exists no other feasible solution which would decrease some criterion without causing a simultaneous increase in at least one other criterion".

One of the benefits of using GAs for multiobjective optimization is that GAs work with a population of individuals, which allows us to find several nondominated solution in a single run. Also, GAs are less susceptible than other techniques to the non-convexity of the search space. In this article, we conduct a study on multiobjective evolutionary optimization of market risk using VaR to quantify the market risk. The approach is very attractive since it is a framework that can be applied to any market risk measure.

The remainder of this paper is organized as follows: section 2 describes the theory behind the portfolio optimization problem. Section 3 explains the basic notions of GAs while section 4 describes how they have been applied to the portfolio optimization problem. In section 5 the data used are presented and analyzed. Section 6 shows the results yielded by the GAs to the optimization problem, and the related conclusions as well as the future lines of work are reported in section 7.

2. PORTFOLIO OPTIMIZATION

The portfolio optimization problem arises from the decisions investors have to make on how to invest their available budgets given a set of financial assets. The investor decides the percentage of the budget in the assets such $w_i \geq 0$, i=1,...,n (where n is the number of available assets and short sales are not permitted) and $w'1 = 1$ is the budget constraint. In our study, a budget of one unit is supposed without loss of generality. The return of the portfolio, $R_p = w'R$, is treated as random variable since the generated assets returns, R, are unknown. The expected return of the portfolio w is $E(R_p) = w'E(R)$.

The portfolio optimization problem lies in minimizing the risk of an investment for a desired level of expected return. Suppose that $\rho(\cdot)$ is a risk measure. Then for a given expected return R^* the optimal portfolio is the solution of the following optimization problem:

$$\begin{aligned} &\min_{w} \rho(R_p) \\ &w'E(R) \geq R^* \\ &w'1 = 1 \\ &w \geq 0 \end{aligned} \qquad (1)$$

where $\rho(\cdot)$ is a function that assigns to each portfolio w its risk, $\rho(w) \in R$. When Problem (1) is solved, we can obtain a numerical relationship between the expected return $E(R_p)$ and $\rho(R_p)$ for different values of the parameter R^*. This is the efficient frontier in which a rational investor is located in function of the risk that he wants to assume.

Traditionally, empirical finance has identified risk with return variability. Portfolio theory of Markowitz proposes a classical measure of dispersion, variance, as the risk measure. Hence Markowitz problem takes $\rho(\cdot)$ equal to the variance, $\rho(R_p) = w'\Omega w$, where Ω is the covariance matrix and the objective function is:

$$\min_{w} w'\Omega w \tag{2}$$

Although some analytical methods are well-known for solving objective function (2), if the problem is extended by introducing more complex risk measures, it requires a new efficient approach. Developing this approach on the basis of classical methods might not be possible due to the irregularity of the objective function and the search space.

The risk definition has changed due to the desire of financial industry and investors of limiting downside risk by putting an upper bound on the maximum loss. In this work, we consider the Value-at-Risk (VaR) as an appropriate risk measure. VaR is defined as the maximum expected loss on an investment over a specified horizon given a confidence level 1-α. Usually α is fixed to be a 5% or 1%. In our study, we took $\rho(\cdot)$ equal to the VaR definition given in Jorion (2001). That is,

$$\rho(R_p) = VaR(R_p) = E(R_p) - q_{\alpha}(R_p) \tag{3}$$

and the objective function is,

$$\min_{w} E(R_p) - q_{\alpha}(R_p) \tag{4}$$

where $q_{\alpha}(R_p)$ is the α-quantile of R_p.

There are several ways of computing $VaR(R_p)$. Some researchers assume $E(R_p)$ to be zero when VaR is calculated daily; then $VaR(R_p) \approx -q_{\alpha}(R_p)$. VaR can also be estimated assuming that the distribution of the returns serie belongs to a parametric family. Approaches to quantify VaR such as delta-normal, delta-gamma or Monte Carlo simulation method rely on the normality assumption or other prespecified distributions. These approaches have several drawbacks, such as the estimation of parameters and whether the distribution fit properly the data in the tail or not (Baixauli, 2004). In our analysis we computed the VaR by historical simulation using Equation (3). Hence, $q_{\alpha}(R_p)$ is the empirical α-quantile of the actual historical data. This specification is valid for any underlying distribution, discrete or continuous, fat or thin-tailed.

The portfolio optimization problem when objective function (2) is considered implies that investor preferences are defined in mean-variance space and Pareto efficient portfolios make up the efficient frontier in this space, which we call σ-efficient frontier. Otherwise, when objective function (4) is considered the investor preferences are expressed as a function of return and maximum loss and a VaR-efficient frontier made up of Pareto efficient portfolios in the mean-VaR space has to be found. As we pointed out, Problem (1) becomes a non-convex and non-differential risk-return optimization problem with the objective function (4). For this reason, we use a multiobjective GA approach to find VaR-efficient portfolios and σ-efficient portfolios by solving the following optimization problems:

$$\begin{aligned} &\min_{w} E(R_p) - q_{\alpha}(R_p) \\ &\max_{w} w'E(R) \\ &w'1 = 1 \\ &w \geq 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &\min_{w} w'\Omega w \\ &\max_{w} w'E(R) \\ &w'1 = 1 \\ &w \geq 0 \end{aligned} \tag{6}$$

It must be highlighted that the inclusion of cardinality constrains, nonlinear and non-differentiable transaction cost structures, buy-in thresholds or round lots would also turn the classical mean-variance problem (6) into a non-convex, non-differential model (Lin, 2008). This would motivate the application of multiobjective Gas for an efficient optimization of portfolio structures. Finally, is worthy to evaluate the differences between σ-efficient portfolios and VaR-efficient portfolios since they are a subset of feasible portfolios.

3. BASICS OF GENETIC ALGORITHMS

Genetic Algorithms (GAs) (Holland, 1975; Goldberg, 1989) are stochastic optimization techniques that mimic the way species evolve in nature. In natural evolution many organisms evolve by means of two mechanisms: natural selection and sexual reproduction.

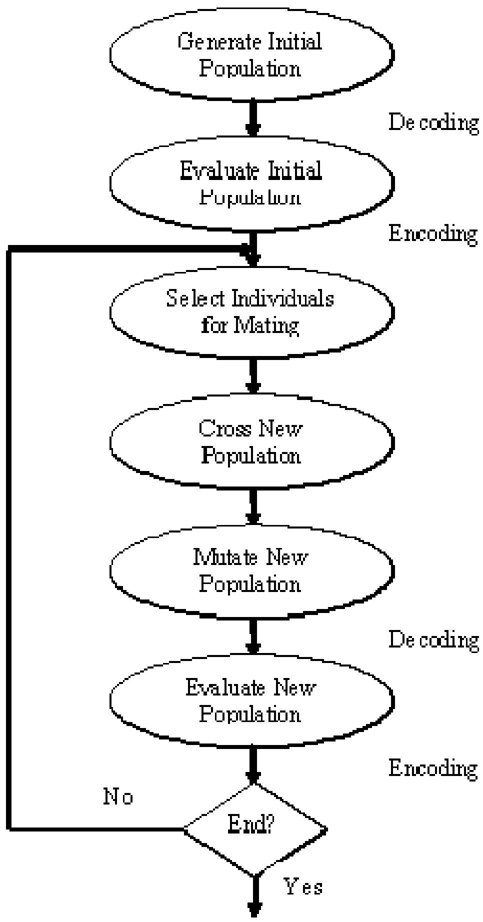
GAs emulate this process by encoding the points of the search space (called individuals) in a chromosome-like shape and evolving a population of them through a number of generations using mechanisms drawn from natural evolution. The better suited to the optimisation problem an individual is, the more chances it has to produce offspring for the next generation. As the generations progress, this results in the prevalence in the population of stronger solution over weaker ones. Thus, the evolution process tends to near optimal solutions.

A GA initiates the process of searching by randomly generating an initial population of possible solutions. The performance of each solution is evaluated using a fitness function, which is a measure of how good the performance of the solution is. Then, a new generation is produced according to the three main operators of the GA: selection, crossover and mutation.

Selection determines which solutions are chosen for mating according to the principal of survival of the fittest (i.e. the better the performance of the solution, the more likely it is to be chosen for mating and therefore the more offspring it produces). In this work we used tournament selection. The tournament selection method works by choosing a group of q individuals randomly from the population and selecting the best individual in terms of fitness from this group.

Crossover allows an improvement in the species in terms of the evolution of new solutions that are fitter than any seen before. The crossover operator combines the features of two parents to create new solutions. One or several crossover points are selected at random on each parent and then, complementary fractions from the two parents are spliced together to form a new chromosome.

Figure 1. Flow Chart of a GA



Mutation reintroduces values that might have been lost through selection or crossover, or creates totally new features. The mutation operator alters a copy of a chromosome. One or more locations are selected on the chromosome and replaced with new randomly generated values. Mutation is used to help ensure that all areas of the search space remain reachable providing higher variation in the chromosomes of each population. It also allows the reintroduction of features that might have been lost during the selection procedure.

The cycle selection-crossover-mutation-evaluation is performed until a termination criterion is met (for instance, a predetermined number of generations).

4. APPLICATION OF GAs TO THE PORTFOLIO OPTIMIZATION PROBLEM

In this section we briefly describe the multiobjective GA framework that we used for portfolio optimization. The GA implementation is based on ECJ (\url{http://cs.gmu.edu/\~{}eclab/projects/ecj}), a research evolutionary computation system in Java developed at George Mason University's Evolutionary Computation Laboratory (ECLab).

4.1. Encoding/decoding

Each individual was encoded as a vector of integers ranging from 0 to 99. Every element of the vector represents the percentage of the budget invested in that particular asset ($w_i^{GA} \geq 0, i = 1,...,n$). Therefore, the length of the vector equals the number of assets available in the portfolio. However, the summation of these weights will not be 1, violating the constraint $w'1 = 1$. This constraint imposes the need of normalizing the vector during the decoding process as follows:

$$w_i = \frac{w_i^{GA}}{W} \tag{7}$$

where w_i represents the new weight invested in asset *i* after normalization, and *W* is the summation of all the elements of the vector, that is $W = \sum_{k=1}^{n} w_k^{GA}$.

4.2. Multiobjective GA

According to the survey on evolutionary multiobjective optimization presented in Coello (2006), a number of algorithms have been designed since the mid-1980s in order to apply evolutionary techniques to multiobjective problems. Roughly, these algorithms could be split into two "generations". The algorithms proposed in the second generation outperformed those of the first generation and are characterized by te use of "elitism", that is, the use of an additional population where the non-dominated individuals found along the evolution are stored. Some of the most representative algorithms from this second generation, to name a few are: Strength Pareto Evolutionary Algorithm (SPEA) (SZitzler, 1999), Strength Pareto Evolutionary Algorithm 2 (SPEA2) (Zitzler *et al*., 2001; Zitzler *et al*., 2002), Pareto Archived Evolution Strategy (PAES) (Knowles, 2000) and Nondominated Sorting Genetic Algorithm II (NSGA-II) (Deb et *al*., 2002).

In this work the SPEA2 package of ECJ was used for the multiobjective aspect of the optimization. The reason for this choice was twofold. On the one hand, SPEA2 and NSGA-II have shown better performance than tha others in various benchmark problems (Zitzler *et al*., 2002). On the other, the on-line avability of the package facilitates the reproducibility of the results presented in this paper.

Figure 2. Algorithm main loop

```
1:     A(0) = ∅
2:     P(0) = init_random();
3:     g = 1;
4:     eval(P(g-1));
5:     eval(A(g-1));
6:     A(g) = save(P(g-1), A(g-1));
7:     truncate(A(g));
8:     if g>g_max then stop;
9:     M(g) = select(A(g));
10:    P(g) = cross&mut(M(g));
11:    g = g+1;
12:    go to Step 4;
```

As mentioned earlier, in GAs the fitness of a solution (e.g. a particular portfolio) is a measure of how good the solution is. That is, in our particular case, the goodness of a portfolio could be measured by how much return we get an how much risk we assume. However, in multiobjective GA the concept of fitness changes. The fitness of an individual it is not anymore how well it solves the problem, but it is based completely on the Pareto optimality concept. The fitness of an individual is a function of how many individuals it dominates and by how many individuals it is dominated (for a more detailed explanation see Zitzler *et al*. (2001) and Zitzler *et al*. (2002)). Thus, nondominated individuals have the highest possible fitness and the rest of individuals are ranked according to their dominance relations.

Other important concept that is often used in GAs is the concept of elitism. In an elitist selection technique the best individuals of the population are automatically selected to go to the next generation without undergoing crossover or mutation. In the context of multiobjective GAs, the use of a subpopulation (usually called archive) where the nondominated individuals are stored along the generations guarantees that nondominated solutions are not lost during the run and that a solution reported as nondominated is nondominated with respect to any other solution generated by our algorithm.

The algorithm main loop is shown in Figure 2. The algorithm works as follows:

- In step 1 and 2 the archive, $A(g)$, where the nondominated solutions are stored and the population, $P(g)$, are initialized. $A(0)$ is an empty set and $P(0)$ is initialized at random.
- In step 3 the generation counter g is set to 1 and then the evolution loop starts.
- In step 4 and 5 the individuals in the population and the archive are evaluated.
- According to this evaluation a new archive is created in step 6 containing all the nondominated individuals found in the union of the previous archive and the population.
- If the size of the resulting archive exceeds the archive size, in step 7 the archive is truncated. This truncation method removes those individuals which are at the minimum distance of another individual. This way the characteristics of the nondominated front are preserved and outer solutions are not lost.
- The termination criterion in step 8 stops the algorithm when the number of generations has been completed.
- In step 9 tournament selection with replacement is performed in the archive set in order to fill the mating pool, $M(g)$.
- The new population, $P(g)$, is created in step 10 by applying crossover and mutation to the mating pool.
- In step 11 the generation counter is increased.

4.3. Individuals Evaluation

In order to establish the dominance relations among the individuals in the population and in the archive all the individuals need to be evaluated. During the evaluation process the return each candidate portfolio generates and its level of risk are computed.

Figure 3. Fitness evaluation

```
ind_n = norm(ind);
R = calculate(hist_data, ind_n);
expR = sum(R[i])/num_obs;
arrange(R);
position = 0.05*num_obs;
quantile = R[position];
VaR = expR - quantile;
ind_fitness[1] = set_obj1(expR);
ind_fitness[2] = set_obj2(1/VaR);
```

The evaluation method is shown in Figure 3. The evaluation method works as follows:

▪ In step 1 the individual, *ind*, is normalized. *ind* is a vector of n integers $(w_1^{GA}, w_2^{GA}, ..., w_n^{GA})$, where n is the number of assets available in the portfolio. The elements of the normalized individual, w_i, are as shown in Equation (7).

▪ In step 2 the historical series of portfolio return is calculated as $\sum_{i=1}^{n} w_i R_{ij} \quad \forall j$, where w_i is the normalized weight assigned to asset i, n is the number of assets available in the portfolio and R_{ij} is the return of asset i at time j.

▪ In step 3 the expected return of the portfolio is calculated as: $E(R) = \frac{1}{T} \sum_{j=1}^{T} R_j$, where T is the number of observations per asset.

▪ In order to calculate the empirical VaR, the vector R is arranged from highest to lowest in step 4.

▪ The position the 0.05-quantile takes is calculated in step 5 as *0.05T* (rounded if necessary).

▪ In step 6 the 0.05-quantile is set to the element in the position *0.05T* in the returns vector.

▪ The VaR is calculated in step 7 as the expected return minus the 0.05-quantile of the historic return series.

▪ In step 8 and 9 the two objective values of *ind* are set to the expected return and the inverse of the VaR (since the GA implemented maximizes the objectives).

4.4. Multiobjective GA Control Parameters

Table 1 shows the control parameters of the multiobjective GA used.

Table 1. GA control parameters

Parameter	Value
Replacement operator	Generational
Selection operator	Tournament selection
Tournament group size	7
Crossover rate	1
Mutation rate	0.05
Population size	1000
Archive size	100
Termination criterion	50 generations

5. DATA ANALYSIS

The data used in this work were extracted from the Bloomberg database. It is a set composed of twelve composite returns indices from USA, Canada, Japan, UK, France, Germany, Spain, Holland and Sweden. We employed weekly data of these indices from

January 1990 until December 2007. This provided us with 939 observations per index. the reason for employing weekly data instead of daily data was that financial managers do not restructure portfolios so frequently. Furthermore we chose weekly data instead of monthly data to avoid inaccurate VaR estimates from small samples.

Table 2. Summary statistics classified by periods

		IN-SAMPLE PERIOD PERIOD 1990-1999			OUT-SAMPLE PERIOD PERIOD 2000-2001		
Country	Index	Mean	Deviation	Skewness	Mean	Deviation	Skewness
USA	DJ Industrial	0.2702	2.036	-0.858	-0.1219	1.163	0.011
	SP500	0.2722	2.040	-0.698	-0.2270	2.396	-0.126
	Nasdaq	0.4152	2.843	-0.829	-0.6781	5.851	-0.164
Canada	SPTSX	0.1433	1.958	-0.997	-0.0805	3.092	-0.510
UK	Footsie 100	0.1940	2.100	-0.091	-0.2338	2.368	-0.244
France	CAC 40	0.2089	2.771	-0.237	-0.2349	3.135	0.121
Germany	DAX	0.2520	2.691	-0.625	-0.2623	3.325	-0.234
Spain	IBEX 35	0.2601	3.077	-0.389	-0.3236	3.192	0.065
Holland	AEX	0.3062	2.462	-0.826	-0.2716	2.733	-0.428
Sweden	OMX 30	0.3226	3.104	-0.121	-0.2843	4.056	0.016
Japan	Nikkei 225	-0.1394	3.225	0.064	-0.5161	3.482	0.349
Europe	Euro Stoxx 50	0.2819	2.433	-0.552	-0.2383	3.012	-0.032

		IN-SAMPLE PERIOD PERIOD 1992-2001			OUT-SAMPLE PERIOD PERIOD 2002-2003		
Country	Index	Mean	Deviation	Skewness	Mean	Deviation	Skewness
USA	DJ Industrial	0.2208	2.047	-0.787	0.0402	3.191	0.160
	SP500	0.1948	2.087	-0.700	-0.0328	3.158	-0.063
	Nasdaq	0.2324	3.701	-0.754	0.0272	3.875	0.010
Canada	SPTSX	0.1504	2.273	-0.973	0.0690	1.968	-0.801
UK	Footsie 100	0.1471	2.151	-0.211	-0.1513	3.189	-0.636
France	CAC 40	0.1843	2.822	-0.238	-0.2484	4.311	-0.149
Germany	DAX	0.2281	2.810	-0.647	-0.2620	5.057	-0.040
Spain	IBEX 35	0.2245	3.074	-0.343	-0.0610	3.712	-0.348
Holland	AEX	0.2361	2.581	-0.781	-0.3796	5.172	-0.219
Sweden	OMX 30	0.3133	3.270	-0.193	-0.2823	3.924	0.017
Japan	Nikkei 225	-0.1449	3.077	0.196	-0.0259	3.459	0.127
Europe	Euro Stoxx 50	0.2525	2.591	-0.503	-0.2940	4.594	-0.220

		IN-SAMPLE PERIOD PERIOD 1994-2003			OUT-SAMPLE PERIOD PERIOD 2004-2005		
Country	Index	Mean	Deviation	Skewness	Mean	Deviation	Skewness
USA	DJ Industrial	0.1941	2.432	-0.438	0.0426	1.454	-0.468
	SP500	0.1643	2.465	-0.497	0.1337	1.412	-0.434
	Nasdaq	0.1857	4.003	-0.580	0.1189	2.127	-0.312
Canada	SPTSX	0.1233	2.360	-0.932	0.3063	1.478	-0.580
UK	Footsie 100	0.0486	2.431	-0.504	0.2255	1.335	-0.242
France	CAC 40	0.0835	3.205	-0.278	0.2828	1.665	-0.720
Germany	DAX	0.1076	3.476	-0.439	0.3000	2.064	-1.032
Spain	IBEX 35	0.1454	3.258	-0.308	0.3050	1.755	-1.313
Holland	AEX	0.1094	3.388	-0.619	0.2589	1.902	-0.690
Sweden	OMX 30	0.1641	3.417	-0.334	0.3854	1.982	-0.655
Japan	Nikkei 225	-0.0902	3.080	0.220	0.4114	2.244	-0.593
Europe	Euro Stoxx 50	0.1244	3.205	-0.478	0.2550	1.769	-1.002

		IN-SAMPLE PERIOD PERIOD 1996-2005			OUT-SAMPLE PERIOD PERIOD 2006-2007		
Country	Index	Mean	Deviation	Skewness	Mean	Deviation	Skewness
USA	DJ Industrial	0.1445	2.438	-0.402	0.1848	1.701	-0.653
	SP500	0.1404	2.479	-0.469	0.1344	1.714	-0.707
	Nasdaq	0.1537	4.022	-0.547	0.1479	2.165	-0.957
Canada	SPTSX	0.1676	2.342	-0.980	0.1876	1.849	-0.933
UK	Footsie 100	0.0827	2.407	-0.556	0.1295	1.992	-0.929
France	CAC 40	0.1777	3.116	-0.367	0.1642	2.207	-0.883
Germany	DAX	0.1650	3.471	-0.491	0.3682	2.284	-0.980
Spain	IBEX 35	0.2062	3.148	-0.401	0.3256	2.121	-1.126
Holland	AEX	0.1302	3.426	-0.635	0.1558	2.150	-0.854
Sweden	OMX 30	0.2011	3.362	-0.385	0.1198	2.861	-1.087
Japan	Nikkei 225	-0.0393	2.961	0.037	-0.0876	2.852	-0.530
Europe	Euro Stoxx 50	0.1654	3.207	-0.530	0.1969	2.053	-0.983

We considered four in-sample periods that consisted of ten consecutive years, two years apart from each other, that is, periods 1990-1999, 1992-2001, 1994-2003 and 1996-2005. For evaluating the out-sample capacity of the selected portfolios we chose one biannual out-sample period for each in-sample period, that is, periods 2000-2001, 2002-2003, 2004-2005 and 2006-2007. The two first two-year periods, January 2000-December 2001 and January 2002-December 2003, are bearish periods whereas the other two-year periods, January 2004-December 2005 and January 2006-December 2007, are bullish periods.

In Table 2 the data are analyzed. It can be observed that the average weekly return on the indices over the initial in-sample period 1990-1999 is positive for all indices except for Nikkei. The average weekly standard deviation in this period is equal to 2.56%. In period 2000-2001 all indices exhibit negative average weekly return. Specifically, the average weekly return on the indices is negative and equal to -0.289% and the standard

deviation is 3.24%. Years 2000 and 2001 are characterized by big losses and high volatility. Years 2002 and 2003 are characterized by minor losses and high volatility. The average weekly return is negative -0.133% and the standard deviation is 3.80%. In the last two-year periods, all indices exhibit positive average weekly return and low volatility. In years 2004 and 2005 the average weekly return is positive, 0.252%, and the standard deviation is 1.76%, whereas in years 2006 and 2007 the average weekly return is positive, 0.168%, and the standard deviation is 2.162%. It can also be observed that significant skewness is prevalent for all periods.

6. EMPIRICAL RESULTS

For starters and in order to test our multiobjective evolutionary approach, we solved Problem (6) using GAs and compared the results obtained with those generated by classical Quadratic Programme (QP). If our approach is good enough the mean square error and the mean absolute error of the deviation of N σ-optimal portfolios computed using QP, w_{QP}^{o}, and GA, w_{GA}^{o}, should be small. That is, Equations (8) and (9) should be close to zero.

$$MSE = \frac{1}{N}\sum_{o=1}^{N}(\sigma(w_{QP}^{o}) - \sigma(w_{GA}^{o}))^2 \tag{8}$$

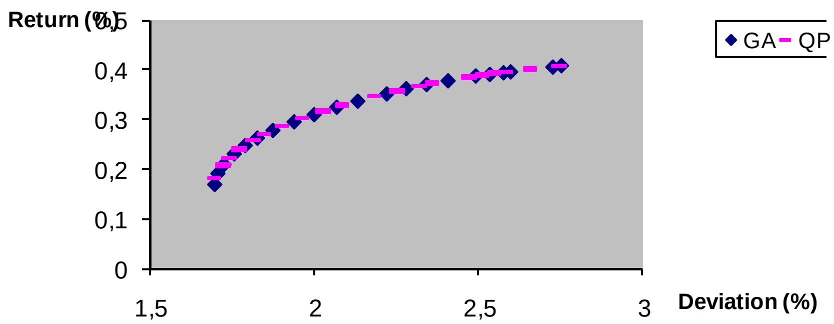
$$MAE = \frac{1}{N}\sum_{o=1}^{N}\left|\sigma(w_{QP}^{o}) - \sigma(w_{GA}^{o}))\right| \tag{9}$$

where, $\sigma(w_{QP}^{o})$ is the standard deviation of the efficient portfolio o computed using classical quadratic program for a prespecified mean return; and $\sigma(w_{GA}^{o})$ is the standard deviation of the efficient portfolio o computed using multiobjective GA.

Figure 4 shows the σ-efficient frontier obtained with QP and GA, respectively. As it is shown both are overlapped and the MSE and the MAE are close to cero. The GA calculates accurately the σ--efficient frontier, as QP does. However, the GA approach is flexible enough to solve Problem (5) while QP is not suitable at all.

So far we have proven that multiobjective GAs can easily solve Problem (6), i.e., Markowicz classical problem. Now we want to demonstrate that multiobjective GAs are able to solve Problem (5), the mean-VaR problem, and also that the solution to the σ-optimal problem does not necessarily solve the VaR problem, since the substitution of σ with some related risk measure as VaR has no sense if optimal portfolios for σ have a VaR value which is close to the optimal VaR.

Figure 4. Mean-variance efficient frontier computed with GA and QP



The portfolio that minimizes the deviation does not minimize VaR. Figures 5 to 8 show both the σ -efficient frontier obtained with QP and the GA VaR-efficient frontier obtained with GAs in the VaR-return space for in-sample periods 90-99, 92-01, 94-03 and 96-05. That is, we have plotted the VaR-efficient frontier obtained using GAs against the VaR values of the σ -optimal portfolios. The vertical axis shows the expected rate of return after a week, that is 5-trading days, in percentage points. The horizontal axis shows VaR values as a percentage of the original portfolio value.

Figure 5. Efficient frontier for mean-VaR portfolios and frontier for mean-VaR with mean-variance efficient frontiers, 1990-1999

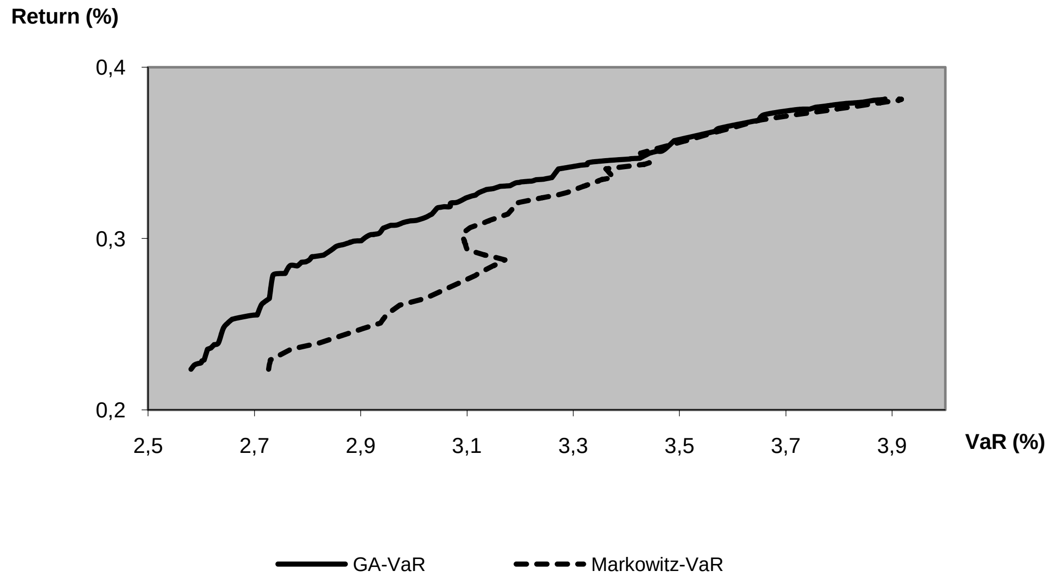


Figure 6. Efficient frontier for mean-VaR portfolios and frontier for mean-VaR with mean-variance efficient frontiers, 1992-2001

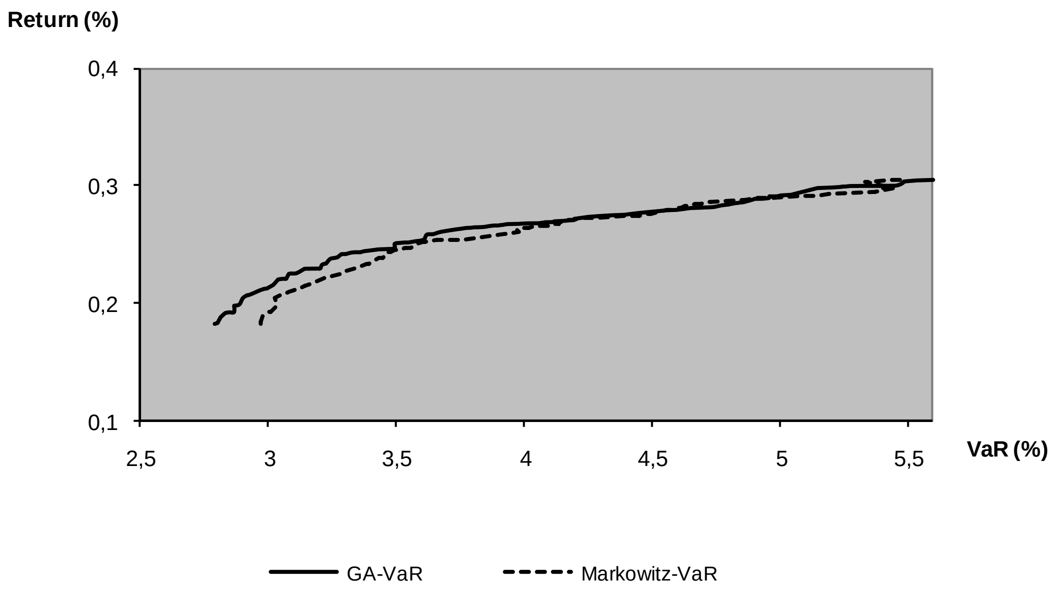
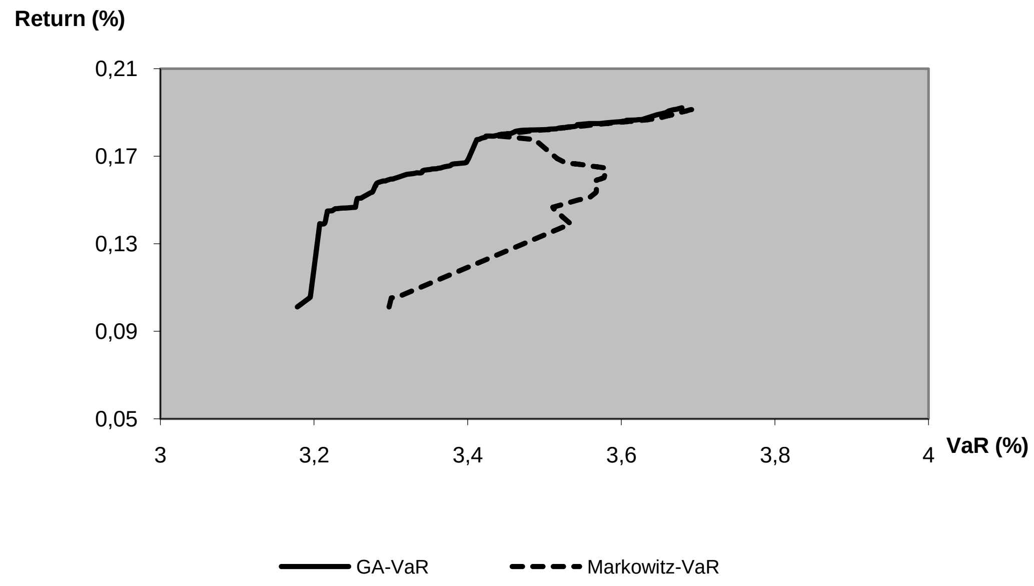
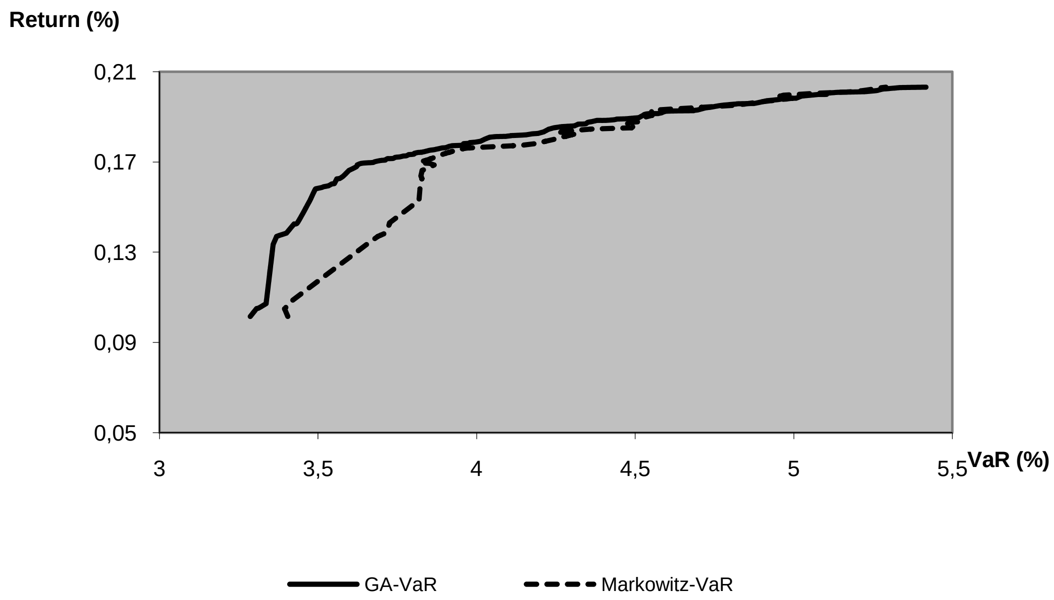


Figure 7. **Efficient frontier for mean-VaR portfolios and frontier for mean-VaR with mean-variance efficient frontiers, 1994-2003**



In period 90-99 (see Figure 5) the higher expected return the more similar the mean-variance and the mean-VaR efficient portfolios. The expected return obtained for portfolios with a VaR less than 3.4% is lower in the case of choosing the portfolio based on σ-efficient portfolios. In period 92-01 (see Figure 6) it can be observed that the differences between the efficient frontiers are not so significant (the gap is negligible). Although it should be noted that the vertical axis represents a larger range of expected returns than for other periods. Again the larger difference in expected returns appears for VaR values below 3.5%. In period 94-03 (see Figure 7) the difference between both efficient frontiers is absolutely relevant for portfolios with VaR below 3.5%. During the period 96-05 (see Figure 8) the differences between the efficient frontiers tend to disappear when portfolios with high expected return are compared. It must be highlighted that all this efficient frontiers have been obtained over ten year periods which included different conditions, high and small volatility periods and bullish and bearish markets. To sum up, we observe differences for all the periods considered. Such fact hints to use the GA algorithm in order to introduce new measures of risk.

Figure 8. Efficient frontier for mean-VaR portfolios and frontier for mean-VaR with mean-variance efficient frontiers, 1996-2005



Following Gaivoronski (2005), in order to quantify the differences between σ-optimal portfolios, w_{σ}^{o}, and VaR-optimal portfolios, w_{VaR}^{o}, we calculated the substitution error given by the largest value of VaR for some expected return. To measure this error, for each expected return value R^{*} we evaluated the VaR of the VaR-optimal portfolio, $VaR(w_{VaR}^{o})$, and the VaR of the σ-optimal portfolios, $VaR(w_{\sigma}^{o})$. That is, we computed,

$$E_{VaR}^{o} - E_{\sigma}^{o} = \left(\frac{R(w_{VaR}^{o})}{VaR(w_{VaR}^{o})} - \frac{R(w_{\sigma}^{o})}{VaR(w_{\sigma}^{o})} \right) \cdot 100 \tag{10}$$

This measure represents the relative improvement of VaR and expected return as percentage. We computed the mean of $E_{VaR}^{o} - E_{\sigma}^{o}$ and the percentage of cases in which the improvement exceed some threshold, given by θ. Furthermore, we obtained the mean squared error and the mean absolute error of the VaR of mean-VaR efficient portfolios and the VaR of the mean-variance efficient portfolios as follows,

$$MSE = \frac{1}{N} \sum_{o=1}^{N} (VaR(w_{\sigma}^{o}) - VaR(w_{VaR}^{o}))^{2} \tag{11}$$

$$MAE = \frac{1}{N} \sum_{o=1}^{N} \left| VaR(w_{\sigma}^{o}) - VaR(w_{VaR}^{o})) \right| \tag{12}$$

where N is the number of portfolios in the efficient frontier.

Table 3 compares the in-sample results. It shows that the mean square error (MSE) from Equation (11) is between 0.036 in 90-99 and 0.023 in 94-03 and the mean absolute error (MAE) from Equation (12) is between 0.16% in 90-99 and 0.10% in 94-03. That is, the annualized values are 8.32% and 5.27%, which represent annualized expected return lost per unit of risk. When we compare the σ-efficient frontier and the VaR-efficient frontier in terms of expected return per percentage of VaR we can observe that the majority of VaR-optimal portfolios are more efficient, if we measure efficiency in terms of Equation (10). Particularly, the percentage of VaR-optimal portfolios that are more efficient than σ-optimal portfolios, that is, $E_{VaR} - E_{\sigma} > 0$ %, goes from 81.21% in 96-05 to 90.33% in 94-03. The mean improvement goes from 0.134% in 96-05 to 0.429% in 90-99, which implies in annualized values 7% to 22.3%.

Table 3. Comparison of in-sample results for efficient frontiers.

	1990-1999	1992-2001	1994-2003	1996-2005
MSE	0.036087	0.02557984	0.02342789	0.02824355
MAE	0.160304	0.14469408	0.10156832	0.14065462
mean E_{VaR}-E_{σ}	0.4295	0.2174	0.1354	0.1348
% E_{VaR}-E_{σ}>-1%	100	100	100	100
% E_{VaR}-E_{σ}>-0.5%	100	100	99.39	100
% E_{VaR}-E_{σ}>0%	86.99	82.47	90.33	81.21
% E_{VaR}-E_{σ}>0.5%	37.66	1.28	0	0
% E_{VaR}-E_{σ}>1%	10.76	0	0	0

Finally, the optimal portfolios obtained using in-sample data were now tested using out-sample data. We analyzed if this efficient behavior remains out-sample. Table 4 summarizes the results.

Table 4. Comparison of out-sample results for efficient frontiers.

	2000-2001	2002-2003	2004-2005	2006-2007
mean E_{VaR}-E_{σ}	0.3438	0.5835	0.1407	0.5046
% E_{VaR}-E_{σ}>-1%	93.72	100	97.58	100
% E_{VaR}-E_{σ}>-0.5%	91.03	84.61	87.31	100
% E_{VaR}-E_{σ}>0%	75.78	64.52	61.32	79.61
% E_{VaR}-E_{σ}>0.5%	40.80	64.52	32.93	58.28
% E_{VaR}-E_{σ}>1%	23.76	45.72	0.60	8.91

As it is shown in Table 4, if we measure efficiency in terms of Equation (10), the percentage of VaR-optimal portfolios that are more efficient than σ-optimal portfolios goes from 61.23% in 04-05 to 75.78% in 00-01. The mean improvement goes from 0.14% in 04-05 to 0.58% in 02-03, which means 7.28% to 30.34% in annualized values. Hence, the mean difference is always positive both in-sample and out-sample and the variance of the improvement is bigger out-sample.

Overall, the results point out the importance of solving the mean-VaR problem using an appropriate method in order to select an efficient portfolio when investors express their market risk in function of the VaR. Multiobjective GAs have proven to be able to solve the problem. Moreover, the time needed to compute around 300 points of the efficient frontier on a 2.8 GHz Celeron CPU with 1 GB RAM is of 60 seconds. This means that the algorithm could handle a massive amount of data (if available) in reasonable computing time.

7. CONCLUSIONS

We have developed a framework for portfolio selection that moves away from convex objective functions or standard mean-variance approach where non-differential restrictions can not be imposed. In our analysis, the risk measure minimized is VaR, which leads to non-convex objective functions. We have compared the mean-variance with mean-VaR approach to measure efficiency of the classical approach when investors are worried about portfolio's potential loss function, that is, the downside risk. We have evaluated optimal VaR-efficient portfolios and optimal σ-efficient portfolios for international stock indices observed weekly over the period 1990-2007. The evaluation has been performed in-sample and out-sample. Furthermore, out-sample period consists on both bullish and bearish markets and high and low volatility periods. Results indicate reliability of VaR-efficient portfolios and significant improve over σ-efficient portfolios. Multiobjective GAs have demonstrated their adequacy for solving this problem in no-time.

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